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Interdimensional Scaling Laws

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We consider in detail the critical behavior of a d-dimensional system that is finite in one of its dimensions. Such a system can exhibit four different types of critical behavior: d-dimensional mean field, d-dimensional critical, $(d-1)$-dimensional mean field, $(d-1)$-dimensional critical. By matching its behavior in the various regions we obtain equalities connecting the critical indices in d and $d-1$ dimensions. Assuming the usual two-dimensional critical indices of the Ising model, these relations lead to the following indices for the three-dimensional Ising model: $\nu=\frac{2}{3}$, $\alpha=0$, $\beta=\frac{3}{8}$, and $\gamma=\frac{5}{4}$.

The usual scaling laws,[1] which relate the various critical exponents for a given model in a given dimensionality d, are well known to be at least an excellent approximation. There also exist "universality" arguments relating in some cases the critical behaviors of different models with the same d. In this note we obtain new scaling laws relating the critical indices for different dimensionalities d. Our basic assumption is that a "thin" system with one small dimension L will behave like a d-dimensional system when L is much larger than the temperature-dependent correlation length $\xi(T)$ and will exhibit $(d-1)$-dimensional behavior when $L \ll \xi(T)$. This change in dimensionality may occur either inside or outside the d-dimensional critical region. By matching the critical behavior in the two cases and assuming that the coefficients have a simple power-law dependence on L we obtain our interdimensional scaling laws. For the three-dimensional Ising model, our relations are easily seen to give $\nu=\frac{2}{3}$, $\alpha=0$, $\beta=\frac{3}{8}$, and $\gamma=\frac{5}{4}$, which compare rather well but *not exactly* with the numerical estimates $\nu=0.64$, $\alpha=\frac{1}{8}$, $\beta=\frac{5}{16}$, and $\gamma=\frac{5}{4}$. Some aspects of the film problem for the Bose gas were treated by one of us.[2] Fisher and co-workers[3,4] have treated the critical behavior of several models of thin systems, including the effect of boundary conditions. However, our point of view is to use the film as a model to relate the critical behaviors in $d-1$ and d dimensions. This is analogous to our use in a former paper[5] of a finite system to get information on the behavior of an "infinite" system.

Let us consider, for definiteness, $d=3$ in the case where there is also a transition for $d=2$. Our main assumption[3] is that only the lengths $\xi(T)$ and L will determine the behavior of the system, thus *assumption* (i) is the following: When $L \gg \xi(T)$ the system will behave as for $d=3$; when $L \ll \xi(T)$ the system will behave as for $d=2$ (this means that the critical exponents will have the $d=2$ values).

This assumption is a natural one in the usual scaling picture, where $\xi(T)$ is the only size-independent characteristic length. It also follows directly from Kadanoff's[1] block picture for scaling: When $L \ll \xi(T)$, the blocks can be chosen so as to form a two-dimensional array. Another way to justify this assumption is to note that the partition function Q may be written[6] as a functional integral of the form

$$Q=\int D\psi\, e^{-\beta F[\psi]},$$

where $\psi(x,y,z)$ is the order parameter and $F[\psi]$ is the free-energy functional. When $L \ll \xi(T)$ and if the boundary conditions on the surfaces do not force ψ to behave otherwise (e.g., if ψ satisfies periodic rather than zero boundary conditions), ψ becomes independent of z (the coordinate perpendicular to the film surface), and the calculation reduces to a two-dimensional one.

We now define a number of parameters which appear in the analysis. The reduced temperature is $\epsilon=(T-T_c)/T_c$, where T_c is the transition temperature of the film. The Ginzburg three-dimensional critical region[7] is denoted by ϵ_3 (assumed $\ll 1$). The Ginzburg critical region for the film in the two-dimensional regime,[7] ϵ_2, is

$$\epsilon_2=\epsilon_3^{1/2}\,\xi(0)/L \qquad (1)$$

[$\xi(0)$ is the coherence length at $T=0$]. The transition between three- and two-dimensional behavior, which by our assumption (i) occurs when $L \cong \xi(T)$, is characterized by $\epsilon=\epsilon_t$, where

$$\epsilon_t(m)=[\xi(0)/L]^2 \qquad (2)$$

in the mean field regime and

$$\epsilon_t(c) = [\bar{\xi}(0)/L]^{1/\nu_3} = [\xi(0)/L]^{1/\nu_3}\epsilon_3^{1-1/2\nu_3} \quad (3)$$

in the critical regime. $\bar{\xi}(0)$ is the constant appearing in the relation $\xi(T) = \bar{\xi}(0)/\epsilon^{\nu_3}$ which holds in the three-dimensional critical region. By matching the critical $\xi(T)$ to the mean field $\xi(T)$ at ϵ_3, one finds $\bar{\xi}(0) = \xi(0)\epsilon_3^{\nu_3-1/2}$, which was used in the second equality of Eq. (3). Defining the thickness parameter A,

$$A \equiv L/\xi(\epsilon_3) = L\epsilon_3^{1/2}/\xi(0) , \quad (4)$$

we get the following relations among the various ϵ's:

$$\epsilon_3/\epsilon_2 = A , \quad (5)$$

$$\epsilon_2/\epsilon_t(m) = A , \quad (6)$$

$$\epsilon_3/\epsilon_t(m) = A^2 , \quad (7)$$

$$\epsilon_t(m)/\epsilon_t(c) = A^{1/\nu_3 - 2} , \quad (8)$$

$$\epsilon_3/\epsilon_t(c) = A^{1/\nu_3} . \quad (9)$$

Two types of behavior can occur in the films.

(a) *Thin films*: $A \ll 1$ [i.e., $\xi(0) \ll L \ll \xi(\epsilon_3)$]. Here, by (5) and (6) we get $\epsilon_t(m) \gg \epsilon_2 \gg \epsilon_3$. Thus as $T \to T_c$ the film will become two dimensional at $\epsilon = \epsilon_t(m)$, long before it has a chance to reach the three-dimensional critical region. After that, at $\epsilon = \epsilon_2$, its behavior will change from two-dimensional mean field to two-dimensional critical.

(b) *Thick films*: $A \gg 1$ [i.e., $L \gg \xi(\epsilon_3)$]. From Eqs. (5), (8), and (9), using the fact that $\frac{1}{2} \leq \nu_3 < 1$, we get $\epsilon_3 \gg \epsilon_2 \gg \epsilon_t(c) \geq \epsilon_t(m)$. As $T \to T_c$, the film will, therefore, first enter the three-dimensional critical region at $\epsilon = \epsilon_3$, and then the two-dimensional critical region at $\epsilon = \epsilon_t(c)$.

From our assumption (i), we can write the correlation length in the two-dimensional critical regime of the film:

$$\xi(\epsilon) \equiv x(L)/\epsilon^{\nu_2} , \quad (10)$$

where x is an unknown function and ν_2 is the appropriate two-dimensional critical index.

In case (a) we match ξ at ϵ_2 between the mean-field and critical regions in two dimensions[8]:

$$\xi(0)/\epsilon_2^{1/2} = x(L)/\epsilon_2^{\nu_2} . \quad (11)$$

Using Eq. (1), we get

$$x(L) \cong \xi(0)\epsilon_3^{(\nu_2-1/2)/2}[\xi(0)/L]^{\nu_2-1/2} . \quad (12)$$

In case (b) we match ξ at $\epsilon_t(c)$ between the critical regions in two and three dimensions:

$$\bar{\xi}(0)/\epsilon_t(c)^{\nu_3} = x(L)/[\epsilon_t(c)]^{\nu_2} . \quad (13)$$

From (3) we now get

$$x(L) \cong \xi(0)\epsilon_3^{(1-1/2\nu_3)\nu_2}[\xi(0)/L]^{(\nu_2-\nu_3)/\nu_3} . \quad (14)$$

In writing down these equations we have ignored any possible difference in T_c between the film and the bulk material. This is, in fact, our *assumption* (ii). The validity of this assumption depends on the boundary conditions obeyed by the film. Fisher[3,4] has found that for the Ising and spherical models with periodic boundary conditions, the relative shift $\delta(L)$ satisfies

$$\delta(L) \lesssim \epsilon_t(c) ,$$

so that in these cases our assumption is justified. A further discussion of this assumption is given below.

We also note that Fisher[3] has already used a general scaling expression for diverging quantities in the film. For our case, his formula [(5.24) of Ref. 3], ignoring T_c shifts, would be

$$\xi(\epsilon) = LF(L^{1/\nu_3}\epsilon) , \quad (15)$$

where $F(Y) \sim Y^{-\nu_2}$ for $y \to 0$ and $F(Y) \sim Y^{-\nu_3}$ for $y \to \infty$. Thus in this formulation $x(L)$ is a pure power law. In the two-dimensional limit our Eqs. (10) and (14) agree with (15). Our Eq. (12), however, contains additional information which will now be used to obtain the new scaling laws.

If $x(L)$ is assumed, as in Eq. (15),[3] to be a simple power law [this is our *assumption* (iii)], we immediately obtain, by equating the exponents of L in Eqs. (12) and (14),

$$\nu_2 - \tfrac{1}{2}\nu_3 - \nu_2\nu_3 = 0 . \quad (16)$$

For the Ising model $\nu_2 = 1$, so (16) leads to the result $\nu_3 = \frac{2}{3}$. Another way to derive Eq. (16), which is perhaps intuitively clearer, is by considering x as a function of ϵ_3 rather than L. ϵ_3 is a small parameter which can be varied for a given L by, say, changing the range of the interactions. Universality predicts that this changes neither the $d = 2$ nor the $d = 3$ critical behavior. Taking into account the dependence of $\xi(0)$ on ϵ_3 (which cancels out at the end of the calculation), both (12) and (14) can be expressed as powers of ϵ_3. These powers can be matched in the same way as done below for powers of L, and the same results are thereby obtained.

Analogous considerations can be made to connect α_2 and α_3, β_2 and β_3, and γ_2 and γ_3 (the critical indices in two and three dimensions of the specific heat, the order parameter, and the susceptibility, respectively). We thus get the following equalities:

$$\alpha_2 - \alpha_3 = \alpha_2\nu_3 , \quad (17)$$

for the Ising model this leads to $\alpha_3 = 0$ (or log);

$$\beta_2 - \beta_3 = \nu_3(\beta_2 - \tfrac{1}{2}) , \quad (18)$$

for the Ising model this leads to $\beta_3 = \frac{3}{8}$;

$$\gamma_3 - \gamma_2 = (1 - \gamma_2)\nu_3 , \quad (19)$$

for the Ising model this leads to $\gamma_3 = \frac{5}{4}$.

It is straightforward to prove the following general statements: (a) The interdimensional scaling laws are exactly satisfied by the Landau classical exponents. (b) If the usual scaling laws[1] hold for two dimensions, they will also hold for three. (c) Our scaling relations are invariant under Fisher's renormalization[9] of the critical indices.

We conclude with a few remarks.

(i) For boundary conditions that give $\delta(L) \gg \epsilon_t(c)$ we can still derive a modified set of interdimensional scaling laws which will now depend on the boundary conditions.[10] Thus, starting from a fixed set of three-dimensional critical exponents (these are, of course, independent of the boundary conditions), we will get two-dimensional exponents which depend on the boundary conditions assumed for the finite three-dimensional system.

(ii) Our considerations can be applied in a quite straightforward way to dynamic critical phenomena.

(iii) The phase transition in two dimensions can be eliminated if the interdimensional scaling law would require, say, ν_2 to be infinite. This is what happens in the spherical model, where $\nu_3 = 1$ and by (16) $\nu_2 = \infty$. Another case which is of considerable interest is when one has a situation where some thermodynamic quantities (e.g., the magnetic susceptibility in the two-dimensional Heisenberg model) become infinite at some temperature but the order parameter remains zero. One rather speculative way to obtain such a result is if in three dimensions the system under consideration has a regular type of second-order transition, but when the interdimensional scaling laws are invoked, they lead in two dimensions to a vanishing value for β_2. The Heisenberg model would conform to this scheme if one had $\beta_3 = \frac{1}{2}\nu_3$.

(iv) We stress that we have completely ignored the possibility of critical indices which vary within the critical region. We always assumed that a single fixed power law holds throughout the critical region.

(v) Our assumptions including those on the amplitude $x(L)$ and the analogous quantities for α, β, and γ can be tested both theoretically and experimentally. The relations, embodied in Eqs. (12) and (14) among these amplitudes and the critical indices, both in two and three dimensions, suggest varying the film thickness as a useful tool for the study of critical phenomena.

(vi) We should like to stress that our considerations apply only when $\epsilon_3 \ll 1$ (which requires long-range forces). The application of our results to realistic (i.e., short-range forces) models requires the universality assumption (independence of critical indices on the range of the force for a given model).

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[8]We note as discussed in Ref. 7 that in Eq. (11) the reduced temperature $\epsilon \equiv (T - T_c)/T_c$ is defined with respect to a corrected T_c, not to the mean field T_c^0. Thus, the difference between T_c and T_c^0 has no effect on our reasoning.

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[10]We have tried to take $\delta(L) \sim L^{-1/(\nu_3+s)}$, with $s > 0$ (presumably $s \ll 1$). This leads to corrections to our scaling laws; s can be fitted to give a better (but still inexact) agreement with the numerical values of the exponents for the Ising model.