PHYSICAL REVIEW A **70**, 010501(R) (2004)

Optimal strategy for a single-qubit gate and the trade-off between opposite types of decoherence

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(Received 21 March 2003; published 22 July 2004)

We study reliable quantum-information processing (QIP) under two different types of environment. The first type is Markovian exponential decay, and the appropriate elementary strategy of protection of qubit is to apply fast gates. The second one is strongly non-Markovian and occurs solely during operations on the qubit. The best strategy is then to work with slow gates. If the two types are both present, one has to optimize the speed of gate. We show that such a trade-off is present for a single-qubit operation in a semiconductor quantum dot implementation of QIP, where recombination of exciton (qubit) is Markovian, while phonon dressing gives rise to the non-Markovian contribution.

DOI: 10.1103/PhysRevA.70.010501 PACS number(s): 03.67.Pp, 03.65.Yz, 03.67.Lx, 78.67.Hc

Already at the early stage of quantum-information theory, the implementation of quantum computing was found to be extremely challenging due to decoherence processes [1]. It was claimed, nevertheless, that a quantum computer could reliably perform quantum algorithms once the error per operation is below some threshold level which is $\sim\delta\approx 10^{-5}$ [2] (for some caveats coming from memory of environment see [3]). In any case, minimizing the error per gate is a reasonable strategy. Usually, one considers the decoherence time of the system τ_d and the time of performing operation (quantum gate) τ_g, so that the error is given by the ratio $\tilde{\delta}=\tau_g/\tau_d$ [4]. This suggests a search for systems with $\tilde{\delta}$ as small as possible, and the obvious strategy to diminish the error is to apply fast gates. One then tacitly assumes that the process of decoherence is independent of running the gate, which means that it is a Markovian process, where the error, to first order, grows linearly in time. Gate speed-up may be achieved by selecting materials to provide favorable spectrum characteristics [5] or by applying techniques reducing unwanted transitions within the register space [6] as well as outside this space (leakage) [7].

The assumption of Markovian character of noise is by no means obvious and, indeed, it has recently been found [3] that due to non-Markovian effects it may prove to be completely invalid. The notion of *minimal decoherence model* was introduced where the error occurs *solely* during gate operation, and it grows for fast gates as $\delta\sim 1/\tau_g^2$ (for the spectral density of the reservoir $\sim\omega^3$). An example of such a model is a degenerate system interacting with a bosonic field at vacuum via dipole interaction. This kind of error favors the counterintuitive strategy of *slowing down* the gate operation.

In real systems, this type of decoherence competes with the Markovian damping, although this effect may be covered by other errors (e.g., leakage). In this paper we indicate that such a competitive effect is essential, e.g., for the solid-state qubit implementation using excitonic (charge) states in quantum dots (QDs) [8], with computational states defined by the absence ($|0\rangle$) or presence ($|1\rangle$) of one exciton in the ground state of the dot, operated by resonant coupling to laser light. In such a system, Markovian decoherence (exponential damping) is related to exciton recombination on 1 ns timescale, while strongly non-Markovian effects result from lattice inertia. The interplay of these two decoherence mechanisms, favoring opposite strategies (fast versus slow gates), leads to a kind of trade-off, resulting in optimal speed of gate for the most reliable operation on the qubit.

To investigate the effect, we analyze decoherence in a general spin-boson system [9,10], and find decoherence to be strongly non-Markovian, fitting into the "minimal decoherence model." Assuming additional, Markovian damping, we obtain the trade-off formula for the error caused by decoherence, averaged over input states of the qubit:

$$\delta=\frac{\gamma_{nM}}{\tau_g^2}+\gamma_M\tau_g \qquad (1)$$

(actually, in the solid state example, singularity of the first term is lifted by the presence of the upper cutoff). The constants $\gamma_{M,nM}$ express the strength of the Markovian and non-Markovian decoherence, respectively. We determine their values for the exciton in a QD with typical parameters and show that the gate duration leading to minimal overall error is of the order of 1 ps. Since level spacing in such a system allows even 100 fs gating [6] the restrictions imposed by non-Markovian phonon mechanisms are decisive for the gate speed. Our result has both practical as well as general implications for QIP: (i) it suggests the proper direction in research towards semiconductor implementation of QIP, (ii) it provides a nontrivial dynamical strategy to minimize decoherence in quantum systems, for a class of reservoirs.

MINIMAL DECOHERENCE IN THE SPIN-BOSON MODEL

We consider a qubit described by means of the spin-boson model. The Hamiltonian H'' of the system plus reservoir is

$$H'' = H_S^0 + H_S(t) + H_{SR} + H_R, \tag{2}$$

where S is our qubit system, R is (bosonic) reservoir. $H_S^0 = (1/2)\Omega\sigma_z$ is the self Hamiltonian of the qubit, $H_S(t) = (1/2)\varepsilon(t)(e^{i\Omega t}\sigma_+ + e^{-i\Omega t}\sigma_-)$ is the gate Hamiltonian (in rotating wave approximation), $\varepsilon(t)$ being the shape of the pulse. $H_R = \Sigma\omega_k a_k a_k^\dagger$ is the self Hamiltonian of the reservoir, where ω_k is the boson energy. Finally, H_{SR} is the interaction with reservoir

$$H_{SR} = \sigma_z \otimes \left[\sum f_k^* a_k + \text{H.c.}\right], \tag{3}$$

where f_k is the coupling constant for the mode k.

In the interaction picture with respect to H_S^0 we get $H' = \varepsilon(t)\sigma_x + H_{SR} + H_R$. Let us now represent H' in the basis of eigenstates of the total system (dressed states). This is achieved by the unitary operation $U = |0\rangle\langle 0| \otimes W - |1\rangle\langle 1| \otimes W^\dagger$, where $W = \exp[\Sigma(f_k^*/\omega_k)a_k - \text{H.c.}]$ and $|0\rangle$, $|1\rangle$ are the eigenvectors of σ_z. In this basis, the Hamiltonian $H = U^\dagger H' U$ up to a constant has the form

$$H = \frac{1}{2}\epsilon(t)\sigma_x \cos\left(2i\sum_k \frac{f_k}{\omega_k}a_k^\dagger + \text{H.c.}\right) + \frac{1}{2}\epsilon(t)\sigma_y \sin\left(2i\sum_k \frac{f_k}{\omega_k}a_k^\dagger + \text{H.c.}\right) + H_R. \tag{4}$$

To the first order in f_k/ω_k we obtain

$$H \simeq \epsilon(t)\sigma_x + \epsilon(t)\sigma_y \otimes \left(i\sum_k \frac{f_k}{\omega_k}a^\dagger + \text{H.c.}\right) + H_R. \tag{5}$$

We define the qubit in terms of the dressed states [10]. With such a choice, the reservoir is decoupled and the interaction term is present only during gate operation. Thus our system may be reduced to the minimal decoherence model: decoherence is not present at all, if the qubit is not active. We now calculate the resulting error and relate it to the speed of gate. Since the regime is strongly non-Markovian we solve the Master equation in the Born approximation and compute the fidelity $F = 1 - \delta$ of a single-qubit operation (see Ref. [3] for details).

The error, averaged over the initial qubit states can be represented as the overlap of two functions

$$\delta = \int \frac{d\omega}{\omega^2} R(\omega) S(\omega). \tag{6}$$

Here, $R(\omega)$ is the spectral density of the reservoir

$$R(\omega) = [n_B(\omega) + 1]\sum_{\mathbf{k}} [\delta(\omega_{\mathbf{k}} - \omega) + \delta(\omega_{\mathbf{k}} + \omega)]|f_k|^2, \tag{7}$$

where $n_B(\omega)$ is the Bose-Einstein distribution. The function $S(\omega)$ fully represents the spectral characteristics of the system and is given by

$$S(\omega) = [\langle\psi|YY^\dagger|\psi\rangle - \langle\psi|Y^\dagger|\psi\rangle\langle\psi|Y|\psi\rangle]_{\text{av}} \approx \frac{1}{3}(|F_-(\omega)|^2 + |F_+(\omega)|^2), \tag{8}$$

where $[\ ]_{\text{av}}$ denotes averaging over the states ψ and

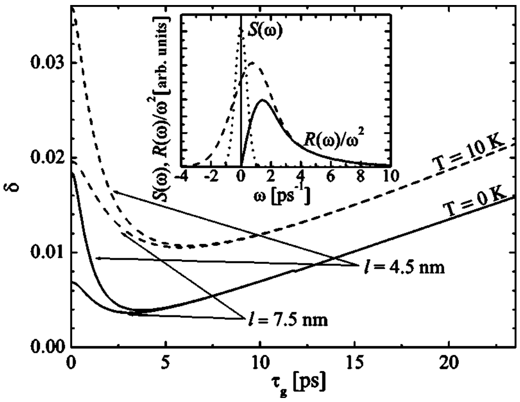

FIG. 1. The total error for an $\alpha = \pi/2$ rotation on a QD qubit, for $T=0$ (solid lines) and $T=10$ K (dashed lines), for two dot sizes ($l_h = 0.8 l_e$ and $l_z = 0.2 l_e$). The Markovian decoherence times are inferred from the experimental data [14]. Inset: Spectral density of the phonon reservoir $R(\omega)$ at these two temperatures and the gate profile $S(\omega)$ for $\alpha = \pi/2$.

$$Y = i(F_+|+\rangle\langle -| + F_-|-\rangle\langle +|), \quad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}},$$

$$F_\pm = \pm\int_{-\infty}^{+\infty} du\, e^{\pm i\phi(u)}\epsilon(u)e^{i\omega u}, \quad \phi(t) = \int_{-\infty}^{t} du\,\epsilon(u).$$

A complete minimization of δ would require full optimization of the pulse shape. However, in order to demonstrate the idea of the trade-off in simple terms, we restrict the discussion to qubit rotations performend by Gaussian pulses, $\varepsilon(t) = \alpha/(\sqrt{2\pi}\tau_g)e^{-1/2(t/\tau_g)^2}$. Here τ_g is the gate duration, while α is the angle determining the gate, e.g., $\alpha = \pi/2$ is $\sqrt{\text{NOT}}$, while $\alpha = \pi$ is σ_x (bit flip).

The functions $|F_\pm(\omega)|^2$ may be written as

$$|F_\pm(\omega)|^2 \approx \alpha^2 e^{-\tau_g^2(\omega \pm \alpha/\sqrt{2\pi}\tau_g)^2}. \tag{9}$$

Although the original minimal decoherence model with its $\sim\omega^3$ dependence appears in many physical situations, other characteristics are obviously possible. In general, as may be seen from Eqs. (6), (7), and (9), for a spectral density $R(\omega) \sim \omega^n$ the error scales with the gate duration as τ_g^{-n+1} and τ_g^{-n+2} at low and high temperatures, respectively. Therefore, for $n > 2$ (typical, e.g., for various types of phonon reservoirs [15]) the error grows for faster gates. Assuming the spectral density of the form $R(\omega) = R_0\omega^3$ (see the QD example below), we obtain

$$\delta_{nM} = \frac{1}{3}\alpha^2 R_0 \tau_g^{-2} \quad \text{at} \quad T = 0. \tag{10}$$

This leading order formula holds for $\delta \ll 1$. Also, if we introduce the upper cutoff, the error will be finite even for an infinitely fast gate [11] (see Fig. 1).

TRADE-OFF BETWEEN TWO TYPES OF DECOHERENCE

As we have shown, in our model the error grows as the speed of gate increases. This could result in obtaining arbitrarily low error by choosing suitably low speed of gates. However, if the system is also subject to other types of noise this becomes impossible. Indeed, assuming an additional contribution growing with rate γ_M, the total error per gate is

$$\delta = \frac{\gamma_{nM}}{\tau_g^2} + \gamma_M \tau_g, \quad \gamma_{nM} = \frac{1}{3}\alpha^2 R_0, \quad \gamma_M = \frac{1}{\tau_r}, \tag{11}$$

where τ_r is the characteristic time of Markovian decoherence (recombination time in the excitonic case). As a result, the overall error is unavoidable and optimization is needed. The above formulas lead to the optimal values of the form (for $T=0$):

$$\delta_{\min} = \frac{3}{2}\left(\frac{2\alpha^2 R_0}{3\tau_r^2}\right)^{1/3} \quad \text{for} \quad \tau_g = \left(\frac{2}{3}\alpha^2 R_0 \tau_r\right)^{1/3}. \tag{12}$$

We will see that the trade-off is also present for finite temperatures.

EXAMPLE: EXCITON IN A QUANTUM DOT

Let us now estimate the error magnitude for the specific semiconductor quantum dot (QD) qubit implementation [8]. The reservoir is then constituted by phonons, the most important branch for our present purpose being the longitudinal acoustical (LA), characterized by linear dispersion, $\omega_k = ck$, coupled via deformation potential (DP) [12]. In fact, the relatively simple model (2) is very accurate for a description of the QD system for ~ 1 ps timescales relevant here, as confirmed by the excellent agreement between the theoretical calculations [13] and experimental results [14]. This is due to the fact that neither the high-frequency optical phonons nor direct or phonon-induced leakage to higher states contribute considerably to the decoherence. Also effects related to piezoelectric coupling to LA and TA phonons may be neglected in weakly piezoelectric systems (e.g., GaAs). Moreover, the qubit control is all-optical, eliminating the need for additional noise-inducing device structures. On the other hand, details of the QD structure (shape, stress, composition) may lead only to quantitative modifications of secondary importance.

The non-Markovian error has a dynamical origin and is related to a which path trace left by exciting the phonon modes rather than to an influence of noise. The spectral density is uniquely defined by the lattice response characteristics: the mode-dependent coupling strength and the density of states. Due to fundamental restrictions (global charge neutrality, translational invariance), the frequency dependence is always strongly super-ohmic [15]. On the other hand, the approximate momentum conservation holding for weakly confined carrier states leads to exponential cutoff of carrier–phonon interaction at high frequencies [16]. In the specific case of DP coupling, the coupling constants are [12]

$$f_k = \frac{1}{2}\sqrt{\frac{k}{2\varrho v c}}\left(\sigma_e \mathcal{F}_k^{(e)} - \sigma_h \mathcal{F}_k^{(h)}\right), \tag{13}$$

where v is the volume of the unit cell, ρ is the crystal density, $\sigma_{e,h}$ are deformation potential constants for the electron and the hole (material parameters are taken as in [15]) and $\mathcal{F}_k^{(e,h)}$ are form factors for the corresponding wave functions (approximated by Gaussians, on the grounds of numerical diagonalization in parabolic confinement [11]),

$$\mathcal{F}_k^{(e,h)} = e^{-1/4(k_\perp^2 l_{e,h}^2 + k_z^2 l_z^2)}, \tag{14}$$

where $l_{e,h}$ and l_z are the wave function widths in the dot plane, for electron, and hole, and in the growth direction, respectively, and $k_\perp$ and k_z are the corresponding components of the phonon wave vector.

Note that the most effectively coupled phonon modes correspond to wavelengths around the dot size. For gapless bosons with linear dispersion, their frequency determines the reservoir memory times and sets up the timescale of non-Markovian effects which, in the case of the QD system, may be interpreted as "dressing" the localized carriers with coherent lattice deformation field (cf. [10]). This kind of dynamics has been described in the limit of very rapid gating pulses [11,15] and has been shown to lower the degree of coherent control over the system and to destroy the coherence of polarization oscillations [13,15]. Due to relatively large dot sizes and low sound speed the cutoff frequency (~ 1 meV) is lower than the transition energy between different carrier states in a small, self-assembled dot and the corresponding 1 ps times are experimentally accessible.

For large enough gate duration only the low-frequency sector contributes and the coupling Eq. (13) may be approximated by

$$f_k \approx \frac{1}{2}(\sigma_e - \sigma_h)\sqrt{\frac{k}{2\varrho v c}}, \tag{15}$$

leading, according to Eq. (7), to

$$R(\omega) \simeq R_0 \omega^3, \quad R_0 = \frac{(\sigma_e - \sigma_h)^2}{16\pi^2 \varrho c^5}. \tag{16}$$

Hence, Eqs. (12) are applicable and one finds for the specific material parameters of GaAs:

$$\tau_g = \alpha^{2/3} 1.47 \text{ ps}, \quad \delta_{\min} = \alpha^{2/3} 0.0035. \tag{17}$$

The full solution within the proposed model, taking into account the phonon cutoff for an anisotropic shape ($l_z < l_{e,h}$), according to Eqs. (14) and (13), and allowing finite temperatures by numerically calculating the spectral density Eq. (7), is shown in Fig. 1. The size-dependent cutoff is reflected by a shift of the optimal parameters for the two dot sizes: larger dots admit slightly faster gates and lead to lower error. Interestingly, the tradeoff becomes more apparent at nonzero temperature.

The presence of the upper cutoff could suggest applying the dynamical decoupling (DD) technique [17] to diminish decoherence. However, for high frequencies many other mechanisms of decoherence become relevant. On the other hand, combining the bounded-control version of DD [18] with the optimization proposed here might lead to some reduction of the resulting error. Such techniques might also be useful for eliminating other sources of noise, not included in our discussion.

Another possibility to reduce the decoherence effect in the QD system might be to encode qubits in decoherence-free subspaces [19]. However, the phonon wavelength corresponding to the optimal range of gating times is comparable to the single dot size, precluding the necessary collective interaction with the whole QD system. Thus, for decoherence effects which do not involve real transitions and taking into account the feasible system geometry and actual nature of phonon coupling, only a small decrease of the minimal decoherence may be expected.

In conclusion, we have exhibited the trade-off between two opposite types of decoherence: usual Markovian damping and dynamically induced non-Markovian decoherence for a realistic super-ohmic reservoir. To protect the qubit, opposite elementary strategies are needed: fast and slow gates, respectively. The minimization of the overall error leads to optimal speed of gate. We have shown that the trade-off is present in a semiconductor implementation of quantum-information processing. The Markovian error is caused by recombination, while the non-Markovian one occurs if the gate operation is not adiabatic with respect to lattice modes. We have evaluated the minimal error in this case for a single qubit gate, showing that the two processes indeed compete. The optimal gating time ($\sim$1 ps) sets up the limit beyond which any further gate speedup is unfavorable. Even at this optimal point the trade-off gives rise to a significant error. For two qubit gates involving single-qubit rotations (as proposed, e.g., in [8]), the present result gives a rough lower bound for the error, which is of 1–2 orders of magnitude higher than that admitted by fault-tolerant schemes [2] known so far ($\simeq 10^{-5}$). However, possible improvements of the latter schemes cannot be excluded. It follows also that diminishing the responsible constants γ_M (e.g., by elimination of radiative losses [20]) and γ_{nM} (by optimizing the system parameters) is the most important task towards semiconductor implementation of a quantum computer. It is also important to explore whether the same effect can occur in other implementations, as well as to what extent the error avoiding techniques may be helpful here.

The authors thank Peter Knight and Martin Plenio for useful feedback to the first version of this paper. L.J. and P.M. are grateful to T. Kuhn and V. M. Axt for discussions. This work was supported by EC Grants EQUIP (IST-1999-11053), RESQ (IST-2002-37559), SQID (IST-1999-11311), QUPRODIS (IST-2001-38877); and by the Polish Government, Grant No. PBZ-MIN-008/P03/2003. P.M. is grateful to the Humboldt Foundation for support.

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