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Phase Transition in the Dicke Model of Superradiance*

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A system of N two-level atoms interacting with a quantized field, the so-called Dicke model of superradiance, is studied. By making use of a set of Glauber's coherent states for the field, the free energy of the system is calculated exactly in the thermodynamic limit. The results agree precisely with those obtained by Hepp and Lieb, who studied the same model using a different method. The exhibition of a phase transition of the system is presented mathematically in an elementary manner in our approach. The generalization to the case of finitely many radiation modes is also presented.

I. INTRODUCTION

The problem of a system of N two-level atoms interacting with a radiation field has been studied by a number of authors.[1–6] In the so-called Dicke model, the atoms are considered to be at fixed positions within a linear cavity of volume V and the separations between the atoms are assumed to be large enough so that the direct interaction among them can be ignored. However, by the fact that the atoms interact with the same radiation field, the atoms cannot be treated as independent. The importance of treating the radiating atoms as a single quantum system was recognized by Dicke[1] who correctly described a coherent spontaneous radiation process of the system. An exact solution for the Hamiltonian of N identical two-level atoms interacting with a single-mode quantized radiation field at resonance was given by Tavis and Cummings.[2] One of the most significant and interesting results regarding the Dicke model was obtained very recently by Hepp and Lieb.[7] They calculated exactly the thermodynamic properties of the system, the free energy in particular, in the thermodynamic limit that N, $V\to\infty$, $N/V=$finite and showed that for a sufficiently large value of the coupling constant between the atoms and the field, the system exhibits a second-order phase transition from normal to superradiance at a certain critical temperature. The results of Hepp and Lieb are rigorous, but the mathematics which they used in arriving at their important results is somewhat abstract.

In this paper, we study the same problem by a rather different method. Employing a set of coherent states of the field defined by Glauber,[8] we find that the partition function of the system can be expressed in terms of an integral which is readily evaluated in the thermodynamic limit. The results, which are in exact agreement with those of Hepp and Lieb, are derived, as will be seen, in a much more straightforward manner than the method used by them. The phase-transition property of the Dicke model is shown, in our approach, to be closely analogous to that of the Curie–Weiss model of ferromagnetism given by the mean-field theory.[9] Other thermodynamic quantities such as those relevant to the study of the photon statistics of the system are also given readily by our method. The generalization to the case in which the field consists of finitely many radiation modes is straightforward.

In Sec. II, the Dicke model of superradiance is briefly reviewed. The free energy of the model is evaluated exactly and analyzed in detail in Sec. III. The generalization to the multimode case is presented in Sec. IV. Finally, a conclusion and a few remarks on other possible generalizations are given in Sec. V.

II. DICKE MODEL OF SUPERRADIANCE

First let us briefly review the Dicke model of superradiance. We consider a system of N identical two level atoms, coupled through dipole interactions with an electromagnetic field in a cavity of volume V. The atoms are kept at fixed position and the dimension V is much smaller than the wavelength of the field so that all atoms see exactly the same field.

The Hamiltonian of the model in the rotating-wave approximation[3,4] is ($\hbar = c = 1$)

$$H = H_0 + H_I , \tag{1}$$

where

$$H_0 = H_{\text{field}} + H_{\text{atoms}} = \sum_s \nu_s a_s^\dagger a_s + \tfrac{1}{2}\omega \sum_{j=1}^{N} \sigma_j^z , \tag{2}$$

$$H_I = \frac{1}{2\sqrt{V}} \left[\left(\sum_s \lambda_s' a_s \right) \left(\sum_{j=1}^{N} \sigma_j^+ \right) + \left(\sum_s \lambda_s' a_s^\dagger \right) \left(\sum_{j=1}^{N} \sigma_j^- \right) \right], \tag{3}$$

and $a_s^\dagger$ and a_s are creation and annihilation operators for the sth mode with frequency ν_s of the electromagnetic field, ω is the energy difference between the two levels of the atoms, and λ' measures the coupling between the field and the atoms. We have also introduced the Pauli spin matrices $\vec{\sigma}_j$'s to describe the two level atoms and

$$\sigma_j^+ = \sigma_j^x + i\sigma_j^y , \quad \sigma_j^- = \sigma_j^x - i\sigma_j^y . \tag{4}$$

In the thermodynamic limit that $N \to \infty$, $V \to \infty$ such that $N/V \equiv \rho =$ finite, we have

$$H_I = \frac{\sqrt{\rho}}{2\sqrt{N}} \left[\left(\sum_s \lambda_s' a_s \right) \left(\sum_{j=1}^{N} \sigma_j^+ \right) + \left(\sum_s \lambda_s' a_s^\dagger \right) \left(\sum_{j=1}^{N} \sigma_j^- \right) \right] . \tag{5}$$

For a single radiation mode of frequency ν, it is convenient to measure the energy in units of the frequency ν, and write

$$H = a^\dagger a + \sum_{j=1}^{N} \left[\tfrac{1}{2}\epsilon \sigma_j^z + (\lambda/2\sqrt{N})(a\sigma_j^+ + a^\dagger \sigma_j^-) \right] , \tag{6}$$

with

$$\epsilon = \omega/\nu , \quad \lambda = \lambda' \sqrt{\rho}/\nu , \tag{7}$$

which is the Hamiltonian of the Dicke model.

III. THERMODYNAMIC PROPERTIES

The thermodynamic functions of the Dicke model described above can be calculated from the canonical partition function $Z(N, T)$:

$$Z(N, T) = \text{Tr} e^{-\beta H} ; \quad \beta = 1/k_B T . \tag{8}$$

A convenient basis to calculate the trace of the partition function is the Glauber's coherent state[8] $|\alpha\rangle$, which has the following properties:

(i) $|\alpha\rangle$ is an eigenstate of the annihilation operator a,

$$a|\alpha\rangle = \alpha|\alpha\rangle ; \quad \langle\alpha|a^\dagger = \langle\alpha|\alpha^* . \tag{9}$$

(ii) The set of all $|\alpha\rangle$'s is complete, and

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1 . \tag{10}$$

Using $\{|\alpha\rangle\}$ for the photon field, we have

$$Z(N, T) = \sum_{s_1 = \pm 1} \cdots \sum_{s_N = \pm 1} \int \frac{d^2\alpha}{\pi} \times \langle s_1 \cdots s_N | \langle\alpha| e^{-\beta H} |\alpha\rangle | s_1 \cdots s_N \rangle , \tag{11}$$

where $\int d^2\alpha$ means $\iint d(\text{Re}\alpha)\, d(\text{Im}\alpha)$, and the sum is taken over all the atomic states.

We shall now obtain an explicit expression for the expectation value with respect to $|\alpha\rangle$ of $e^{-\beta H}$, which turns out to be very simple for the Dicke model. Let us first write the Hamiltonian in the form

$$H = \sum_{j=1}^{N} \left[\frac{a^\dagger}{\sqrt{N}} \frac{a}{\sqrt{N}} + \frac{\epsilon}{2} \sigma_j^z + \frac{\lambda}{2} \left(\frac{a^\dagger}{\sqrt{N}} \sigma_j^- + \frac{a}{\sqrt{N}} \sigma_j^+ \right) \right] . \tag{12}$$

It should be noted that H is a function of N. In the following discussion we shall write H_N for H whenever we wish to emphasize the dependence of H on N. If we write $b = a/\sqrt{N}$ and $b^\dagger = a^\dagger/\sqrt{N}$, the commutation relation of the operators b and $b^\dagger$ is given by

$$[b, b^\dagger] = (1/N)[a, a^\dagger] = 1/N . \tag{12a}$$

Using Eq. (12a), we shall show that in the thermodynamic limit the field operators appearing in each term of the expansion of $e^{-\beta H}$ can be replaced by the same operators arranged in the so called antinormal order, in which all the a's occur to the right-hand side of $a^\dagger$'s. It follows then that the expectation value $\langle\alpha|e^{-\beta H}|\alpha\rangle$ becomes

$$\langle\alpha|e^{-\beta H}|\alpha\rangle = \exp\left\{ -\beta\left[\alpha^*\alpha + \sum_{j=1}^{N} \left(\frac{\epsilon}{2} \sigma_j^z + \frac{\lambda}{2\sqrt{N}} (\alpha^*\sigma_j^- + \alpha\sigma_j^+) \right) \right] \right\} , \tag{12b}$$

for it is known that[8] the expectation value of any operator in the antinormal order such as $(a^\dagger)^r a^s$ is simply given by substituting α for a, and α^* for $a^\dagger$, namely,

$$\langle\alpha|(a^\dagger)^r a^s|\alpha\rangle = (\alpha^*)^r \alpha^s .$$

Our proof showing that the field operators in $e^{-\beta H_N}$ can be replaced by the same operators arranged in the antinormal order which we shall give in the following is based upon the following two assumptions.

Assumption i: The limits as $N \to \infty$ of the field operators $a/\sqrt{N}$ and $a^\dagger/\sqrt{N}$ exist.

Assumption ii: The order of the double limit in the exponential series

$$\lim_{N\to\infty} \lim_{R\to\infty} \sum_{r=0}^{R} (-\beta H_N)^r / r!$$

can be interchanged. We hope to provide a rigorous justification of the above assumptions in a future publication.

Consider first the limits of $a/\sqrt{N}$ and $a^\dagger/\sqrt{N}$ to be different from zero. Bearing in mind the two assumptions above, let us consider the exponential series

$$e^{-\beta H_N} = \sum_r (-\beta H_N)^r / r! \qquad (12c)$$

and suppose that we take sufficiently many but finite number of terms from the right-hand side of (12c). The field operators appearing in a typical term is of the form

$$b^\dagger b b b^\dagger \cdots b b^\dagger ,$$

where the number of factors is at most of the order of the highest power of H in (12c). To see how it can be arranged into the antinormal order we give a simple example

$$b^\dagger b b^\dagger b b^\dagger b = b^\dagger b^\dagger b^\dagger b b b + (3/N) b^\dagger b^\dagger b b + (1/N^2) b^\dagger b .$$

Each interchange of b and $b^\dagger$ introduces an extra term with an extra factor $1/N$ due to the commutation relation (12a). The number of interchanges required to put all the field operators into the antinormal order is clearly finite. By taking N sufficiently large, all the terms with the extra factors $1/N$, $1/N^2$, etc., can be made to drop out. Thus, in effect, the field operators appearing in each term of the expansion of $e^{-\beta H}$ in any order can be replaced by the same operators arranged in the antinormal order. Therefore, (12b) follows.

On the other hand if b and $b^\dagger$ tend to zero in the limit $N \to \infty$, the Hamiltonian becomes simply

$$H = a^\dagger a + \sum_{j=1}^{N} \frac{\epsilon}{2} \sigma_j^z .$$

The free energy per atom $f(T)$ of the system is then given by

$$-\beta f(T) = \ln[2\cosh(\tfrac{1}{2}\beta\epsilon)] .$$

Thus, the field does not contribute to the free energy and the ordering of the field operators makes no difference to the calculation.

We now proceed from Eq. (12b) and define

$$h_j = (\tfrac{1}{2}\epsilon)\sigma_j^z + (\lambda/2\sqrt{N})(\alpha^* \sigma_j^- + \alpha\sigma_j^+) . \qquad (13)$$

Using Eq. (12) and the property $[h_i, h_j] = 0$, we can reduce the integrand in (11) to

$$\begin{aligned} \langle s_1 \cdots s_N | \langle \alpha | e^{-\beta H} | \alpha \rangle | s_1 \cdots s_N \rangle &= e^{-\beta|\alpha|^2} \langle s_1 \cdots s_N | \exp\Big(-\beta \sum_{j=1}^{N} h_j\Big) | s_1 \cdots s_N \rangle \\ &= e^{-\beta|\alpha|^2} \langle s_1 \cdots s_N | \prod_{j=1}^{N} e^{-\beta h_j} | s_1 \cdots s_N \rangle \\ &= e^{-\beta|\alpha|^2} \prod_{j=1}^{N} \langle s_j | e^{-\beta h_j} | s_j \rangle . \end{aligned} \qquad (14)$$

From (14) and (11), we have

$$\begin{aligned} Z(N, T) &= \int \frac{d^2\alpha}{\pi} \sum_{s_1 = \pm 1} \cdots \sum_{s_N = \pm 1} e^{-\beta|\alpha|^2} \Big(\prod_{j=1}^{N} \langle s_j | e^{-\beta h_j} | s_j \rangle \Big) \\ &= \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} (\langle +1 | e^{-\beta h} | +1 \rangle + \langle -1 | e^{-\beta h} | -1 \rangle)^N \\ &= \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} (\mathrm{Tr}\, e^{-\beta h})^N , \end{aligned} \qquad (15)$$

where

$$h = (\tfrac{1}{2}\epsilon)\sigma^z + (\lambda/2\sqrt{N})(\alpha^*\sigma^- + \alpha\sigma^+) . \qquad (16)$$

Writing in the matrix form, the operator h becomes

$$h = \begin{pmatrix} \frac{1}{2}\epsilon & \lambda\alpha/\sqrt{N} \\ \lambda\alpha^*/\sqrt{N} & -\frac{1}{2}\epsilon \end{pmatrix} , \qquad (17)$$

whose eigenvalues μ's satisfy the secular equation

$$\begin{vmatrix} \frac{1}{2}\epsilon - \mu & \lambda\alpha/\sqrt{N} \\ \lambda\alpha^*/\sqrt{N} & -\frac{1}{2}\epsilon - \mu \end{vmatrix} = 0 \qquad (18)$$

or

$$\mu = \pm(\tfrac{1}{2}\epsilon)(1 + 4\lambda^2|\alpha|^2/\epsilon^2 N)^{1/2} . \qquad (19)$$

Hence substituting (19) into (15), we get

$$\begin{aligned} Z(N, T) &= \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} (e^{-\beta|\mu|} + e^{\beta|\mu|})^N \\ &= \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} \times \{2\cosh[(\tfrac{1}{2}\beta\epsilon)(1 + 4\lambda^2|\alpha|^2/\epsilon^2 N)^{1/2}]\}^N . \end{aligned} \qquad (20)$$

We note that the integrand in (20) is real and depends only upon $|\alpha|$ and the integral (20) converges as $|\alpha| \to \infty$. The free energy per atom $f(T)$ is obtained from $Z(N, T)$ by the usual formula

$$f(T) = \lim_{N\to\infty} [(1/\beta N) \ln Z(N, T)] . \qquad (21)$$

To evaluate (20), we write

$$\int \frac{d^2\alpha}{\pi} = \int_0^\infty r\,dr \int_0^{2\pi} \frac{d\theta}{\pi} = 2\int_0^\infty r\,dr . \qquad (22)$$

Equation (20) then becomes

$$Z(N, T) = 2\int_0^\infty r\,dr\, e^{-\beta r^2} \times \{2\cosh[(\tfrac{1}{2}\beta\epsilon)(1 + 4\lambda^2 r^2/\epsilon^2 N)]\}^N . \qquad (23)$$

Let $y = r^2/N$, and write

$$Z(N, T) = N\int_0^\infty dy\, e^{-N\beta y}\, 2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1 + (4\lambda^2/\epsilon^2)y]^{1/2}\} \qquad (24)$$

$$= N\int_0^\infty dy \exp\Big\{N\Big[-\beta y+\ln\Big(2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1+(4\lambda^2/\epsilon^2)y]^{1/2}\}\Big)\Big]\Big\}. \tag{25}$$

By Laplace's method,[10] integral (25) is given by

$$Z(N,T)=N\frac{C}{\sqrt{N}}\max_{0\le y\le\infty}\exp\Big\{N\Big[-\beta y+\ln\Big(2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1+(4\lambda^2/\epsilon^2)y]^{1/2}\}\Big)\Big]\Big\}. \tag{26}$$

Let

$$\phi(y)=-\beta y+\ln\Big(2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1+(4\lambda^2/\epsilon^2)y]^{1/2}\}\Big), \tag{27}$$

then

$$\phi'(y)=-\beta+(\beta\lambda^2/\epsilon)[1+(4\lambda^2/\epsilon^2)y]^{-1/2}\times\tanh\{(\tfrac{1}{2}\beta\epsilon)[1+(4\lambda^2/\epsilon^2)y]^{1/2}\}. \tag{28}$$

Putting $\phi'(y)=0$, we get

$$(\epsilon/\lambda^2)\eta=\tanh[(\tfrac{1}{2}\beta\epsilon)\eta], \tag{29}$$

where

$$\eta=[1+(4\lambda^2/\epsilon^2)y]^{1/2}, \quad 1\le\eta<\infty. \tag{30}$$

We note that in the region $0\le x<\infty$, the function $\tanh x$ is a monotonically increasing function of x as x increases and that its slope is a monotonically decreasing function of x and $\tanh x<1$ for $x<\infty$. For $\lambda^2<\epsilon$, Eq. (29) has no solution in the region $1\le\eta<\infty$. Thus, for $\lambda^2<\epsilon$, it is easy to see that the function

$$e^{-\beta y}2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1+(4\lambda^2/\epsilon^2)y]^{1/2}\}$$

is a monotonically decreasing function of y with the maximum of the function at $y=0$ equal to $2\cosh\frac{1}{2}\beta\epsilon$. Thus, the free energy per particle $f(T)$ is given by

$$-\beta f(T)=\lim_{N\to\infty}(1/N)\ln Z(N,T)=\ln[2\cosh(\tfrac{1}{2}\beta\epsilon)]. \tag{31}$$

For $\lambda^2>\epsilon$, the solution of Eq. (29) depends on the value of β. For $\beta<\beta_c$, where β_c is given by

$$(\epsilon/\lambda^2)=\tanh(\tfrac{1}{2}\beta_c\epsilon), \tag{32}$$

Eq. (29) has no solution in the region $1\le\eta<\infty$ and thus

$$-\beta f(T)=\ln[2\cosh(\tfrac{1}{2}\beta\epsilon)] \tag{33}$$

as in Eq. (31). For $\beta>\beta_c$, Eq. (29) has one (and only one) solution in the region $1\le\eta<\infty$ given by

$$(\epsilon/\lambda^2)\eta_0=\tanh[(\tfrac{1}{2}\beta\epsilon)\eta_0]. \tag{34}$$

Writing $(\epsilon/\lambda^2)\eta_0=2\sigma$ or $(\epsilon/\lambda^2)[1+(4\lambda^2/\epsilon^2)y_0]^{1/2}=2\sigma$, we have

$$y_0=\lambda^2\sigma^2-\epsilon^2/4\lambda^2$$

and

$$-\beta f(T)=\ln(\exp(-\beta y_0)\,2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1+(4\lambda^2/\epsilon^2)y_0]^{1/2}\}) \tag{35}$$

or

$$-\beta f(T)=\ln[2\cosh(\beta\lambda^2\sigma)]-\beta\lambda^2\sigma^2+\beta\epsilon^2/4\lambda^2, \tag{36}$$

where

$$2\sigma=\tanh(\beta\lambda^2\sigma)\ne0.$$

Thus, to summarize, we have, for (i), $\lambda<\epsilon^2$, no phase transition occurs in the system at any temperature. For (ii), $\lambda>\epsilon^2$, there is a critical temperature T_c given by

$$\epsilon/\lambda^2=\tanh(\tfrac{1}{2}\beta_c\epsilon), \quad \beta_c=1/kT_c$$

at which the system changes discontinuously from one state to another. The average number of photons in the two different states (the states at $T>T_c$ being called the normal state and the state $T<T_c$ being called the superradiant state) can be easily computed. More generally, let us consider the quantities $\langle(a^\dagger a/N)^r\rangle$ defined by

$$\left\langle\left(\frac{a^\dagger a}{N}\right)^r\right\rangle=\frac{\mathrm{Tr}(a^\dagger a/N)^r e^{-\beta H}}{\mathrm{Tr}e^{-\beta H}}. \tag{37}$$

It follows from our formulation that $\langle(a^\dagger a/N)^r\rangle$ is given by

$$\left\langle\left(\frac{a^\dagger a}{N}\right)^r\right\rangle=\int_0^\infty y^r dy\exp\Big\{N\Big[-\beta y+\ln\Big(2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1+(4\lambda^2/\epsilon^2)y]^{1/2}\}\Big)\Big]\Big\}\times\left(\int_0^\infty dy\exp\Big\{N\Big[-\beta y+\ln\Big(2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1+(4\lambda^2/\epsilon^2)y]^{1/2}\}\Big)\Big]\Big\}\right)^{-1}. \tag{38}$$

By Laplace's method, it immediately follows that for (i) $\lambda^2<\epsilon$ and (ii) $\lambda^2>\epsilon$, $\beta<\beta_c$

$$\langle(a^\dagger a/N)^r\rangle=\delta_{r,0}. \tag{39}$$

For (iii), $\lambda^2>\epsilon$, $\beta>\beta_c$, however, $\langle(a^\dagger a/N)^r\rangle$ is given by

$$\langle(a^\dagger a/N)^r\rangle=y_0^r=(\lambda^2\sigma^2-\epsilon^2/4\lambda^2)^r, \tag{40}$$

where σ is the root of the equation

$$2\sigma = \tanh(\beta\lambda^2\sigma) \neq 0 \;. \quad (41)$$

The above results, which are in exact agreement with those of Hepp and Lieb, are derived, as we have seen, in a manner which is straightforward and elementary compared to the method used by Hepp and Lieb. The close analogy between the Dicke model of superradiance and the Curie-Weiss model of ferromagnetism[7] is also more apparent from our formulation.

In Sec. IV, we shall deal with the case of finitely many radiation modes.

IV. GENERALIZATION TO MULTIMODE CASE

The Hamiltonian of the Dicke model, in the case of m radiation modes of frequencies $\nu_1, \nu_2, \ldots, \nu_m$, is given by

$$H = \sum_{s=1}^{m} \nu_s a_s^\dagger a_s + \sum_{i=1}^{N} \left\{ \frac{\omega}{2}\sigma_i^z + \frac{1}{2\sqrt{N}} \times \left[\left(\sum_{s=1}^{m} \lambda_s a_s \right) \sigma_i^+ + \left(\sum_{s=1}^{m} \lambda_s a_s^\dagger \right) \sigma_i^- \right] \right\}, \quad (42)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the coupling constants. In analogy with Eq. (20), the partition function of the system is now given by

$$Z = \int \cdots \int \frac{d^2\alpha_1}{\pi} \frac{d^2\alpha_2}{\pi} \cdots \frac{d^2\alpha_m}{\pi} \exp(-\beta |\nu_1\alpha_1^2 + \nu_2\alpha_2^2 + \cdots + \nu_m\alpha_m^2|) \times \left(2\cosh\{(\tfrac{1}{2}\beta\omega)[1 + (4/\omega^2 N)|\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_m\alpha_m|^2]^{1/2}\}\right)^N , \quad (43)$$

where

$$\begin{aligned} |\nu_1\alpha_1^2 + \nu_2\alpha_2^2 + \cdots + \nu_m\alpha_m^2| &= (\nu_1 x_1^2 + \nu_2 x_2^2 + \cdots + \nu_m x_m^2) + (\nu_1 y_1^2 + \nu_2 y_2^2 + \cdots + \nu_m y_m^2) , \\ |\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_m\alpha_m|^2 &= (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m)^2 + (\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_m y_m)^2 , \\ d^2\alpha_i &= dx_i\, dy_i \;. \end{aligned} \quad (44)$$

We now define a new set of variables $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$ in place of the original variables $x_1, x_2, \ldots, x_m$ by the following relations:

$$\begin{aligned} \lambda x^{(1)} &= \lambda_{11}x_1 + \lambda_{12}x_2 + \cdots + \lambda_{1m}x_m , \\ \lambda x^{(2)} &= \lambda_{21}x_1 + \lambda_{22}x_2 + \cdots + \lambda_{2m}x_m , \\ &\cdots \\ &\cdots \\ \lambda x^{(m)} &= \lambda_{m1}x_1 + \lambda_{m2}x_2 + \cdots + \lambda_{mm}x_m \;. \end{aligned} \quad (45)$$

A new set of y variables is also defined by the same set of relations. The elements in the first row of the determinant

$$\Delta = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \cdots \lambda_{1m} \\ \lambda_{21} & \lambda_{22} & \cdots \lambda_{2m} \\ \lambda_{m1} & \lambda_{m2} & \cdots \lambda_{mm} \end{vmatrix} \quad (46)$$

are chosen to be equal to the respective coupling constants $\lambda_1, \lambda_2, \ldots, \lambda_m$ so that the expression $|\lambda_1\alpha_1 + \cdots + \lambda_m\alpha_m|^2$ in Eq. (43) reduces simply to $\lambda^2(x^{(1)2} + y^{(1)2})$.

We now wish to choose the remaining elements λ_{ij} in such a way that when $\nu_1 x_1^2 + \nu_2 x_2^2 + \cdots + \nu_m x_m^2$ is expressed in terms of the new set of variables $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$, the coefficients of the cross-product terms $x^{(1)}x^{(j)}$, $j = 2, 3, \ldots, m$ vanish.

Consider the solutions of (45). We have

$$x_1 = \frac{1}{\Delta} \begin{vmatrix} \lambda x^{(1)} & \lambda_{12} \cdots \lambda_{1m} \\ \lambda x^{(2)} & \lambda_{22} \cdots \lambda_{2m} \\ \lambda x^{(m)} & \lambda_{m2} \cdots \lambda_{mm} \end{vmatrix} ,$$

$$x_2 = \frac{-1}{\Delta} \begin{vmatrix} \lambda_{11} & \lambda x^{(1)} \cdots \lambda_{1m} \\ \lambda_{21} & \lambda x^{(2)} \cdots \lambda_{2m} \\ \lambda_{m1} & \lambda x^{(m)} \cdots \lambda_{mm} \end{vmatrix} , \text{ etc.} \quad (47)$$

If we substitute (47) into $\nu_1 x_1^2 + \nu_2 x_2^2 + \cdots + \nu_m x_m^2$, it is easy to see that the coefficient of $x^{(1)}x^{(j)}$, $j \neq 1$ is given by

$$(2/\Delta^2)(\nu_1 A_{11}A_{j1} + \nu_2 A_{12}A_{j2} + \cdots + \nu_m A_{1m}A_{jm}) , \quad (48)$$

where A_{kl} is the signed cofactor of λ_{kl} in the determinant Δ. If we now choose

$$A_{1j} = c\lambda_{1j} \prod_{\substack{k=1 \\ k\neq j}}^{m} \nu_k , \quad j = 1, 2, \ldots, m \quad (49)$$

where c is any arbitrary constant, then (48) becomes

$$(2/\Delta^2)\nu_1\nu_2 \cdots \nu_m c(\lambda_{11}A_{j1} + \lambda_{12}A_{j2} + \cdots + \lambda_{1m}A_{jm}) = 0 , \quad (50)$$

because the quantity inside the bracket vanishes for $j \neq 1$. It is interesting to note that the substitution defined by (49) is all that is needed for our purpose and it is not necessary for us to determine the individual λ_{ij}, because the new variables $x^{(2)}$,

$x^{(3)}, \ldots, x^{(m)}$ and $y^{(2)}, y^{(3)}, \ldots, y^{(m)}$ occurring only in the exponential factor in (43) can now be integrated independently of the variables $x^{(1)}$ and $y^{(1)}$, giving rise to a constant which is of no physical interest. Using the substitution (49), $\nu_1 x_1^2 + \nu_2 x_2^2 + \cdots + \nu_m x_m^2$ becomes

$$\nu_1 x_1^2 + \nu_2 x_2^2 + \cdots + \nu_m x_m^2 = (1/\Delta^2)(\nu_1 A_{11}^2 + \nu_2 A_{22}^2 + \cdots + \nu_m A_{1m}^2)\lambda^2 x^{(1)^2} + \text{terms involving } x^{(i)^2}, x^{(i)}x^{(j)}, \quad i,j \neq 1. \tag{51}$$

We have similar expression for $\nu_1 y_1^2 + \nu_2 y_2^2 + \cdots$.

The coefficient of $x^{(1)^2}$ is

$$\frac{\lambda^2}{\Delta^2}\nu_1 A_{11}^2 + \nu_2 A_{12}^2 + \cdots + \nu_m A_{1m}^2 = \lambda^2 \frac{\nu_1\nu_2\cdots\nu_m(\lambda_1^2\nu_2\nu_3\cdots\nu_m + \lambda_2^2\nu_1\nu_3\cdots\nu_m + \cdots)}{(\lambda_1^2\nu_2\nu_3\cdots\nu_m + \lambda_2^2\nu_1\nu_3\cdots\nu_m + \cdots)^2} = \lambda^2\left(\frac{\lambda_1^2}{\nu_1} + \frac{\lambda_2^2}{\nu_2} + \cdots + \frac{\lambda_m^2}{\nu_m}\right)^{-1}. \tag{52}$$

Thus if we define

$$\lambda^2 = \lambda_1^2/\nu_1 + \lambda_2^2/\nu_2 + \cdots + \lambda_m^2/\nu_m, \tag{53}$$

then

$$Z = C\iint \frac{dx^{(1)}}{\sqrt{\pi}}\frac{dy^{(1)}}{\sqrt{\pi}} \exp[-\beta(x^{(1)^2} + y^{(1)^2})]\left(2\cosh\{(\tfrac{1}{2}\beta\omega)[1 + (4\lambda^2/\omega^2 N)(x^{(1)^2} + y^{(1)^2})]^{1/2}\}\right)^N$$

$$= C\int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2}\left(2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1 + (4\lambda^2/\omega^2 N)|\alpha|^2]^{1/2}\}\right)^N \quad \text{if we let } \alpha = (x^{(1)}, y^{(1)}), \tag{54}$$

where C is a constant given by the integrals over $dx^{(i)}dy^{(i)}$, $i = 2, \ldots, m$, times the Jacobian of the variable transformation. The constant C is of no physical importance as long as m is finite, for the free energy per atom $f(T)$ is given by

$$f(T) = \lim_{N\to\infty}[-(1/\beta N)\ln Z_1], \tag{55}$$

where

$$Z_1 = \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} \times\left(2\cosh\{(\tfrac{1}{2}\beta\epsilon)[1 + (4\lambda^2/\omega^2 N)|\alpha|^2]^{1/2}\}\right)^N. \tag{56}$$

We have reduced the multimode case to the single-mode case [with λ^2 in Eq. (56) given by Eq. (53)] for which the analysis of Sec. III applies.

The generalization to the multimode case was briefly mentioned by Hepp and Lieb in their paper but no explicit formula was given by them.

V. CONCLUSION

In conclusion, we have outlined an alternative approach to the thermodynamics of the Dicke model of superradiance which is considerably simpler than the formulation of Hepp and Lieb. In our approach, the phase transition is presented mathematically in an elementary manner. We have also used the same approach to analyze the phase transition property of the Dicke model in the case of many radiation modes by reducing the multimode case to the equivalent single-mode case. Some further generalizations are under consideration: (i) to relax the very unrealistic restriction of the model that the dimension of the cavity is much smaller than the wavelength of the electromagnetic field even in the thermodynamic limit, and (ii) to generalize the model to include the effects due to the motions of the atoms. These and others will be published in a later paper.

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