Operations and single-particle interferometry

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Interferometry of single particles with internal degrees of freedom is investigated. We discuss the interference patterns obtained when an internal state evolution device is inserted into one or both the paths of the interferometer. The interference pattern obtained is not uniquely determined by the completely positive maps (CPMs) that describe how the devices evolve the internal state of a particle. By using the concept of gluing of CPMs, we investigate the structure of all possible interference patterns obtainable for given trace preserving internal state CPMs. We discuss what can be inferred about the gluing, given a sufficiently rich set of interference experiments. It is shown that the standard interferometric setup is limited in its abilities to distinguish different gluings. A generalized interferometric setup is introduced with the capacity to distinguish all gluings. We also connect to another approach using the well known fact that channels can be realized using a joint unitary evolution of the system and an ancillary system. We deduce the set of all such unitary "representations" and relate the structure of this set to gluings and interference phenomena.

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I. INTRODUCTION

Single-particle interferometry has been widely used to demonstrate quantum mechanical phenomena. The central question in his investigation is how interference phenomena are affected when arbitrary operations are applied to the internal degrees of freedom of the particle. Only quite recently has this question received explicit attention in the literature [1–3]. These types of studies are relevant since the transition to general operations provides a richer structure in the interference phenomena. Furthermore, general operations may give more realistic descriptions of interference experiments where noise and decoherence effects cannot be neglected [4].

It has been shown [1] that the interference patterns obtained in an interferometer are not uniquely determined by the operations applied. This calls for an investigation of what interference effects are compatible with a given pair of operations. By applying the concept of gluing [5] of completely positive maps (CPMs), we will see that it is the choice of gluing that determines the interference effects. We are thus able to describe all the interference effects compatible with given operations.

We also investigate another intuitively reasonable approach to implementing operations in an interferometer, which has been used in other investigations [1–3]. This uses the well known fact that operations can be realized using joint unitary evolution with the system and an ancillary system. Here we investigate the relation between this approach and the gluing concept in order to clarify how the choice of joint unitary evolution affects the interference.

The above questions treat the problem of what interference patterns are compatible with given channels. We also turn the question around and ask what information the interference experiments can reveal about the gluing. It is shown that the ordinary interferometric setup has only a limited capacity to determine the gluing. However, it is shown that it is possible to construct a generalized interferometer for which there is a bijective correspondence between gluings and interference effects. In Ref. [5] a complete characterization of all possible trace preserving gluings of given channels was developed. The generalized interferometer provides us with a way to determine these gluings. As such it opens up for experimental investigations of these types of problems.

The structure of this article is the following. In Sec. II the model for the two-path interferometer is introduced. Here we also make the basic questions of this investigation more precise. In Sec. III the interferometer is discussed in terms of the gluing concept. By application of the theory developed in Refs. [5–7], all possible trace preserving gluings are expressed. In Sec. IV we deduce all possible interference effects compatible with given channels. Moreover, we investigate what can be inferred about an unknown gluing by performing interference experiments. In Sec. V, a generalization of the interferometric setup is introduced. It is shown that this generalized interferometer has the power to determine arbitrary unknown trace preserving gluings of two arbitrary known channels. Section VI connects the unitary representation approach with the gluing approach, by translating results from Refs. [6,7] to the present context. In Sec. VII all unitary representations of given channels are deduced. The structure of this set is investigated in terms of gluings, which makes it possible to select arbitrary gluings of a channel and an identity channel by a choice of unitary representation. In Sec. VIII the nature of the nonuniqueness of interference effects and gluings is discussed. The conclusions are presented in Sec. IX.

II. THE TWO-PATH INTERFEROMETER

The spatial degree of freedom of the interferometer is modeled as a two-dimensional Hilbert space $\mathcal{H}_s$, spanned by $|1\rangle$ and $|2\rangle$, which correspond to the particle being localized in paths 1 and 2, respectively. The internal Hilbert space is

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denoted $\mathcal{H}_I$ and the total Hilbert space is $\mathcal{H}_s \otimes \mathcal{H}_I$.

The interferometer consists of three parts: First, a "preparation stage," consisting of a 50-50 beam splitter that creates a superposition of the particle in the two paths; second, an "interaction stage," where the state of the particle is affected; Last, the "measurement stage," where a variable phase shifter is inserted into one of the paths, followed by a second beam splitter, and finally a detector that determines the presence or nonpresence of the particle in one of the outgoing paths.

We regard the preparation stage of the interferometer only as a way to create special types of states on the two paths. If the particle is initially in path 2 and the internal state is represented by the density operator ρ_I, the first beam splitter creates a state of the form $\rho_i = |\psi\rangle\langle\psi| \otimes \rho_I$, where $|\psi\rangle = (1/\sqrt{2})(|1\rangle + |2\rangle)$. This first beam splitter, as well as the second beam splitter, is modeled by the unitary operator $U_{bs} = (1/\sqrt{2})(|1\rangle\langle 1| + |1\rangle\langle 2| - |2\rangle\langle 1| + |2\rangle\langle 2|)$.

In the interaction stage the total state ρ_i of the particle may change into some new state ρ_f. This state is thereafter analyzed in the measurement stage. We return to the interaction stage below and focus for a moment on the measurement stage. The phase shifter is described by the unitary operator $U_{ps} = |1\rangle\langle 1| + e^{i\chi}|2\rangle\langle 2|$, where χ is a real number. The probability of finding the particle in path 1, after the second beam-splitter, is [8,9]

$$
\begin{aligned}
p_1 &= \mathrm{Tr}[(|1\rangle\langle 1| \otimes \hat{1}_I) U_{bs} U_{ps} \rho_f U_{ps}^\dagger U_{bs}^\dagger] \\
&= \frac{1}{2} + |E| \cos[\arg(E) - \chi],
\end{aligned} \tag{1}
$$

where $E = \langle 1|\mathrm{Tr}_I(\rho_f)|2\rangle$, and where $\hat{1}_I$ is the identity operator on $\mathcal{H}_I$. Thus, the effect of the measurement stage is to measure the off-diagonal element of the reduced density operator of the spatial degree of freedom, in the $\{|1\rangle, |2\rangle\}$ basis. The absolute value $|E|$ and the argument $\arg(E)$ determine the visibility and the phase shift, respectively, of the interference pattern.

In the interaction stage some operation acts on the total state of the particle. Here the words "operation" and "channel" are synonymous with a trace preserving completely positive map [10]. The operation is described by a channel Φ_{tot} that maps the initial total state ρ_i to the final total state $\rho_f = \Phi_{tot}(\rho_i)$.

Suppose we have a device that can evolve the internal state of a particle sent through it. The action of this device is described by the channel Φ_1. What is the interference pattern if this device is inserted into path 1? One may be tempted to answer that the interference pattern should be uniquely determined by the channel Φ_1. This is, however, not the case [1]. The channel Φ_1 does not provide sufficient information to determine the interference pattern. The root of this phenomenon is that the total channel Φ_{tot} is not uniquely determined by Φ_1 [5]. One way to put this is to say that the internal state channel Φ_1 is not a "complete" description of the evolution device when it is to act in a path of an interferometer. The following example may clarify the situation.

What is the channel describing a phase shifter? Since the only effect of the phase shifter is to add an overall phase, it is the identity CPM. If we prepare particles, let them pass a phase shifter, and then measure the state of the outgoing particles, the phase shifter has no measurable effect. However, when inserted into the interferometer, the effect of the phase shifter is visible as a constant phase shift in the interference pattern. Hence, the channel describing the phase shifter, regarded as a device on its own, is not a sufficient description of the phase shifter when acting inside an interferometer.

In this investigation, we wish to find all possible interference patterns compatible with given internal state evolution channels. We approach this problem from two different directions. The first approach is to note that the total channel Φ_{tot} can be regarded as a subspace preserving gluing [5] of Φ_1 acting in path 1 and the identity CPM acting in path 2. We also consider more general situations with a nontrivial evolution device in each path of the interferometer. These questions are discussed in Secs. III–V, where we also discuss what can be inferred about unknown gluings from interference experiments.

The second approach is to use the well known fact that channels can be realized using joint unitary evolution of the system and an ancillary system [10] as

$$
\Phi_1(\rho_I) = \mathrm{Tr}_a(U_{Ia} \rho_I \otimes |a\rangle\langle a| U_{Ia}^\dagger), \tag{2}
$$

where $\mathcal{H}_a$ is the Hilbert space of the ancillary system and $|a\rangle$ is a normalized state of the ancilla. A reasonable method to create an operation Φ_{tot} would be the following: Let U_{sIa} be the unitary operator acting on $\mathcal{H}_s \otimes \mathcal{H}_I \otimes \mathcal{H}_a$ as

$$
U_{sIa} = |1\rangle\langle 1| \otimes U_{Ia} + |2\rangle\langle 2| \otimes \hat{1}_I \otimes \hat{1}_a. \tag{3}
$$

In words, this means that if the particle passes path 1 the ancilla interacts with the particle. If it passes path 2, then nothing happens. The total evolution of the particle would then be

$$
\rho_f = \Phi_{tot}(\rho_i) = \mathrm{Tr}_a(U_{sIa} \rho_i \otimes |a\rangle\langle a| U_{sIa}^\dagger). \tag{4}
$$

If we assume that the initial total state is created with a beam splitter $\rho_i = |\psi\rangle\langle\psi| \otimes \rho_I$, the interference is determined by [2]

$$
E(\rho_I) = \frac{1}{2} \mathrm{Tr}(\langle a|U_{Ia}|a\rangle \rho_I). \tag{5}
$$

E is a function from the set of internal state density operators ρ_I to the set of complex numbers. We refer to this function as the *interference function*.

At first sight this procedure may seem as a straightforward way to calculate the interference phenomenon caused by a given channel Φ_1. However, the operator U_{Ia}, which we use to represent the internal state evolution device, is not unique. There exist several unitary operators that realize Φ_1 via Eq. (2). The choice of U_{Ia} affects the interference effect, as the following example shows.

Suppose we have a channel Φ_1 and a representation U_{Ia} of this channel which gives a nontrivial interference function E. Suppose the internal Hilbert space is of dimension $N < +\infty$. It follows that there exists some Kraus representation of Φ_1

with at most N^2 elements [6]. Hence, there exist operators V_k on $\mathcal{H}_I$ such that $\Phi_1(\rho_I)=\sum_{k=1}^{N^2} V_k\rho_I V_k^\dagger$. Assume an ancilla system with Hilbert space $\mathcal{H}_a$ of dimension N^2+1. Let $\{|a\rangle,|a_1\rangle,\ldots,|a_{N^2}\rangle\}$ be an orthonormal basis of $\mathcal{H}_a$. On $\mathcal{H}_I\otimes\mathcal{H}_a$ one can construct the following operator:

$$U'_{Ia} = \hat{1}_I\otimes\hat{1}_a - \hat{1}_I\otimes|a\rangle\langle a| - \sum_{j,l=1}^{N^2} V_jV_l^\dagger\otimes|a_j\rangle\langle a_l| + \sum_{j=1}^{N^2} V_j\otimes|a_j\rangle\langle a| + \sum_{j=1}^{N^2} V_j^\dagger\otimes|a\rangle\langle a_j|. \tag{6}$$

One can verify that U'_{Ia} is a unitary operator, and also that Φ_1 is obtained if U'_{Ia} is inserted into Eq. (2), instead of U_{Ia}. Hence, U_{Ia} and U'_{Ia} realize the same CPM Φ_1. A global unitary operator U'_{sIa} can be constructed, as in Eq. (3), but with U'_{Ia} replaced by U_{Ia}. With U'_{sIa}, a modified global operation Φ'_{tot} can be constructed via Eq. (4). For this new operation the interference function satisfies $E'(\rho_I)=0$ for every ρ_I. Hence, there are no interference fringes for any input state. In other words, we have constructed two evolution devices which give the same internal state evolution, but which nevertheless give rise to two different interference effects. This example shows that we may choose to set the visibility to zero. In Sec. VII it is shown that the choice of U_{Ia} may affect the interference in more general ways. This may be of relevance for studies like [2], where the relative phase for CPMs is defined in terms of unitary representations.

III. GLUINGS

In this section we introduce the main tool, gluing of channels, which we will use to analyze the interferometer. We here give a brief overview of the concepts developed in Refs. [5–7], and translate two results from Ref. [5], which will be needed in the subsequent analysis.

A device, whose effect on the internal state of a particle sent through it, is described by a channel Φ_1. Likewise, another device is described by a channel Φ_2. These devices are inserted, one in each path of the interferometer. The question is, what is the "global" channel Φ_{tot} that describes the total operation the single particle has experienced when passing the two devices? The total Hilbert space of the interferometer can be decomposed into the orthogonal subspaces $\mathrm{Sp}\{|1\rangle\}\otimes\mathcal{H}_I$ and $\mathrm{Sp}\{|2\rangle\}\otimes\mathcal{H}_I$, each representing pure states localized in one of the paths. (Sp denotes the linear span.) If the particle is localized in path 1, then channel Φ_1 operates on the internal degree of freedom. If the particle is localized in path 2, then channel Φ_2 is effected. The set of all trace preserving gluings of the two channels Φ_1 and Φ_2 is precisely the set of all possible total channels Φ_{tot} compatible with Φ_1 and Φ_2 [5]. The set of trace preserving gluings of two channels is the same as the set of *subspace preserving* (SP) gluings of these two channels [5].

In the present context, the set of SP channels has a rather simple conceptual interpretation. With respect to the two paths of the interferometer, a global channel Φ_{tot} is SP if and only if it causes no transport of probability weight between the two paths. More precisely, the channel Φ_{tot} is subspace preserving if and only if $\mathrm{Tr}[|1\rangle\langle 1|\otimes\hat{1}_I\Phi_{tot}(\rho)]=\mathrm{Tr}(|1\rangle\langle 1|\otimes\hat{1}_I\rho)$ for all density operators ρ on $\mathcal{H}_s\otimes\mathcal{H}_I$.

The following proposition is a translation of a general result on SP gluings [5] to the situation considered here.

Proposition 1. Let Φ_1 be a channel with linearly independent Kraus representation $\{V_n\}_{n=1}^N$ and let Φ_2 be a channel with linearly independent Kraus representation $\{W_m\}_{m=1}^M$. All trace preserving gluings of Φ_1 and Φ_2 can be written

$$\Phi_{tot}(\rho) = |1\rangle\langle 1|\otimes\sum_{n=1}^{N} V_n\langle 1|\rho|1\rangle V_n^\dagger + |2\rangle\langle 2|\otimes\sum_{m=1}^{M} W_m\langle 2|\rho|2\rangle W_m^\dagger + |1\rangle\langle 2|\otimes\sum_{n,m} C_{n,m}V_n\langle 1|\rho|2\rangle W_m^\dagger + |2\rangle\langle 1|\otimes\sum_{n,m} C_{n,m}^* W_m\langle 2|\rho|1\rangle V_n^\dagger \tag{7}$$

for all density operators ρ on $\mathcal{H}_s\otimes\mathcal{H}_I$, where the matrix $C=[C_{n,m}]_{n=1,m=1}^{N,M}$ satisfies the condition

$$I_N \geqslant CC^\dagger, \tag{8}$$

where I_N is the $N\times N$ identity matrix. Moreover, Eq. (7) defines a bijection between the set of trace preserving gluings and the set of $N\times M$ matrices C that satisfy Eq. (8).

We will in the following refer to the matrix C, of the above proposition as the *gluing matrix*. Note that the choice of linearly independent Kraus representations does not affect the set of gluings. The Kraus representations play only the role of a "reference" in terms of which we can describe the gluing using the gluing matrix. When changing the linearly independent Kraus representations, the new and the old gluing matrices are related as $C'=U_1CU_2^\dagger$, where U_1 and U_2 are unitary matrices relating the old Kraus representations to the new ones [6].

It is to be noted that the above proposition is not formulated correctly from a technical point of view. It is stated that the CPMs Φ_1 and Φ_2 are glued. To be correct we should first construct a CPM $\bar{\Phi}_1$ acting on density operators on $\mathrm{Sp}\{|1\rangle\}\otimes\mathcal{H}_I$. If the original CPM has Kraus representation $\{V_n\}_n$, then $\bar{\Phi}_1$ has Kraus representation $\{|1\rangle\langle 1|\otimes V_n\}_n$. Similarly, one can construct $\bar{\Phi}_2$. To be correct, it is $\bar{\Phi}_1$ and $\bar{\Phi}_2$ that are glued. However, since the difference between Φ_1 and $\bar{\Phi}_1$ is purely technical, we do not make any distinction between them here.

The set of channels given by Proposition 1 is rather "allowing" in the sense that it includes cases where the two devices may interact or share correlated resources during the operation. If one wishes to model two *independent* devices, restrictions have to be imposed on the set of gluings. In Ref. [7] the concept of *subspace locality* has been introduced. Subspace locality is intended to describe a total operation Φ_{tot} which is composed from two independent operations, each acting on one location, without any need for communication or sharing of correlated resources, and where the two locations are associated with orthogonal subspaces, rather than a tensor product decomposition. The following proposition is a translation of a general result on subspace local gluings [5], to the present conditions. The set of subspace local gluings are called *local subspace preserving* (LSP) gluings.

Proposition 2. Let Φ_1 be a channel with linearly independent Kraus representation $\{V_n\}_{n=1}^N$ and let Φ_2 be a channel with linearly independent Kraus representation $\{W_m\}_{m=1}^M$. All LSP gluings of Φ_1 and Φ_2 can be written

$$\Phi_{tot}(\rho) = |1\rangle\langle 1| \otimes \sum_{n=1}^{N} V_n\langle 1|\rho|1\rangle V_n^\dagger + |2\rangle\langle 2| \otimes \sum_{m=1}^{M} W_m\langle 2|\rho|2\rangle W_m^\dagger + |1\rangle\langle 2| \otimes V\langle 1|\rho|2\rangle W^\dagger + |2\rangle\langle 1| \otimes W\langle 2|\rho|1\rangle V^\dagger \qquad (9)$$

for all density operators ρ on $\mathcal{H}_s \otimes \mathcal{H}_I$, where

$$V = \sum_{n=1}^{N} c_{1,n}V_n, \quad W = \sum_{m=1}^{M} c_{2,m}W_m, \qquad (10)$$

where the vectors $c_1=[c_{1,n}]_{n=1}^N$ and $c_2=[c_{2,m}]_{m=1}^M$ satisfy the conditions

$$\|c_1\|^2 = \sum_n |c_{1,n}|^2 \leqslant 1, \quad \|c_2\|^2 = \sum_m |c_{2,m}|^2 \leqslant 1.$$

Moreover, if a total channel Φ_{tot} can be written as above, then it is a LSP gluing of Φ_1 and Φ_2.

Note that the vectors c_1 and c_2 are not uniquely determined by the LSP gluing, but the gluing matrix $C=c_1c_2^\dagger$ is.

The most simple example of a gluing is the gluing of two identity channels (which is also an example of an LSP gluing). The total CPM is

$$\Phi_{tot}(\rho) = |1\rangle\langle 1| \otimes \langle 1|\rho|1\rangle + |2\rangle\langle 2| \otimes \langle 2|\rho|2\rangle + re^{i\phi}|1\rangle\langle 2| \otimes \langle 1|\rho|2\rangle + re^{-i\phi}|2\rangle\langle 1| \otimes \langle 2|\rho|1\rangle. \qquad (11)$$

In this case the gluing matrix is reduced to a single complex number $c=re^{i\phi}$, with $0\leqslant r\leqslant 1$. Although the two channels are identity channels, there is still a freedom in the choice of gluing. Suppose the input state of channel (11) is $\rho_i = |\psi\rangle\langle\psi| \otimes \rho_I$ with $|\psi\rangle=(1/\sqrt{2})(|1\rangle+|2\rangle)$. The output state is $\rho_f=|1\rangle\langle 1|\otimes\rho_I+|1\rangle\langle 2|\otimes\rho_I+re^{i\phi}|1\rangle\langle 2|\otimes\rho_I+re^{-i\phi}|2\rangle\langle 1|\otimes\rho_I$. The smaller r, the smaller is the "coherence" between the two paths. Although we have two identity channels we may nevertheless completely destroy the coherence by setting $r=0$. Hence, in this case the effect of the gluing is a relative phase shift and some degree of destruction of coherence between the two paths.

IV. DETERMINING THE GLUING

So far we have considered only the structure of the set of gluings on the two paths of the interferometer. We now turn to the interference effects caused by these channels. Here we obtain expressions for all possible interference effects compatible with given channels. Moreover, we investigate what interference experiments may tell us about unknown gluings.

To make the analysis as clear as possible, we assume the input states to be of the form $\rho_i=|\psi\rangle\langle\psi|\otimes\rho_I$ with $|\psi\rangle=(1/\sqrt{2})(|1\rangle+|2\rangle)$. This is the type of states created by the beam splitter, as described in the Introduction.

Assume channels Φ_1 and Φ_2 and consider the possible total channels Φ_{tot} given by Proposition 1. One can deduce the interference function E to be

$$E(\rho_I) = \frac{1}{2}\mathrm{Tr}(R\rho_I), \qquad (12)$$

where

$$R = \sum_{n,m=1}^{N,M} C_{n,m}W_m^\dagger V_n, \qquad (13)$$

with $C_{n,m}$ as in Proposition 1. In the more restrictive case of LSP gluings, given by Proposition 2, one obtains

$$R = \sum_{n,m=1}^{N,M} c_{1,n}c_{2,m}^{*}W_m^\dagger V_n, \qquad (14)$$

with the vectors c_1 and c_2 as in Proposition 2.

We have now found all the possible interference effects compatible with two given channels. As seen, all possible choices of interference effects can be reached by some choice of gluing matrix C. As seen the interference effects are determined by the gluings, not the channels *per se*.

Since the gluing determines the interference effect this means that the interference experiment gives us information about the gluing at hand. This means that, if we have an unknown gluing, we might possibly use the interferometer to reveal what gluing we have. In the following we investigate to what extent this is possible.

Assuming the internal state channels Φ_1 and Φ_2 are known, what can be said about the gluing from the interference experiments? Since the interference function E is linear, it follows that E is determined by its values on a set of internal states forming a basis of $\mathcal{L}(H_I)$, where $\mathcal{L}(H_I)$ denotes the set of all linear operators on $\mathcal{H}_I$. If $\{|n\rangle\}_{n=1}^N$ is an ON-basis of $\mathcal{H}_I$, then the set of density operators $\{|n\rangle\langle n|\}_n \cup \{|\psi_{nn'}\rangle\langle\psi_{nn'}|,|\chi_{nn'}\rangle\langle\chi_{nn'}|\}_{n,n':n>n'}$, where $|\psi_{nn'}\rangle=(1/\sqrt{2})(|n\rangle+|n'\rangle)$, $|\chi_{nn'}\rangle=(1/\sqrt{2})(|n\rangle+i|n'\rangle)$, is such a basis. Given such a set of interference experiments, the function E, and by that the operator R, can be determined. But the task is not to find R, but the gluing matrix C. From Eq. (13) it can be seen that if $\{W_m^\dagger V_n\}_{n,m=1}^{N,M}$ is a linearly independent set; then the coefficients $C_{n,m}$ are determined by R. Hence, we can conclude the following.

Proposition 3. Let the CPM Φ_{tot} be a trace preserving gluing of two channels with linearly independent Kraus representations $\{V_n\}_{n=1}^N$ and $\{W_m\}_{m=1}^M$, respectively. If the set $\{W_m^\dagger V_n\}_{n,m=1}^{N,M}$ is linearly independent, then the gluing matrix C is uniquely determined by the interference function E.

It is not always necessary to run the experiment over a basis of density operators of $\mathcal{L}(H_I)$. All information attainable is extracted for a set of density operators spanning the subspace $\mathrm{Sp}\{W_m^\dagger V_n\}_{n,m=1}^{N,M}$. Note also that Proposition 3 is about the specific type of setup considered here. As is shown in Sec. V one can construct generalized interference experiments that give more information. One may further note that this proposition gives only a sufficient condition. It is an open question whether or not it is also a necessary condition. The condition (8) may possibly cause some cases to be uniquely determined in spite of a linearly dependent set $\{W_m^\dagger V_n\}_{n,m=1}^{N,M}$. Additional constraints, such as restriction to LSP gluings, may possibly help to determine the gluing.

The following examples illustrate various situations that may arise. If one of the devices to be glued is the identity CPM, then $R=\sum_{m=1}^{M} c_{1,1}c_{2,m}^{*}W_m^\dagger$. Since the set $\{W_m\}_{m=1}^{M}$ is linearly independent, it follows that the gluing matrix (which now is a $1\times M$ matrix) with $C_{1,m}=c_{1,1}c_{2,m}^{*}$ is uniquely determined. Hence, the gluing, in sense of the gluing matrix, is uniquely determined. We can conclude the following.

Proposition 4. Let Φ_{tot} be a trace preserving gluing of a channel Φ_1 and an identity channel. The gluing matrix C of Φ_{tot}, with respect to some linearly independent Kraus representation of the channel Φ_1, is uniquely determined by the interference function E.

Although this is a special case it is a rather important one. Physically it corresponds to a situation where we have a "black box" inserted into one of the paths of the interferometer. Using the interferometer we can investigate evolution caused by this black box. What Proposition 3 tells us is that the ordinary interferometer is sufficient to fully explore this black box, with respect to the gluing property. These aspects will be discussed further in Sec. VIII.

As a second example consider devices Φ_1 and Φ_2, with linearly independent Kraus representations $\{|\psi_1\rangle\langle n|\}_{n=1}^{N}$ and $\{|\psi_2\rangle\langle m|\}_{m=1}^{N}$, respectively. Both $|\psi_1\rangle$ and $|\psi_2\rangle$ are normalized, and $\{|n\rangle\}_{n=1}^{N}$ is some orthonormal basis of $\mathcal{H}_I$. These two devices have the effect of taking arbitrary internal states to the pure states $|\psi_1\rangle\langle\psi_1|$ and $|\psi_2\rangle\langle\psi_2|$, respectively. If $|\psi_1\rangle=|\psi_2\rangle$ then $W_m^\dagger V_n=|m\rangle\langle n|$. The set $\{|m\rangle\langle n|\}_{m,n=1}^{N}$ is linearly independent and the gluing can be completely determined. If, on the other hand, the two output states $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal, then $W_m^\dagger V_n=0$, and nothing can be inferred about the gluing. One may note that in this case the interference function E is identically zero and there are no interference fringes.

There are cases when it is possible to partially infer the gluing matrix. Let both channels Φ_1 and Φ_2 have the linearly independent Kraus representation $\{|n\rangle\langle n|\}_{n=1}^{N}$. This corresponds to devices that set all off-diagonal elements in the $\{|n\rangle\}_{n=1}^{N}$ basis to zero, but leave the diagonal elements intact. One finds that $W_m^\dagger V_n=\delta_{mn}|n\rangle\langle n|$. Hence, the diagonal elements $C_{n,n}$ can be determined, but not the off-diagonal elements. This example also demonstrates that it is not always necessary to run the interference experiments on a basis of density operators spanning the whole of $\mathcal{L}(H_I)$. Here it is sufficient to run the experiment for a set spanning the subspace $\mathrm{Sp}\{|n\rangle\langle n|\}_{n=1}^{N}$.

With these examples we clearly see that this interferometer cannot distinguish all gluings. Moreover, we see that its abilities to recognize the gluings depend on which channels are glued. Although the interferometer is sufficient in the special case given by Proposition 4, it is problematic as an experimental tool if one wishes to investigate what gluings of general type are present in an evolution mechanism.

In the above examples one may recognize a distant analogy with the problem of an undefined noncyclic geometric phase, because of vanishing visibility when the interfering states are orthogonal. In Refs. [11–13] the concept of an off-diagonal geometric phase is introduced, which in some sense extracts more phase information. In Sec. V a generalized interferometer is introduced, which has the ability to completely determine arbitrary gluings; this seems vaguely analogous to the idea behind the off-diagonal geometric phase. Note, however, that the geometric phase is based on given initial states, while here we consider channels.

V. GENERALIZED INTERFEROMETRY

It is disturbing that the interferometric setup has only a limited capacity to determine the gluing. Here it is shown that there exists a generalization of the interference setup, with the capacity to completely determine any trace preserving gluing of any pair of channels.

The standard two-path interferometer determines a detection probability as a function of a variable phase shift in one of the paths. This variable phase shift can be regarded as a family of unitary operators acting on the internal state. This suggests a generalization, namely, to find the probability as a function of all unitary operators acting on one of the paths, not only the subfamily of phase shifts.

In very much the same way as described in Sec. II we consider a setup with a beam splitter creating an input state $\rho_i=|\psi\rangle\langle\psi|\otimes\rho_I$, followed by an interaction stage with two evolution devices acting according to some gluing. Then follows a variable unitary operator U in one path, acting on the total state as $|1\rangle\langle 1|\otimes U+|2\rangle\langle 2|\otimes\hat{1}_I$. Finally, there is the second beam splitter and a measurement of location of the particle. Much as in Sec. II, one finds that the probability of finding the particle in path 1, after the final beam splitter, is

$$p_1=\frac{1}{2}+|G(U,\rho_I)|\cos\{\arg[G(U,\rho_I)]\},\tag{15}$$

$$G(U,\rho_I)=\frac{1}{2}\sum_{n,m=1}^{N,M}C_{n,m}\mathrm{Tr}(W_m^\dagger U V_n\rho_I).\tag{16}$$

Although not needed in principle, it may be convenient to add a variable phase shifter to obtain $p_1=\frac{1}{2}+|G(U,\rho_I)|\cos\{\arg[G(U,\rho_I)]-\chi\}$. This means that for a specific choice of ρ_I and U one performs ordinary interference experiments to determine $G(U,\rho_I)$. We call G the *generalized interference function*. One may note that $E(\rho_I)=G(\hat{1}_I,\rho_I)$.

One may wonder if it is not possible to generalize this setup even further. What if another unitary operator U' is applied to the second path? Moreover, one may apply unitary operators $\overline{U}$ and $\overline{U'}$ to the two paths *before* the action of the two evolution devices. However, this does not provide any more information than does G. The generalized interferometer, as described above, has the power to distinguish all trace preserving gluings of two known channels.

Lemma 1 and Proposition 5 below are formulated in slightly more general settings than in the rest of this investigation. Here we allow the internal state channels to have output on a Hilbert space $\mathcal{H}_T$ different from the input Hilbert space $\mathcal{H}_S$. We say that the channels have *source space* $\mathcal{H}_S$ and *target space* $\mathcal{H}_T$ [6]. This means that the interferometer might start with one type of system on the input side, but end in another type of system on the output side. Propositions 1 and 2 both remain true under this generalization, with the modification that the total channel Φ_{tot} has source space $\mathcal{H}_s \otimes \mathcal{H}_S$ and target space $\mathcal{H}_s \otimes \mathcal{H}_T$. The variable unitary operator U in the generalized interferometer, as described above, operates on the target space.

Lemma 1. Let $\{V_k\}_{k=1}^K$ and $\{W_{k'}\}_{k'=1}^K$ be two bases (not necessarily orthonormal) of $\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$. The set of linear maps $\{\eta_{kk'}\}_{k,k'=1}^K$, where the elements are defined as $\eta_{kk'}(Q) = W_{k'}^\dagger Q V_k$, $\forall\, Q \in \mathcal{L}(\mathcal{H}_T)$, is a basis of $\mathcal{L}(\mathcal{L}(\mathcal{H}_T), \mathcal{L}(\mathcal{H}_S))$.

In this lemma $\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$ denotes the set of all linear mappings from $\mathcal{H}_S$ to $\mathcal{H}_T$. The proof of this lemma is very similar to a proof in Ref. [6]. There it is proved that the set $\{\phi_{kk'}\}_{k,k'=1}^K$, defined by $\phi_{kk'}(Q) = V_k Q V_{k'}^\dagger$, is a basis of $\mathcal{L}(\mathcal{L}(\mathcal{H}_S), \mathcal{L}(\mathcal{H}_T))$, if $\{V_k\}_k$ is a basis of $\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$.

Proposition 5. Let the CPM Φ_{tot} be a trace preserving gluing of two channels Φ_1 and Φ_2. The gluing matrix C of Φ_{tot}, with respect to some linearly independent Kraus representations of the channels Φ_1 and Φ_2, is uniquely determined by the generalized interference function G.

The procedure described here can be said to be a process tomography of the channel Φ_{tot} [14–17], but with some *a priori* information on the process; since we already have the information on which channels are glued, and wish to determine the gluing.

Proof. The function $G(U, \rho_I)$ can be written $G(U, \rho_I) = \frac{1}{2}\mathrm{Tr}[F(U)\rho_I]$, with $F(U) = \sum_{n,m=1}^{N,M} C_{n,m} W_m^\dagger U V_n$. For each fixed U the operator $F(U)$ can be determined, given the values of $G(U, \rho_I)$ on a set of density operators forming a basis of $\mathcal{L}(\mathcal{H}_S)$.

It is always possible to find a basis of $\mathcal{L}(\mathcal{H}_T)$ consisting of unitary operators [18]. Since F is a linear map, it is determined by how it maps such a basis. Hence, if G is known, the function F is known.

Both $\{V_n\}_{n=1}^N$ and $\{W_m\}_{m=1}^M$ are linearly independent. From these we construct two bases $\{\tilde{V}_n\}_{n=1}^K$ and $\{\tilde{W}_m\}_{m=1}^K$ of $\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$, by adding linearly independent elements. We add these elements in such a way that the first $N(M)$ elements are $\{V_n\}_{n=1}^N$ $(\{W_m\}_{m=1}^M)$. The unknown matrix C is extended such that $C_{m,n} = 0$ if $m > M$ or if $n > N$. With these extensions all the conditions of Lemma 1 are satisfied. Hence, the matrix C is uniquely determined, since it is formed by the expansion coefficients of F, with respect to the basis $\{\eta_{kk'}\}_{k,k'=1}^K$. ■

One can note another approach to constructing an interferometer to determine the gluing matrix. In this alternative setup the initial internal state ρ_I is fixed, and instead there are two variable local unitary operators: $\bar{U}$ before and U after the evolution devices. This arrangement results in another interference function $\bar{G}_{\rho_I}(U, \bar{U})$. If both the variable unitary operators act in path 1, then $\bar{G}_{\rho_I}(U, \bar{U}) = \sum_{nm} C_{n,m} \mathrm{Tr}(W_m^\dagger U V_n \bar{U} \rho_I)$. With an appropriate choice of the initial internal state ρ_I, the function $\bar{G}$ can determine arbitrary gluings. By using the following lemma, which is stated without proof, one can show that acceptable initial states have nonsingular density operators.

Lemma 2. Let ρ be a density operator on $\mathcal{H}$. There exists a set of unitary operators $\{U_k\}_{k=1}^K$ such that $\{U_k \rho\}_{k=1}^K$ is a basis of $\mathcal{L}(\mathcal{H})$ if and only if ρ is nonsingular.

One may note that the maximally mixed state is an acceptable choice of initial state, while a pure state is not.

Although the function $\bar{G}$ or other similar constructions in principle give the same information as G, there may be other aspects that may make one preferable compared to the others. Apart from the question of difficulties of experimental realization, there are also questions about statistics and sensitivity to errors. These questions are not addressed here, but below we will see another type of consideration where the choice of setup does matter.

We here relate the material in this and the previous section to some measures introduced in Ref. [1]. These measures relate, through a certain construction, the visibility in an interferometer to Kraus representations of two given channels inserted into the interferometer. The dependence on the choice of Kraus representations one can recognize as the different choices of LSP gluings of the given channels. That the gluings are LSP can be seen by comparing the construction in Ref. [1] with Proposition 7, in the next section. In Ref. [1] the *coherent fidelity* $\mathcal{F}_c$ between two Kraus representations is defined as the visibility in the ordinary interferometer, when the initial internal state is maximally mixed. In the language used here, $\mathcal{F}_c$ is the visibility caused by a LSP gluing of the two given channels. Hence, $\mathcal{F}_c(\Phi_1, \Phi_2, C) = 2|E(1/N\hat{1}_I)|$, where C is a LSP gluing matrix $C = c_1 c_2^\dagger$, with respect to some arbitrary choices of linearly independent Kraus representations. In Ref. [1] the maximal coherent fidelity is defined as the the maximum of $\mathcal{F}_c$ over all possible pairs of Kraus representations of the two channels. This can be recognized as the maximum of $2|E(1/N\,\hat{1}_I)|$ over all possible LSP gluings of Φ_1 and Φ_2. In Ref. [1] it is also determined what is the closest unitary channel to a given Kraus representation of a channel. The closest unitary operator is defined as the one giving the largest visibility for the maximally mixed state as input state, when the operation acts in one path and the unitary operator acts in the other path. The maximal visibility so reached can be recognized as the maximum of $2|G(U, 1/N\,\hat{1}_I)|$ over all unitary U, for a fixed LSP gluing of the channel and the identity channel.

Using the generalized interferometer one might define several different measures in the same spirit as in Ref. [1]. In doing this, one must be aware that the setup may matter in a nontrivial way. We have seen that the two setups leading to G and $\bar{G}$ are equivalent in their abilities to determine gluings. However, when defining measures based on maximizing visibilities, these two setups, as well as other constructions, may give different answers. As an example one may consider $A(\Phi_1, \Phi_2, C) = \sup_{U,\rho_I}|G(U, \rho_I)|$, which corresponds to the maximal visibility over all unitary shifts and initial internal states. If we restrict to LSP gluings one can deduce that $A(\Phi_1, \Phi_2, C) = \frac{1}{2}\sup_{\|\psi\|=1}\|V|\psi\rangle\|\|W|\psi\rangle\|$, with V and W as in Proposition 9. One may consider another setup, which is the same as the construction leading to $\overline{G}$, with the only modification that we also admit variations of the initial internal state. The corresponding interference function is $\widetilde{G}(U,\overline{U},\rho_I)=\frac{1}{2}\Sigma_{nm} C_{nm}\mathrm{Tr}(W_m^{\dagger}UV_n\overline{U}\rho_I)$. Clearly, knowledge of $\widetilde{G}$ is sufficient to determine the gluing. In this sense, $\widetilde{G}$ is equivalent to G. In analogy with the function A one may consider $B(\Phi_1,\Phi_2,C)=\sup_{U,\overline{U},\rho_I}|\widetilde{G}(U,\overline{U},\rho_I)|$. One can show that, in the case of LSP gluings, $B(\Phi_1,\Phi_2,C)=\frac{1}{2}\sup_{\|\psi\|=1}\|V|\psi\rangle\|\sup_{\|\chi\|=1}\|W|\chi\rangle\|=\frac{1}{2}\|V\|\|W\|$. There exist LSP gluings for which $A(\Phi_1,\Phi_2,C)\neq B(\Phi_1,\Phi_2,C)$. One example is if both Φ_1 and Φ_2 have Kraus representation $\{|1\rangle\langle 1|,|2\rangle\langle 2|\}$, where $\{|1\rangle,|2\rangle\}$ is an orthonormal basis of a two-dimensional $\mathcal{H}_I$. We assume that the LSP gluing is such that $V=|1\rangle\langle 1|$ and $W=|2\rangle\langle 2|$. In this case $A(\Phi_1,\Phi_2,C)=\frac{1}{4}$ and $B(\Phi_1,\Phi_2,C)=\frac{1}{2}$. Hence, for these types of questions the choice of interference setup matters.

VI. UNITARY REPRESENTATION OF GLUINGS

In this section we connect the gluing approach with the approach using unitary channels acting on combinations of the system and ancillary systems. In Refs. [6,7] it has been shown that for a special class of CPMs the property of being SP or LSP can be characterized in terms of unitary actions on system-ancilla combinations. The present setting of a two-path interferometer belongs to this special class of CPMs.

The following proposition is a translation of a proposition in Ref. [6] to the specific condition considered here.

Proposition 6. A channel Φ_{tot} is SP on $(\mathrm{Sp}\{|1\rangle\}\otimes\mathcal{H}_I,\mathrm{Sp}\{|2\rangle\}\otimes\mathcal{H}_I)$ if and only if there exists an ancilla space $\mathcal{H}_a$, a normalized state $|a\rangle\in\mathcal{H}_a$, and unitary operators U_1 and U_2 on $\mathcal{H}_I\otimes\mathcal{H}_a$ such that

$$\Phi_{tot}(\rho)=\mathrm{Tr}_a(U\rho\otimes|a\rangle\langle a|U^{\dagger}) \tag{17}$$

for all density operators ρ on $\mathcal{H}_s\otimes\mathcal{H}_I$, where

$$U=|1\rangle\langle 1|\otimes U_1+|2\rangle\langle 2|\otimes U_2. \tag{18}$$

Note that every trace preserving gluing of two channels is a SP gluing. Vice versa, every SP channel is a trace preserving gluing of two channels [5]. The two channels that are glued are Φ_1 and Φ_2, which are obtained from U_1 and U_2, respectively, through Eq. (2). To see this, note that if the particle is localized in path 1 with internal state ρ_I, then the result of the mapping Φ_{tot} is again localized in path 1, but with the new internal state $\Phi_1(\rho_I)$.

The following gives a similar construction for LSP channels, and is a translation of a proposition in Ref. [7] to the present context.

Proposition 7. A channel Φ_{tot} is LSP on $(\mathrm{Sp}\{|1\rangle\}\otimes\mathcal{H}_I,\mathrm{Sp}\{|2\rangle\}\otimes\mathcal{H}_I)$ if and only if there exist Hilbert spaces $\mathcal{H}_{a1}$,$\mathcal{H}_{a2}$, normalized vectors $|a1\rangle\in\mathcal{H}_{a1}$, $|a2\rangle\in\mathcal{H}_{a2}$, a unitary operator U_1 on $\mathcal{H}_I\otimes\mathcal{H}_{a1}$, and a unitary operator U_2 on $\mathcal{H}_I\otimes\mathcal{H}_{a2}$ such that

$$\Phi_{tot}(\rho)=\mathrm{Tr}_{a1,a2}(U\rho\otimes|a1\rangle\langle a1|\otimes|a2\rangle\langle a2|U^{\dagger}), \tag{19}$$

for all density operators ρ on $\mathcal{H}_s\otimes\mathcal{H}_I$, where

$$U=|1\rangle\langle 1|\otimes U_1\otimes\hat{1}_{a2}+|2\rangle\langle 2|\otimes U_2\otimes\hat{1}_{a1}. \tag{20}$$

Comparing Propositions 6 and 7, one can see the difference. For SP gluings the system of interest interacts with one and the same ancilla system, while for the LSP gluings there are two ancillary systems. If the particle passes path 1, it interacts only with ancilla 1, while leaving ancilla 2 untouched, and the other way around if the particle passes path 2.

In the special case of a gluing of a channel and an identity channel, Eqs. (19) and (20) are unnecessarily complicated. In this case all possible gluings, which necessarily are LSP, can be reached using only one ancillary system. In the next section we will see that every such gluing can be written as in Eq. (4) with a joint unitary operator as in Eq. (3), for a suitably chosen ancillary space.

VII. UNITARY REPRESENTATION OF CHANNELS

As exemplified in the Introduction, one may use a joint unitary evolution with an ancilla system to implement a channel in one of the paths of the interferometer. It was also shown that the choice of unitary representation may affect the interference effects. From Secs. III and IV we know that it is the gluing that determines the interference effects. Moreover, from the previous section we know that every gluing can be expressed through such unitary representations. Hence, there must exist some connection between the choice of unitary representation and the resulting gluing. The material in the previous sections does not provide us with any explicit relation between the unitary representations and the resulting gluing. Here we establish such a relation, in the special case of gluings of a channel and an identity channel. Ultimately we will obtain a strategy to determine which gluing a given unitary representation gives rise to. Vice versa, if we have a specific gluing of a channel and an identity channel which we wish to implement, we will have means to select unitary operators that create precisely this gluing. This may be of use in theoretical investigations as well as in design of actual physical realizations.

We wish to find the relation between the unitary representation of a channel Φ_1 and the LSP gluing Φ_{tot} which this representation gives rise to, as described in the Introduction. To do this we first deduce an expression for the set of all unitary representations of a given channel Φ_1, which then is related to the LSP gluings. The only limiting assumption is that the Hilbert space of the internal degree of freedom and the ancillary Hilbert space are finite dimensional. The strategy to be used is that every unitary representation U_{Ia} can be decomposed into two complementary partial isometries R and W, where R, say, contains the "gluing information." By using this decomposition, an equivalence relation can be defined on the set of unitary representations, which tells if these can be distinguished or not in the interferometer. The equivalence classes correspond to the different LSP gluings.

Let Φ_1 be a trace preserving CPM. Let $\mathcal{H}_a$ be finite dimensional, and let $|a\rangle\in\mathcal{H}_a$ be normalized. We let $\mathbb{U}(\Phi_1,\mathcal{H}_a,|a\rangle)$ denote the set of all unitary operators U_{Ia} which represent Φ_1 via Eq. (2).

The *Kraus number* $K(\Phi_1)$ of a CPM Φ_1 is the number of operators in a linearly independent Kraus representation of Φ_1 [6]. One can see that $\mathbb{U}(\Phi_1,\mathcal{H}_a,|a\rangle)$ is empty if $K(\Phi_1)>\dim(\mathcal{H}_a)$.

If an operator $R\in\mathcal{L}(\mathcal{H})$ satisfies $R^\dagger R=P_i$ and $RR^\dagger=P_f$, where P_i and P_f are projectors onto two subspaces of $\mathcal{H}$, then R is a *partial isometry* [19]. We say that the projector P_i projects onto the *initial space* of R. Likewise we say that P_f projects onto the *final space* of R. One may note that the subspaces onto which P_i and P_f project are of the same dimension. In the following we let $P_i^\perp$ denote the complementary projector to P_i and similarly with $P_f^\perp$ and P_f.

Lemma 3. Let $\{V_k\}_{k=1}^K$ be a linearly independent Kraus representation of a trace preserving CPM Φ_1. Let $\{|a_1\rangle,\ldots,|a_K\rangle\}$ be an orthonormal set of K elements in an at least K-dimensional space $\mathcal{H}_a$. Then the operator

$$R=\sum_{k=1}^{K} V_k\otimes|a_k\rangle\langle a| \tag{21}$$

is a partial isometry.

To prove this lemma one has to show that

$$P_i=\hat{1}\otimes|a\rangle\langle a|,\quad P_f=\sum_{kk'} V_kV_{k'}^\dagger\otimes|a_k\rangle\langle a_{k'}| \tag{22}$$

are projectors. We state without proof the following lemma.

Lemma 4. Let U be a unitary operator on $\mathcal{H}$. Let P_i and P_f be projectors onto two subspaces of equal dimension. If P_fUP_i is a partial isometry, then $P_f^\perp UP_i^\perp$ is a partial isometry and

$$U=P_fUP_i+P_f^\perp UP_i^\perp. \tag{23}$$

Here we introduce some notation. Let $\mathcal{H}_a$ be at least K dimensional. Let $\mathbb{A}_K$ denote the set of all ordered K-tuples $(|a_1\rangle,\ldots,|a_K\rangle)$ of pairwise orthonormal elements in $\mathcal{H}_a$. Note that two elements $\overline{a},\overline{a'}\in\mathbb{A}_K$ are equal if and only if $|a_k\rangle=|a_k'\rangle$, $k=1,\ldots,K$.

Let $\{V_k\}_{k=1}^K$ be a linearly independent Kraus representation of some channel. Given $\overline{a}\in\mathbb{A}_K$, let $\mathcal{R}_{\overline{a}}$ denote the range of the operator R as defined in Eq. (21). By Lemma 3, it follows that R is a partial isometry. The initial space of R is $\mathcal{H}\otimes\mathrm{Sp}\{|a\rangle\}$ and the final space is $\mathcal{R}_{\overline{a}}$. Let $\mathbb{W}_{\overline{a}}$ denote the set of partial isometries on $\mathcal{H}_I\otimes\mathcal{H}_a$ with initial space $(\mathcal{H}_I\otimes\mathrm{Sp}\{|a\rangle\})^\perp$ and final space $\mathcal{R}_{\overline{a}}^\perp$.

Proposition 8. Let $\{V_k\}_{k=1}^K$ with $K=K(\Phi_1)$ be a linearly independent Kraus representation of the channel Φ_1. Let $\mathcal{H}_a$ be at least K dimensional and let $|a\rangle\in\mathcal{H}_a$ be normalized. Then

$$U_{Ia}=W+\sum_{k=1}^{K} V_k\otimes|a_k\rangle\langle a| \tag{24}$$

defines a bijection between the set $\mathbb{U}(\Phi_1,\mathcal{H}_a,|a\rangle)$ and the set of all pairs $(\overline{a},W)$ with $\overline{a}\in\mathbb{A}_K$ and $W\in\mathbb{W}_{\overline{a}}$.

Proof. First it is proved that if $\overline{a}\in\mathbb{A}_K$ and $W\in\mathbb{W}_{\overline{a}}$ then the operator U_{Ia} defined by Eq. (24) belongs to $\mathbb{U}(\Phi_1,\mathcal{H}_a,|a\rangle)$. One can verify that U_{Ia}, so defined, is unitary since it is a sum of two complementary partial isometries. Moreover, one can verify that U_{Ia} represents Φ_1 via Eq. (2). Hence, $U_{Ia}\in\mathbb{U}(\Phi_1,\mathcal{H}_a,|a\rangle)$.

It has to be shown that if $U_{Ia}\in\mathbb{U}(\Phi_1,\mathcal{H}_a,|a\rangle)$ then there exist $\overline{a}\in\mathbb{A}_K$ and $W\in\mathbb{W}_{\overline{a}}$, which give U_{Ia} via Eq. (24). Let $\{|b_l\rangle\}_{l=1}^N$ be an arbitrary orthonormal basis of $\mathcal{H}_a$. It follows that $\{W_l\}_{l=1}^N$ with $W_l=\langle b_l|U_{Ia}|a\rangle$ is a Kraus representation of Φ_1. Let $\{V_k\}_{k=1}^K$ be a linearly independent Kraus representation of Φ_1. It is well known [20] that any two Kraus representations can be connected through a unitary matrix, where the Kraus representation with the smaller number of elements is padded with zero operators in such a way that the two sets have the same number of elements. Note that the set $\{W_l\}_{l=1}^N$ has at least as many elements as $\{V_k\}_{K=1}^K$, since the last is a linearly independent Kraus representation [6]. The existence of a unitary matrix connecting padded sets of Kraus operator is equivalent to the existence of an $N\times K$ matrix M such that $W_l=\sum_{k=1}^K M_{lk}V_k$, for all $l=1,\ldots,N$, and such that $M^\dagger M=I_K$, where I_K denotes the $K\times K$ identity matrix. Define the set $\{|a_k\rangle\}_{k=1}^K$ by $|a_k\rangle=\sum_{l=1}^N M_{lk}|b_l\rangle$ for $k=1,\ldots,K$. One can verify that $\{|a_k\rangle\}_{k=1}^K$ is an orthonormal set. Let P_i and P_f be defined as in Eq. (22). Using the fact that Φ_1 is trace preserving, one can verify that $P_fU_{Ia}P_i=R$, with R defined as in Eq. (21). Since $\{|a_k\rangle\}_{k=1}^K$ and $\{V_k\}_{k=1}^K$ satisfy the properties required by Lemma 3, it follows that R is a partial isometry. By Lemma 4 it follows that U_{Ia} can be written as in Eq. (24) with $W=P_f^\perp U_{Ia}P_i^\perp$. By Lemma 4 it follows that W is a partial isometry with the correct initial and final spaces.

Finally, it has to be shown that if two pairs $(\overline{a},W)$ and $(\overline{a'},W')$ are different, then the corresponding operators U_{Ia} and U_{Ia}' are different. Assume these two pairs are mapped to the same U. Then

$$W-W'=\sum_{k=1}^{K} V_k\otimes(|a_k'\rangle-|a_k\rangle)\langle a|. \tag{25}$$

The operator $W-W'$ maps elements in $\mathcal{H}_I\otimes\mathrm{Sp}\{|a\rangle\}$ to the zero element. Similarly, $\sum_{k=1}^K V_k\otimes(|a_k'\rangle-|a_k\rangle)\langle a|$ maps elements in $\mathcal{H}_I\otimes\mathrm{Sp}\{|a\rangle\}^\perp$ to the zero element. Hence, from Eq. (25) it follows that $W-W'=0$ and $\sum_{k=1}^K V_k\otimes(|a_k'\rangle-|a_k\rangle)\langle a|=0$. Let $|\chi\rangle\in\mathcal{H}_a$ be arbitrary. By applying $\langle\chi|$ "from the left" and $|a\rangle$ "from the right" onto the last expression, one obtains $\sum_{k=1}^K(\langle\chi|a_k'\rangle-\langle\chi|a_k\rangle)V_k=0$. By linear independence of $\{V_k\}_{k=1}^K$, and the arbitrariness of $|\chi\rangle$ it follows that $\overline{a'}=\overline{a}$. Hence, no two distinct pairs are mapped to the same unitary operator. ■

Using the interferometric setup, as described in the Introduction, two unitary representations U_{Ia} and U_{Ia}' are distinguishable in the interferometer, if and only if the corresponding interference functions E and E' are different. From Eq. (5) and Proposition 8 it follows that the interference function is $E(\rho_I)=\frac{1}{2}\sum_{k=1}^K\langle a_k|a\rangle\mathrm{Tr}(V_k\rho_I)$. Because of the linear independence of $\{V_k\}_{k=1}^K$, two unitary representations are distinguishable if and only if the corresponding vectors $(\langle a_k|a\rangle)_{k=1}^K$ and $(\langle a_k'|a\rangle)_{k=1}^K$ are different. Since $\{V_k\}_{k=1}^K$ is a linearly independent Kraus representation of Φ_1, it follows that $(\langle a_k|a\rangle)_{k=1}^K$ can be identified with the $1\times K(\Phi_1)$ gluing matrix C. As shown in Sec. IV, the gluing matrix is uniquely determined by the interference function E, for this type of gluing. From this it follows that two unitary representations U_{Ia} and U'_{Ia} are distinguishable in the interferometer if and only if they correspond to different LSP gluings of the channel Φ_1 and the identity channel. Another way to put this is to say that $\mathbb{U}(\Phi_1, \mathcal{H}_a, |a\rangle)$ can be equipped with an equivalence relation $\sim$. Two unitary representations are equivalent, $U_{Ia} \sim U'_{Ia}$, if $(\langle a_k|a\rangle)_{k=1}^K = (\langle a'_k|a\rangle)_{k=1}^K$. As we have seen, this is equivalent to being indistinguishable by the interferometer, which is the same as saying that they correspond to the same LSP gluing of Φ_1 and the identity channel.

Since the gluing matrix C, in the present case, is only a row (or column) matrix, it can be regarded as a vector. If this vector C satisfies $\|C\|=1$, we say that the LSP gluing is *maximal*.

Using Eqs. (3) and (4), it is possible to define a mapping $\mathcal{M}$ from the set $\mathbb{U}(\Phi_1, \mathcal{H}_a, |a\rangle)$ to the set of LSP gluings of the channel Φ_1 and the identity channel.

Proposition 9. Let Φ_1 be a channel. Let $\mathcal{H}_a$ be finite dimensional and $|a\rangle \in \mathcal{H}_a$ normalized.

(1) If $\dim(\mathcal{H}_a) < K(\Phi_1)$ then $\mathbb{U}(\Phi_1, \mathcal{H}_a, |a\rangle)$ is empty.

(2) If $\dim(\mathcal{H}_a) = K(\Phi_1)$ then $\mathcal{M}$ defines a bijection between the set of equivalence classes under $\sim$ and the set of maximal LSP gluings of Φ_1 and the identity channel.

(3) If $\dim(\mathcal{H}_a) > K(\Phi_1)$ then $\mathcal{M}$ defines a bijection between the set of equivalence classes under $\sim$ and the set of LSP gluings of Φ_1 and the identity channel.

In essence this proposition says that if the dimension of the Hilbert space of the ancilla is equal to the Kraus number of the channel Φ_1, then we reach precisely the maximal LSP gluings through the unitary representations. If the dimension of the ancillary Hilbert space is strictly larger than the Kraus number, then we reach all LSP gluings of Φ_1 and the identity channel.

Proof. The first statement follows since $K(\Phi_1)$ is the minimal number of elements in any Kraus representation of Φ_1 [6].

For the second statement, assume $U_{Ia} \in \mathbb{U}(\Phi_1, \mathcal{H}_a, |a\rangle)$. By using Proposition 8 and Eqs. (3) and (4), one finds that the gluing matrix is $C = [\langle a_k|a\rangle]_{k=1}^K$. Since $\mathcal{H}_a$ is K dimensional, it follows that $\{|a_k\rangle\}_{k=1}^K$ is an orthonormal basis of $\mathcal{H}_a$. Since $|a\rangle$ is normalized, $\Sigma_k |c_k|^2 = \Sigma_{k=1}^K |\langle a_k|a\rangle|^2 = 1$. Hence, the gluing is maximal. One can see that all elements in an equivalence class are mapped to the same gluing. Moreover, two elements from different equivalence classes are mapped to different gluings.

It has to be shown that every maximal gluing can be reached via $\mathcal{M}$. Suppose we have a maximal gluing with gluing matrix C. If we regard the gluing matrix as a vector, it follows that $C \in \mathbb{C}^K$, such that $\|C\|=1$. It is always possible to find an orthonormal basis $\overline{a} = \{|a_k\rangle\}_{k=1}^K$ of $\mathcal{H}_a$, such that $C_k = \langle a_k|a\rangle$. Let U_{Ia} be defined from $\{|a_k\rangle\}_{k=1}^K$, through Eq. (24), for some arbitrary choice of $W \in \mathbb{W}_{\overline{a}}$.

For the case $\dim(\mathcal{H}_a) > K(\Phi_1)$ one can reason very similarly as above, with the modification that $\{|a_k\rangle\}_{k=1}^K$ spans a proper subspace of $\mathcal{H}_a$. This implies that the gluing matrix does not have to be maximal, and we can reach all the LSP gluings. ■

VIII. DISCUSSION

As we have demonstrated it is not the internal state channels *per se* that determine the interference pattern, but their gluings. Even if it assumed that the devices are acting independently of each other (LSP gluing), there remains an arbitrariness in the interference pattern, which corresponds to the nonuniqueness of LSP gluings. Here we concentrate on the special case of gluings of a channel Φ_1 and an identity channel. Such gluings can be described as pairs (Φ_1, V), where V is as in Eq. (10). One way to understand the nonuniqueness is to describe the state of the particle in the interferometer in terms of an occupation number representation. This describes the occupation states of the two paths, rather than the location of the particle. It is sufficient to extend the Hilbert space of the internal degree of freedom with one additional dimension spanned by a "vacuum state," which describes the nonpresence of the particle in that path [7]. The total extended Hilbert space is the tensor product of two such extended Hilbert spaces. In the case of a trace preserving gluing of a channel and an identity channel, the corresponding channel in the occupation number representation can be written as a product channel $\tilde{\Phi}_1 \otimes \tilde{I}_2$, where $\tilde{I}_2$ is the identity channel acting on operators on the extended Hilbert space of the empty path. The channel $\tilde{\Phi}_1$ takes the form [7]

$$\tilde{\Phi}_1(\tilde{\rho}) = \sum_k V_k \tilde{\rho} V_k^\dagger + V\tilde{\rho}|0\rangle\langle 0| + |0\rangle\langle 0|\tilde{\rho}V^\dagger + |0\rangle\langle 0|\tilde{\rho}|0\rangle\langle 0|, \tag{26}$$

where $|0\rangle$ is the vacuum state of the path in which $\tilde{\Phi}_1$ acts. As seen, the extended channel $\tilde{\Phi}_1$ contains the same information as the pair (Φ_1, V). To every trace preserving gluing of the channel Φ_1 and the identity channel, there corresponds a channel $\tilde{\Phi}_1$. The channel $\tilde{\Phi}_1$ describes not only what the machine does with a particle present in the input, but also what is does with superpositions of the particle and the vacuum state. For more details concerning this occupation number approach the reader is referred to Refs. [5,7].

In Sec. IV (Proposition 4) we saw that the ordinary interferometric setup has the power to determine trace preserving gluings of a channel and an identity channel. Hence, it can determine the operator V in Eq. (26). In other words, the interferometer has the capacity to reveal more about the global evolution than direct measurements, as pointed out in Ref. [1]. However, the equivalent description in terms of $\tilde{\Phi}_1$ suggests that another strategy is possible, at least in principle. If the evolution device is subjected to a process tomography on the *extended* Hilbert space, the channel $\tilde{\Phi}_1$ would be revealed and hence provide the same information as the interference experiments would. This would correspond to preparing states including linear combinations of the particle in some internal state and the vacuum state. Similarly, the measurements performed on the output has to be sufficiently rich on the extended state space. Leaving aside the question of how such states actually would be produced, and how such measurements would be performed, this means that the interferometer is not really necessary to determine trace preserving gluings of a channel and the identity channel. The same information could, in principle, be obtained with direct measurement on the output states, provided the input states and the measurements are sufficiently general on the extended Hilbert space.

IX. CONCLUSIONS

Two-path single-particle interferometry of particles with an internal degree of freedom is investigated. Given internal state evolution devices, whose action is characterized by trace preserving completely positive maps (channels), we ask how the interference phenomena are affected when such devices are inserted into the paths of the interferometer. We investigate the nonuniqueness of the interference patterns for given internal state evolution channels. This question is approached from two points of view. The first is to use the concept of gluing of completely positive maps developed in Ref. [5]. It is found that the possible interference effects are determined by the gluings, rather than the internal state channels *per se*. Using the gluing approach we deduce all possible interference effects compatible with given channels.

In the second approach we make use of the fact that channels can be realized using joint unitary evolution on a system and an ancillary system. By this approach we connect to other investigations in the literature [1–3] in which joint unitary evolution is used in interferometers. The choice of joint unitary evolution used to realize a given channel is not unique. Although two such unitary operators realize the same channel, they may cause different interference phenomena when the machine is inserted into one of the paths of the interferometer. We investigate which gluing each choice of unitary representation gives rise to, and hence which interference pattern. Conversely, if one wishes to construct a specific gluing we determine the possible choices of unitary representations which give the desired gluing. This may be of use in the design of actual physical implementations of this type of channel.

In previous work [5] the set of all possible trace preserving gluings of given pairs of channels has been deduced. Here we extend this work by investigating how interferometers can be used to analyze which gluing is actually present. It is shown that the standard interferometer in general has a limited capacity to determine the gluing. Several gluings give rise to identical interference phenomena. Due to these limitations we here introduce a generalized interferometer. It is shown that this setup has the capacity to distinguish all possible trace preserving gluings of arbitrary channels. As such this provides a tool for experimental investigations of which gluings are present in actual evolutions.

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