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# Low-Lying Energy Spectrum of a One-Dimensional Disordered System*

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The energy spectrum of a particle in the presence of random fixed finite-ranged repulsive potentials is investigated at low energy. The resulting asymptotically exact density of states is derived. It exhibits an exponential dependence on energy. This result is shown to be equivalent, for the case of $\delta$-function potentials, to that of Frisch and Lloyd (1960) in the limit of low energy.

## I. INTRODUCTION

In this paper, we study the one-dimensional system of noninteracting particles in the presence of random fixed finite-ranged repulsive potentials. The low-lying spectrum is investigated. This region is interesting not only in its own right, but because knowledge of it is essential in the study of Bose–Einstein condensation in the one-dimensional system (see following paper).

Section II treats the case of $\delta$-function repulsive potentials of strength $V_0$, in the limit where $V_0$ is infinitely large. The wave functions are then simply localized in the respective cells between two potentials. The cumulative density of states can then be derived, by considering the probability distribution functions of the cell lengths.

In Sec. III, we consider $\delta$-function potentials of arbitrary strength. In the region of small energy, we use the assumption that the system is equivalent to that of independent "big" cells in the middle of only "small" cells. The wave functions are then localized near the big cells. The reason for this assumption will be discussed.

In Sec. IV, the resulting low-lying density of states for arbitrary $V_0$ is shown to be exponentially decreasing for small energy, with the coefficient determined by a set of integral equations.

Section V shows that the rigorous results of Frisch and Lloyd[1] yield identical energy dependence at low energy, whereas Sec. VI shows that the coefficient from Frisch and Lloyd[1] is the same as what we obtain in Sec. IV. This gives an indirect justification for our assumption.

Appendix A discusses the probability distribution function of the cell lengths. Appendix B generalizes the result to arbitrary finite-ranged potentials.

## II. ENERGY SPECTRUM FOR $\delta$ FUNCTION WITH STRENGTH $V_0 \to \infty$

We consider a system of independent particles, each of which satisfies a Schrödinger equation on an interval of length $L$,

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)+V\psi(x)=E\psi(x)\ . \tag{2.1}$$

For most of this paper, we restrict ourselves to $\delta$-function potentials, i.e.,

$$V=\sum_{i=1}^{\nu-1} V_0\delta(x-x_i)\ . \tag{2.2}$$

(See Appendix B for the generalization to any finite-ranged repulsive potentials.) There are $\nu-1$ potentials at fixed but random positions $x_i$. Unless otherwise stated, we use fixed boundary conditions throughout, i.e.,

$$\psi(x)=0 \quad \text{at } x=0 \text{ and } x=L\ . \tag{2.3}$$

Because of the $\nu-1$ potentials, the total length $L$ will be divided into $\nu$ one-dimensional cells, the $i$th from the left having length $L^i$ (see Fig. 1). The essential task is then to solve the Schrödinger equation, subject to the above conditions.

We start from the case where $V_0 \to \infty$. The energy levels are simply

$$E_{is}=\frac{\hbar^2}{2m}\frac{\pi^2 s^2}{(L^i)^2}=\frac{\hbar^2 K_{is}^2}{2m}, \qquad i=1,\ldots,\nu,\quad s=1,2,\ldots \tag{2.4}$$

where

$$K_{is}\equiv \pi s/L^i\ , \tag{2.5}$$

with corresponding eigenfunctions

$$\psi_{is}(x)=(2/L^i)^{1/2}\sin K_{is}(x-x_{i-1}),\quad x_{i-1}\le x\le x_i$$
$$=0,\qquad x\ge x_i \text{ or } x\le x_{i-1}\ . \tag{2.6}$$

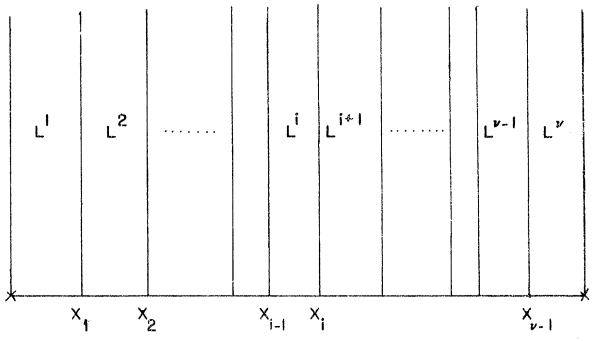


FIG. 1. Labeling of intervals between potentials.

The cumulative density of states $N(E)$, defined as the number of energy levels below $E$, is given by

$$N(E) = \sum_{i=1}^{\nu} \sum_{s=1}^{\infty} \Theta(E - E_{is})$$

$$= \sum_{i=1}^{\nu} \sum_{s=1}^{\infty} \left( E - \frac{\hbar^2 \pi^2}{2m} \frac{s^2}{(L^i)^2} \right) , \qquad (2.7)$$

where

$$\Theta(x) = 1, \quad x > 0$$

$$= 0, \quad x < 0$$

is the step function. Equation (2.7) gives the cumulative density of states for a particular distribution of lengths $\{L^i\}$. To find the density of states for a random system, we have to know the probability distribution function of the lengths. This is

$$P(L^1 L^2 \cdots L^\nu)$$

$$\equiv (\text{Probability distribution function of specific cell lengths being } L^1, L^2, \ldots, L^\nu)$$

$$= [(\nu - 1)!/L^{\nu-1}\, \delta(L^1 + L^2 + \cdots + L^\nu - L)] . \qquad (2.8)$$

Since the distribution is random, all positions are equally probable. The constant in front of the $\delta$ function comes from the normalization condition

$$\int_0^L \cdots \int_0^L P(L^1, L^2, \ldots, L^\nu)\, dL^1 dL^2 \cdots dL^\nu = 1.$$

In the limit where $\nu \to \infty$, (2.8) is equivalent to

$$P(L^1, L^2, \ldots, L^\nu) = \rho^\nu \prod_{i=1}^{\nu} e^{-\rho L^i} , \qquad (2.9)$$

where $\rho \equiv \nu/L$ is the number of potentials (impurities) per unit length. $\nu, L \to \infty$, but $\rho$ is finite. [For a more detailed discussion of $P(L^1, L^2, \ldots, L^\nu)$, see Appendix A.]

The density of states of a random system is given by its ensemble average, because, as we show, the standard deviation is negligible:

$$\langle N(E) \rangle = \sum_s \sum_i \int \Theta \left( E - \frac{\hbar^2 \pi^2}{2m} \frac{s^2}{(L^i)^2} \right) \times P(L^1, L^2, \ldots, L^\nu)\, dL^1 \ldots dL^\nu$$

$$= \sum_s \nu \int_0^\infty \Theta \left( E - \frac{c^2 s^2}{(L^i)^2} \right) \rho e^{-\rho L^i} dL^i$$

$$= \nu \sum_s e^{-\rho c s/\sqrt{E}}$$

$$= \nu/(e^{\rho c/\sqrt{E}} - 1) = \nu n(E) , \qquad (2.10)$$

where

$$c \equiv \hbar \pi/(2m)^{1/2}$$

and $n(E) \equiv$ average cumulative density of states per unit cell. The standard deviation

$$(\Delta N)^2 = \langle N^2(E) \rangle - \langle N(E) \rangle^2$$

$$= \sum_i \sum_j \sum_s \sum_t \int\int \Theta(E - E_{is})\, \Theta(E - E_{jt})\, \rho e^{-\rho L^i} dL^i\, \rho e^{-\rho L^j} dL^j$$

$$- \left( \sum_i \sum_s \int \Theta(E - E_{is})\, \rho e^{-\rho L^i} dL^i \right) \left( \sum_j \sum_t \int \Theta(E - E_{jt})\, \rho e^{-\rho L^j} dL^j \right)$$

$$= \frac{\nu(2e^{\rho c/\sqrt{E}} - 1)}{(e^{\rho c/\sqrt{E}} - 1)^2} \equiv \nu u(E) . \qquad (2.11)$$

Hence

$$\frac{\Delta N}{\langle N(E) \rangle} = \left( \frac{\nu\, u(E)}{\nu^2 n^2(E)} \right)^{1/2} = \frac{1}{\sqrt{\nu}} \frac{[u(E)]^{1/2}}{n(E)} \to 0$$

$$\text{as } \nu \to \infty . \qquad (2.12)$$

Therefore the density of states is given by the ensemble-average value (2.10). This result has been derived earlier by Bychkov and Dykhne.[2] Notice that for $E \to \infty$, (2.10) reduces to

$$\langle N(E) \rangle \to (L/\hbar\pi)(2mE)^{1/2} , \qquad (2.13)$$

which is the same as the case with no potentials. For $E \to 0$,

$$\langle N(E) \rangle \to \nu \exp[-\rho\hbar\pi/(2mE)^{1/2}] , \qquad (2.14)$$

or $\langle N(E) \rangle$ is exponentially small.

## III. ENERGY STATES FOR GENERAL $V_0$

In this section we study the energy states of the system for $\delta$-function potentials of arbitrary strength $V_0$. First notice that the wave function is that of a free particle between potentials (see Fig. 2):

$$\psi_i(x) = A_i \cos\kappa(x - x_i L_i) + B_i(\kappa_0/\kappa) \sin\kappa(x - x_{i-1}),$$

$$x_{i-1} \le x \le x_i$$

$$\psi_{i+1}(x) = A_{i+1} \cos\kappa(x - x_i) + B_{i+1}(\kappa_0/\kappa) \sin\kappa(x - x_i),$$

$$x_i \le x \le x_{i+1} \qquad (3.1)$$

where

$$A_i \equiv \psi_i(x_{i-1}^+), \quad B_i \equiv (1/\kappa_0)\psi_i'(x_{i-1}^+) ,$$

$$\kappa_0 \equiv 2mV_0/\hbar^2 , \qquad (3.2)$$

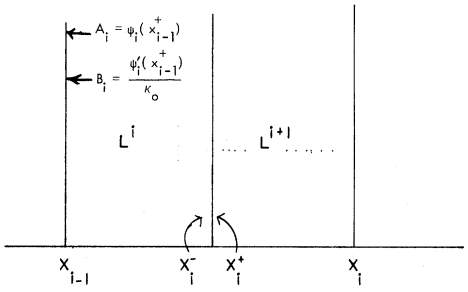


FIG. 2. Definitions of $A_i$, $B_i$, $x_i^+$, $x_i^-$.

and

$$E \equiv \hbar^2\kappa^2/2m \tag{3.3}$$

is the energy of the system. The wave function must be continuous:

$$(\psi_{i+1})_{x=x_i^+} - (\psi_i)_{x=x_i^-} = 0 . \tag{3.4}$$

Integrating the Schrödinger equation across $x_i$ yields

$$\begin{aligned} &-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V\psi(x) = E\psi(x), \\ &\frac{1}{\psi}\frac{d\psi}{dx}\bigg|_{x_i^+} - \frac{1}{\psi}\frac{d\psi}{dx}\bigg|_{x_i^-} = \frac{2mV_0}{\hbar^2} = \kappa_0 . \end{aligned} \tag{3.5}$$

Substituting (3.1) in (3.4) and (3.5), we get

$$A_{i+1} = A_i \cos\kappa L^i + B_i(\kappa_0/\kappa)\sin\kappa L^i , \tag{3.6}$$

$$B_{i+1} = A_i[\cos\kappa L^i - (\kappa/\kappa_0)\sin\kappa L^i] + B_i[\cos\kappa L^i + (\kappa_0/\kappa)\sin\kappa L^i] . \tag{3.7}$$

These relations are well known (see, e.g., Borland[3] and Hori[4]).

We now study the low-lying states of the system. From Sec. II, ($V_0 \to \infty$), we know that the lowest states are localized in the big cells. Hence we use the assumption that in the low-energy region, the system is equivalent to that of independent big cells in the middle of only small cells. This is possible because, as we shall show, the wave functions stay localized near the big cells. Further justification comes from the fact that the resulting density of states agrees with that of Frisch and Lloyd.[1] We have not, however, rigorously proved that this assumption is correct.

Suppose $j$ is the big cell under consideration. Assume that all the other cells are small, such that

$$\kappa L^i \ll 1, \qquad i \neq j . \tag{3.8}$$

Since we are interested in small energy, we shall also use the condition

$$\kappa/\kappa_0 \ll 1. \tag{3.9}$$

(2.6) and (2.7) can then be simplified:

$$\begin{aligned} &A_{i+1} = A_i + B_i(\kappa_0 L^i) , \\ &B_{i+1} = A_i + B_i(1+\kappa_0 L^i), \quad i \neq j . \end{aligned} \tag{3.10}$$

We can also define $A_i' B_i'$ from the right-hand side of the system (Fig. 3):

$$\begin{aligned} (\text{wave function}) = \bar{\psi}_i(y) = A_i' \cos\kappa_i(y - y_i) \\ + B_i'(\kappa_0/\kappa)\sin\kappa(y - y_i) , \end{aligned} \tag{3.11}$$

where $y$ is the distance measured from the right-hand boundary of the system,

$$\begin{aligned} &y = L - x , \\ &y_i = L - x_i , \end{aligned} \tag{3.12}$$

$$A_i' = \bar{\psi}_i(y_i^-), \quad B_i' = (1/\kappa_0)\bar{\psi}_i'(y_i^-) . \tag{3.13}$$

Relations equivalent to (3.10) can be obtained:

$$\begin{aligned} &A_{i-1}' = A_i' + B_i'(\kappa_0 L^i), \\ &B_{i-1}' = A_i' + B_i'(\kappa_0 L^i + 1), \quad i \neq j . \end{aligned} \tag{3.14}$$

Using (3.10) and (3.14), we will now show that the wave function is localized near the big cell $j$. Let us now start from the left-hand boundary of the system. Fixed boundary condition gives $A_1 = 0$ and $B_1 = B^L$ (so far an undetermined constant). Using (3.10), we obtain

$$A_2 = B^L(\kappa_0 L^1) \quad B_2 = B^L(1 + \kappa_0 L^1) . \tag{3.15}$$

We can continue the process, finding all the $A_i$, $B_i$ by iteration. Since (3.10) is linear,

$$\begin{aligned} &A_i = B^L a_i(L^1 \cdots L^{i-1}), \\ &B_i = B^L b_i(L^1 \cdots L^{i-1}) , \quad i = 1, 2, \ldots, j \end{aligned} \tag{3.16}$$

where $a_i$ and $b_i$ are functions of $L^1 L^2 \cdots L^{i-1}$ only.

We want to prove now the following theorem.

*Theorem* ($L$):

(i) $|B_{i+1}| \geq |B_i|(1 + \kappa_0 L^i)$;

(ii) $B_{i+1}$ has the same sign as $B^L$,

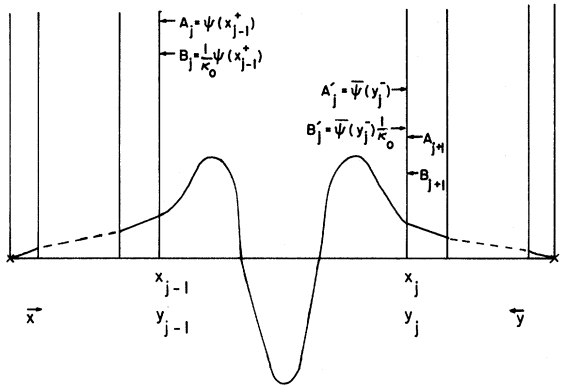


FIG. 3. Schematic of localized wave function. $y = L - x$, $\bar{\psi}(y) = \psi(x)$.

$$i = 1, 2, \ldots, j-1.$$

*Proof.* First eliminate $B_i$ or $A_i$ from (3.10):

$$B_{i+2} = B_{i+1}(2+\kappa_0 L^{i+1}) - B_i \ , \tag{3.17}$$

$$A_{i+2} = A_{i+1}(1+L^{i+1}/L^i + \kappa_0 L^{i+1}) - A_i L^{i+1}/L^i,$$

$$i = 1, 2, \ldots, j-2 \ . \tag{3.18}$$

We will prove Theorem ($L$) by mathematical induction. Since

$$B_2 = B^L(1+\kappa_0 L^1),$$

the theorem is certainly true for $i=1$. Suppose it is true for all $i \le i'$; then

$$B_{i'+2} = B_{i'+1}(1+\kappa_0 L^{i'+1}) + (B_{i'+1} - B_{i'}) \ .$$

Since $B_{i'}$ and $B_{i'+1}$ have the same sign as $B_1^L$ and $|B_{i'+1}| > |B_{i'}|$, $B_{i'+2}$ has the same sign as $B^L$, and

$$|B_{i'+2}| \ge |B_{i'+1}|(1+\kappa_0 L^{i'+1}) \ .$$

Therefore, Theorem ($L$) is proved.

Similarly, using (3.18), we can prove the following theorem.

*Theorem* ($L'$):

(i) $|A_{i+1}| \ge |A_i|(1+\kappa_0 L^i)$ ;

(ii) $A_{i+1}$ has the same sign as $B^L$,

$$i = 1, 2, \ldots, j-1 \ .$$

Hence, from (3.1) we see that the amplitude of the wave function increases exponentially from the left-hand boundary to cell $j$.

By the same reasoning, we can show, using (3.14), the following theorem.

*Theorem* ($R$):

(i) $|A'_{i-1}| \ge |A'_i|(1+\kappa_0 L^i)$;

(ii) $|B'_{i-1}| \ge |B'_i|(1+\kappa_0 L^i)$;

(iii) $A'_{i-1}$ and $B'_{i-1}$ have the same sign as $B'_\nu \equiv B^R$ and $A'_\nu = 0$ for $i = j+1, j+2, \ldots, \nu$.

Therefore, the amplitude of the wave function increases exponentially from the right-hand boundary to the cell $j$.

Let us now write the wave function in the cell $j$ (Fig. 3):

$$\psi_j(x) = A_j \cos\kappa(x - x_{j-1}) + B_j(\kappa_0/\kappa)\sin\kappa(x - x_{j-1}) \ .$$

But the wave function can also be written

$$\bar{\psi}_j(y) = A'_j \cos\kappa(y - y_j) + B'_j(\kappa_0/\kappa)\sin\kappa(y - y_j)$$

$$= A'_j \cos\kappa(L^j - x + x_{j-1}) + B'_j(\kappa_0/\kappa)\sin\kappa(L^j - x + x_{j-1}) \ .$$

Comparing the coefficients of the above equations, we obtain

$$A_j = A'_j \cos\kappa L^j + B'_j(\kappa_0/\kappa)\sin\kappa L^j \ ,$$

$$B_j = A'_j(\kappa/\kappa_0)\sin\kappa L^j - B'_j \cos\kappa L^j \ . \tag{3.19}$$

Since $A_j$ and $B_j$ are multiples of $B^L$, $A'_j$ and $B'_j$ are multiples of $B^R$. (3.19) allows the ratio $B^L/B^R$ to be determined. By dividing the two equations in (3.19), an equation for $\kappa$ can be found;

$$\tan\kappa L^j = -\frac{(\kappa/\kappa_0)(A_j/B_j) + (\kappa/\kappa_0)(A'_j/B'_j)}{1 - [(\kappa/\kappa_0)(A'_j/B'_j)(\kappa/\kappa_0)(A_j/B_j)]} \ . \tag{3.20}$$

$A_j/B_j$ and $A'_j/B'_j$ are always positive; $\kappa/\kappa_0$ is small. Hence, we can solve for $\kappa L^j$:

$$\kappa L^j = s\pi - \frac{\kappa}{\kappa_0}\frac{A_j}{B_j} - \frac{\kappa}{\kappa_0}\frac{A'_j}{B'_j}, \qquad s = 1, 2, \ldots \ . \tag{3.21}$$

The wave function is

$$\psi(x) = A_i \cos\kappa(x - x_i) + B_i(\kappa_0/\kappa)\sin\kappa(x - x_i), \qquad x_i \le x \le x_{i+1}.$$

No new wave functions are obtained by replacing $s$ by $-s$. Also, the solution $s=0$ gives $\psi = 0$, which is physically uninteresting.

We still have one degree of freedom, viz., $B^L$ or $B^R$. It can be determined by normalization,

$$\int_0^L \psi(x)\psi(x)\,dx = 1 \ .$$

We can find an expression for $A_j/B_j$ and $A'_j/B'_j$. Using (3.10), we obtain

$$\frac{B_j}{A_j} = 1 + \frac{1}{\kappa_0 L^{j-1} + A_{j-1}/B_{j-1}} \ . \tag{3.22}$$

We can iterate the process, giving the continued fraction[5]

$$\frac{B_j}{A_j} = 1 + \frac{1}{\kappa_0 L^{j-1}+}\ \frac{1}{1+}\ \frac{1}{\kappa_0 L^{j-2}+}\ \frac{1}{1+\cdots}\ \frac{1}{\kappa_0 L^1} \ . \tag{3.23}$$

Similarly,

$$\frac{B'_j}{A'_j} = 1 + \frac{1}{\kappa_0 L^{j+1}+}\ \frac{1}{1+}\ \frac{1}{\kappa_0 L^{j+2}+\cdots}\ \frac{1}{\kappa_0 L^\nu} \ . \tag{3.24}$$

(3.23) and (3.24) are both positive continued fractions. Both are convergent.

*Proof.* The positive continued fraction

$$F = f_0 + \frac{1}{f_1+}\ \frac{1}{f_2+}\ \frac{1}{f_3+}\ \frac{1}{f_4+\cdots}$$

is convergent if, and only if, at least one of the series is divergent[5]:

$$f_1 + f_3 + f_5 + \cdots \ , \qquad f_2 + f_4 + f_6 + \cdots \ .$$

For (3.23), the second series is $1+1+1+\cdots$, which is divergent. Hence (3.23) is convergent. Similarly, (3.24) is convergent.

*In conclusion,* if $j$ is the only big cell, in the presence of only small ones, the low-lying state is localized in the neighbor of the cell $j$, with cor-

responding energy level [see (3.21)]

$$\kappa_{js} = \frac{s\pi}{(L^j + a_j^L + a_j^R)}, \tag{3.25}$$

where

$$a_j^L = \frac{1}{\kappa_0}\frac{A_j}{B_j} \text{ and } a_j^R = \frac{1}{\kappa_0} = \frac{A_j'}{B_j'} \tag{3.26}$$

are positive continued fractions and

$$\kappa_{js}/\kappa_0 \ll 1 \text{ and } \kappa_{js} l \ll 1,$$

where $l$ is the average cell length.

In reality, the system consists of many big cells. We will, however, treat them independently. This is possible because, as we have shown, the low-lying states are localized near the respective cells. More detailed analysis shows that the overlap between states localized near different cells is indeed small. We have, hence, a tight-binding picture for disordered systems.

Arguments for localization of all states have been given by Borland[6] and Halperin.[7] We have illustrated this for low-lying states.

## IV. DENSITY OF STATES FOR ARBITRARY $V_0$

The cumulative density of states of a particle in repulsive δ-function potentials of arbitrary strength $V_0$ is, by definition,

$$N(E) = \sum_m \Theta(E - E_m\{L_i\}),$$

where $E_m\{L_i\}$ are the various energy levels as a function of the distribution of cells $\{L_i\}$. For a random system, the cumulative density of states is given by its ensemble average

$$\langle N(E)\rangle = \int \sum_m \Theta(E - E_m\{L^i\}) \times P(L^1, L^2, \ldots, L^\nu)\, dL^1 \ldots dL^\nu$$

because, just as in the discussion of the $V_0 \to \infty$ limit, the standard deviation is negligible. For small $E$ (the region in which we are interested), only small $E_m\{L^i\}$ contribute to $\langle N(E)\rangle$. Three levels can be described by the large interval $j$ and the neighboring intervals as in (3.25). If we use (3.25) for all levels, the higher ones will be treated incorrectly, but do not contribute in any case. Hence

$$\langle N(E)\rangle = \nu \sum_s \int_0^\infty \cdots \int_0^\infty \Theta(E - c^2 s^2/(L^j + a_j^L + a_j^R)^2) e^{-\rho L^j} \rho\, dL^j \times \rho e^{-\rho L^{j-1}} \cdots \rho e^{-\rho L^1} dL^{j-1} \cdots dL^1 \rho e^{-\rho L^{j-1}} \cdots \rho e^{-\rho L^\nu} dL^{j+1} \cdots dL^\nu . \tag{4.1}$$

The factor $\nu$ enters because all the cells are equivalent after ensemble averaging.

The integration over $L^j$ can be easily done:

$$\langle N(E)\rangle = \nu \sum_{s=1} e^{-\rho c s/\sqrt{E}} \langle e^{\rho a_j^L}\rangle \langle e^{\rho a_j^R}\rangle = \langle N_\infty(E)\rangle W^2, \tag{4.2}$$

where $\langle N_\infty(E)\rangle$ = cumulative density of states when $V_0 \to 0$,

$$\langle e^{\rho a_j^L}\rangle = \int \exp[\rho a_j^L (L^1, L^2, \ldots, L^{j-1})] \times P(L^1 \cdots L^{j-1})\, dL^1 dL^2 \cdots dL^{j-1} \tag{4.3}$$

and

$$\langle e^{\rho a_j^R}\rangle = \int \exp[\rho a_j^R (L^{j+1}, L^{j+2}, \ldots, L^\nu)] \times P(L^{j+1} \cdots L^\nu)\, dL^{j+1} \cdots dL^\nu . \tag{4.4}$$

Of course, $\langle e^{\rho a_j^R}\rangle = \langle e^{\rho a_j^L}\rangle \equiv W$ by symmetry, and $W$ is independent of $i$.

We now proceed to calculate $W$. First, we obtain

$$a_j^L = 1/\kappa_0 t_j^L, \tag{4.5}$$

where, using (3.22),

$$t_j^L \equiv \frac{B_j}{A_j} = 1 + \frac{t_{j-1}^L}{1 + (\kappa_0 L^{j-1}) t_{j-1}^L} . \tag{4.6}$$

Define

$$x_j = t_j^L - 1, \quad \infty > x_j \geq 0 \tag{4.7}$$

$$x_j = \frac{1 + x_{j-1}}{1 + (1 + x_{j-1})\kappa_0 L^{j-1}} \equiv x(L^{j-1} L^{j-2} \cdots L^1) . \tag{4.8}$$

Let $Q(x_j)$ be the probability distribution function for $x_j$; then

$$Q(x_0) = \int_0^\infty \cdots \int_0^\infty \delta(x_j(L^{j-1}, \ldots, L^1) - x_0) \rho e^{-\rho L^{j-1}} \cdots \rho e^{-\rho L^1} dL^{j-1} \cdots dL^1 \tag{4.9}$$

$$= \int_0^\infty dx' \, \delta(x_{j-1} - x') \int_0^\infty \delta\left(\frac{x_{j-1} + 1}{1 + (1 + x_{j-1})\kappa_0 L^{j-1}} - x_0\right) \rho e^{-\rho L^{j-1}} \cdots \rho e^{-\rho L^1} dL^{j-1} \cdots dL^1$$

$$= \int_0^\infty \int_0^\infty \delta\left(\frac{x' + 1}{1 + (1 + x')\kappa_0 L^{j-1}} - x_0\right) \rho e^{-\rho L^{j-1}} dL^{j-1} Q(x')\, dx' , \tag{4.10}$$

where we have interchanged the order of integrations and used the fact that

$$Q(x') = \int \delta(x_{j-1} - x')\, \rho e^{-\rho L^{j-2}} \cdots dL^{j-2} \cdots dL^1 .$$

The distribution function for $x_{j-1}$ must have the same form as that for $x_j$. After integrating over $L^{j-1}$ in (4.10), we obtain

$$Q(x_0) = \int_0^\infty Q(x')\, dx' \exp\left[ -\frac{\rho}{\kappa_0}\left(\frac{1}{x_0} - \frac{1}{x'+1}\right)\right] \frac{\rho}{\kappa_0 x_0^2}, \quad 0 \le x_0 \le 1 \tag{4.11}$$

$$= \int_{x_0-1}^\infty Q(x')\, dx' \exp\left[ -\frac{\rho}{\kappa_0}\left(\frac{1}{x_0} - \frac{1}{x'+1}\right)\right] \frac{\rho}{\kappa_0 x_0^2}, \quad 1 \le x_0 < \infty . \tag{4.12}$$

Define

$$Q(x_0) \equiv \frac{e^{-\gamma/x_0}}{x_0^2} R(x_0) , \tag{4.13}$$

where $\gamma \equiv \rho/\kappa_0$. The equations (4.11) and (4.12) may be written

$$R(x_0) = \gamma \int_0^\infty \frac{R(x') \exp[-\gamma/x' + \gamma/(x'+1)]}{x'^2} dx' ,$$

$$0 \le x_0 \le 1 \tag{4.14}$$

$$= \gamma \int_{x_0-1}^\infty \frac{R(x') \exp[-\gamma/x' + \gamma/(x'+1)]}{x'^2} dx' ,$$

$$1 \le x_0 \le \infty . \tag{4.15}$$

Equations (4.14) and (4.15) and the normalization condition

$$\int_0^\infty Q(x)\, dx = \int_0^\infty \frac{e^{-\gamma/x}}{x^2} R(x)\, dx = 1 \tag{4.16}$$

determine $R(x)$. Thus we find

$$W = \langle e^{\rho a_j^L} \rangle = \langle e^{\rho/\kappa_0 (1+x_j)} \rangle$$

$$\equiv \int_0^\infty e^{\gamma/(1+x_j)} Q(x_j)\, dx_j$$

$$= \int_0^\infty e^{\gamma/(1+x)} \left(\frac{e^{-\gamma/x}}{x^2}\right) R(x)\, dx . \tag{4.17}$$

$W$ is independent of $E$ and is a function of $\gamma$ only.

From (4.2), the cumulative density of states has the same exponential dependence on energy as the $V_0 = \infty$ case. Only the multiplying constant is altered. That is, for small $E$ we may write

$$N(E) = [1/J(\gamma)]\, e^{-\rho c/\sqrt{E}} , \tag{4.18}$$

where

$$J(\gamma) \equiv 1/W^2 .$$

We have not obtained a closed expression for $J$ from the integral equations (4.14) and (4.15). However, it is possible to obtain solutions for small $\gamma$ ("low density of impurities") and large $\gamma$ ("high density of impurities"). We only present the results of these calculations.[8] They are

$$J(\gamma) = 1 - 2\gamma + O(\gamma^2 \ln\gamma), \quad \gamma \ll 1 \tag{4.19}$$

$$= (\tfrac{1}{2}\pi)^{1/2} \gamma^{1/4} e^{-\alpha\sqrt{\gamma}}, \quad \gamma \gg 1 \tag{4.20}$$

$$\alpha = 1 + \frac{1}{2}\int_0^\infty \frac{1 + (1+u)e^{-u}}{\sqrt{u}\,(1-e^{-u})^{3/2}} e^{-u} du$$

$$= 1.257\ldots . \tag{4.21}$$

## V. RESULTS OF FRISCH AND LLOYD AT LOW ENERGY

The density of states of this model has been discussed previously by many authors using the method of node counting.[1,3,9,10] In Frisch and Lloyd,[1] the cumulative density of states $N(\hbar^2\kappa^2/2m)$ is given by solving the following integral equations:

$$T(\vartheta) = u'(z)\, e^{\rho\pi/2\kappa} \frac{1}{\nu} \langle N(\hbar^2\kappa^2/2m)\rangle - \int_{-\infty}^{\vartheta} \frac{T(\xi - \kappa_0)}{u(\xi)} d\xi ,$$

$$u(\xi) = \exp\left(\frac{\rho}{\kappa} \tan^{-1}\frac{\xi}{\kappa}\right), \quad -\infty < z < \infty . \tag{5.1}$$

This is Eq. (18) of Frisch and Lloyd. Equation (16) of Frisch and Lloyd is

$$(1/\nu)\langle N(\hbar^2\kappa^2/2m)\rangle = (1/\rho)\lim_{\vartheta\to\pm\infty} \vartheta^2 T(\vartheta) , \tag{5.2}$$

and Eq. (13) of Frisch and Lloyd is

$$\int_{-\infty}^{\infty} T(\vartheta)\, d\vartheta = 1 . \tag{5.3}$$

We have used our notation. We want to simplify (5.1)–(5.3) at low energy. Put

$$T(\vartheta) \equiv \frac{1}{\nu}\langle N(\hbar^2\kappa^2/2m)\rangle\, e^{\rho\pi/2\kappa} u'(\vartheta) F(\vartheta) . \tag{5.4}$$

Then (5.1) becomes

$$F(\vartheta) = 1 - \int_{-\infty}^{\vartheta} \frac{u'(\xi - \kappa_0)}{u(\xi)} F(\xi - \kappa_0)\, d\xi . \tag{5.5}$$

$F(\vartheta) > 0$ since $T(\vartheta) > 0$ (see Frisch and Lloyd). Equation (5.3) becomes

$$1 = (1/\nu)\langle N(\hbar^2\kappa^2/2m)\rangle\, e^{\pi\rho/2\kappa} \int_{-\infty}^{\infty} u'(\vartheta) F(\vartheta)\, d\vartheta \tag{5.6}$$

and (5.2) becomes

$$(1/\nu)\langle N(\hbar^2\kappa^2/2m)\rangle = \lim_{\vartheta\to\pm\infty} (1/\nu)\langle N(\hbar^2\kappa^2/2m)\rangle$$

$$\times e^{\rho\pi/2\kappa}\frac{\vartheta^2}{\vartheta^2+\kappa^2}\exp\left(\frac{\rho}{\kappa}\tan^{-1}\frac{\vartheta}{\kappa}\right)F(\vartheta)\ ,$$

since

$$u'(\xi)=\frac{\rho}{\xi^2+\kappa^2}\exp\left(\frac{\rho}{\kappa}\tan^{-1}\frac{\xi}{\kappa}\right)\ .$$

Therefore

$$\begin{aligned}1&=e^{\rho\pi/2\kappa}e^{-\pi\rho/2\kappa}F(-\infty)\\&=e^{\rho\pi/2\kappa}e^{\pi\rho/2\kappa}F(+\infty)\ ,\end{aligned}\qquad(5.7)$$

$$F(-\infty)=1,\quad F(+\infty)=e^{-\rho\pi/\kappa}\ .$$

Equation (5.6) can be further simplified:

$$\frac{1}{\nu}\langle N(\hbar^2\kappa^2/2m)\rangle e^{\pi\rho/2\kappa}\left(\int_{-\infty}^{0}d\vartheta\,\frac{\rho}{\vartheta^2}\,e^{-\rho/\vartheta}F(\vartheta)e^{-\pi\rho/2\kappa}+\int_0^{\infty}d\vartheta\,\frac{\rho}{\vartheta^2}e^{-\rho/\vartheta}F(\vartheta)\,e^{\pi\rho/2\kappa}\right)=1$$

as $\kappa\to 0$. The first term in the large parentheses is zero because of $e^{-\pi\rho/2\kappa}$ if $F(\vartheta)$ exists as $\kappa\to 0$. Therefore

$$(1/\nu)\langle N(h^2\kappa^2/2m)\rangle=e^{-\rho\pi/\kappa}(1/J)\ ,\qquad(5.8)$$

$$J=\int_0^{\infty}d\vartheta\ (\rho/\vartheta^2)e^{-\rho/\vartheta}F(\vartheta)\ .\qquad(5.9)$$

Equation (5.5) can be further simplified in the limit $\kappa\to 0$:

$$\frac{u'(\xi-\kappa_0)}{u(\xi)}=\frac{\rho}{(\xi-\kappa_0)+\kappa^2}\exp\left[\frac{\rho}{\kappa}\left(\tan^{-1}\frac{\xi-\kappa_0}{\kappa}-\tan^{-1}\frac{\xi}{\kappa_0}\right)\right],$$

$$\lim_{\kappa\to 0}\frac{\rho}{\kappa}\left(\tan^{-1}\frac{\xi-\kappa_0}{\kappa}-\tan^{-1}\frac{\xi}{\kappa}\right)=-\infty,\qquad \kappa_0>\xi>0$$

$$=\rho\left(\frac{1}{\xi}-\frac{1}{\xi-\kappa_0}\right),\qquad \xi<0\ \text{ or }\ \xi>\kappa_0$$

$$F(\vartheta)=1-\int_{-\infty}^{\vartheta}d\xi\ \frac{\rho}{(\xi-\kappa_0)^2}\exp\left[\rho\left(\frac{1}{\xi}-\frac{1}{\xi-\kappa_0}\right)\right]F(\xi-\kappa_0)\ ,\qquad \vartheta\le 0\qquad(5.10)$$

$$F(\vartheta)=1-\int_{-\infty}^{0}d\xi\ \frac{\rho}{(\xi-\kappa_0)^2}\exp\left[\rho\left(\frac{1}{\xi}-\frac{1}{\xi-\kappa_0}\right)\right]F(\xi-\kappa_0),\qquad 0\le\vartheta\le\kappa_0\qquad(5.11)$$

$$F(\vartheta)=F(\kappa_0)-\int_{\kappa_0}^{\vartheta}d\xi\ \frac{\rho}{(\xi-\kappa_0)^2}\exp\left[\rho\left(\frac{1}{\xi}-\frac{1}{\xi-\kappa_0}\right)\right]F(\xi-\kappa_0),\qquad \kappa_0<\vartheta\ .\qquad(5.12)$$

Introduce the notation

$$\vartheta=\kappa_0 t\ \text{ and }\ F(\vartheta)=F(\kappa_0 t)=f(t)\ ,\qquad(5.13)$$

and $\gamma=\rho/\kappa_0$, as before. Equations (5.10)–(5.12) and (5.9) become

$$f(t)=1-\int_{-\infty}^{t}d\tau\ \frac{\gamma}{(\tau-1)^2}\exp\left[\gamma\left(\frac{1}{\tau}-\frac{1}{\tau-1}\right)\right]f(\tau-1)\ ,\qquad t\le 0\qquad(5.14)$$

$$f(t)=1-\int_{-\infty}^{0}d\tau\ \frac{\gamma}{(\tau-1)^2}\exp\left[\gamma\left(\frac{-1}{\tau}-\frac{1}{\tau-1}\right)\right]f(\tau-1)\ ,\qquad 0\le t\le 1\qquad(5.15)$$

$$f(t)=f(1)-\int_{1}^{t}d\tau\ \frac{\gamma}{(\tau-1)^2}\exp\left[\gamma\left(\frac{1}{\tau}-\frac{1}{\tau-1}\right)\right]f(\tau-1)\ ,\qquad 1\le t<\infty\qquad(5.16)$$

and

$$J=\gamma\int_0^{\infty}dt\,(e^{-\gamma/t}/t^2)f(t)\ .\qquad(5.17)$$

Equations (5.14)–(5.16) determine $f(t)$, and (5.17) and (5.8) give the cumulative density of states.

Notice that (5.8) has the same exponential form as (4.2). $J$ is a function of $\gamma$ only.

## VI. EQUIVALENCE OF OUR RESULT AND THAT OBTAINABLE FROM FRISCH AND LLOYD

We want to prove

$$W^2=1/J\qquad(6.1)$$

from (4.17) and (5.17). First write (5.16) as follows:

$$f(t)=\left\{f(1)-\gamma\int_0^\infty d\tau\,\frac{\gamma}{(\tau-1)^2}\exp\left[\gamma\left(\frac{1}{\tau}-\frac{1}{\tau-1}\right)\right]f(\tau-1)\right\}$$

$$=\gamma\int_t^\infty d\tau\,\frac{\gamma}{(\tau-1)^2}\exp\left[\gamma\left(\frac{1}{\tau}-\frac{1}{\tau-1}\right)\right]f(\tau-1)\,,\quad 1\le t<\infty\,. \tag{6.2}$$

The term in the large brace is equal to

$$f(\infty)=\lim_{\kappa\to 0}e^{-\pi\rho/\kappa}=0\,,$$

from (5.7). Therefore,

$$f(t)=\gamma\int_t^\infty d\tau\,\frac{\gamma}{(\tau-1)^2}\exp\left[\gamma\left(\frac{1}{\tau}-\frac{1}{\tau-1}\right)\right]f(\tau-1)\,,\quad 1\le t<\infty. \tag{6.3}$$

Similarly, (5.15) becomes

$$f(t)=\gamma\int_1^\infty d\tau\,\frac{\gamma}{(\tau-1)^2}\exp\left[\gamma\left(\frac{1}{\tau}-\frac{1}{\tau-1}\right)\right]f(\tau-1)\,,\quad 0\le t\le 1\,. \tag{6.4}$$

Equations (6.3) and (6.4) are just Eqs. (5.15) and (5.14) with the trivial notation change $x'=\tau-1$. Therefore, $f(t)$ and $R(t)$ can at most differ by a constant

$$f(t)=AR(t)\,. \tag{6.5}$$

Now, using (4.16) we obtain

$$J=\gamma\int_0^\infty dt\,\frac{e^{-\gamma/t}}{t^2}f(t)=A\gamma\int_0^\infty dt\,\frac{e^{-\gamma/t}}{t^2}R(t)=A\gamma\,.$$

Therefore, using (4.15) we find

$$W=\frac{R(1)}{\gamma}=\frac{f(1)}{A\gamma}=\frac{f(1)}{J}. \tag{6.6}$$

Equation (6.1) becomes

$$\frac{f^2(1)}{J^2}=\frac{1}{J}\quad\text{or}\quad\frac{J}{f(1)}=f(0)\,, \tag{6.7}$$

where we note that $f(1)=f(0)$ from (5.15). Define

$$\tilde f(t)=f(t)/f(1),\quad t\ge 0\,. \tag{6.8}$$

Then

$$\tilde f(t)=1,\qquad 0\le t\le 1 \tag{6.9}$$

$$=1-\gamma\int_1^t d\tau\,\frac{1}{(\tau-1)^2}\times\exp\left[\gamma\left(\frac{1}{\tau}-\frac{1}{\tau-1}\right)\right]\tilde f(\tau-1),\quad t\ge 1 \tag{6.10}$$

and (6.7) becomes

$$f(0)=\tilde J\equiv\gamma\int_0^\infty dt\,(e^{-\gamma/t}/t^2)\tilde f(t)\,. \tag{6.11}$$

First calculate $\tilde J$ by iterating (6.9) and (6.10):

$$\tilde f(t)=1+\sum_{i=1}^{n}(-\gamma)^i\int_i^t d\tau_1\,\phi(\tau_1)\int_{i-1}^{\tau_1-1}d\tau_2\,\phi(\tau_2)\cdots\int_1^{\tau_{i-1}-1}d\tau_i\,\phi(\tau_i)\,,\qquad n+1\ge t\ge n\,,$$

$$n=1,\ 2,\ 3,\ \ldots \tag{6.12}$$

$$\phi(\tau)=\frac{\exp\{\gamma[1/\tau-1/(\tau-1)]\}}{(\tau-1)^2}\,. \tag{6.13}$$

Then

$$\tilde J=\sum_{m=0}^{\infty}\tilde Q_m\,, \tag{6.14}$$

$$\tilde Q_m=\gamma(-\gamma)^m\int_m^\infty dt\,\frac{e^{-\gamma/t}}{t^2}\int_m^t d\tau_1\phi(\tau_1)\times\int_{m-1}^{t_1-1}d\tau_2\,\phi(\tau_2)\cdots\int_1^{t_{m-1}-1}\phi(\tau_m)d\tau_m\,, \tag{6.15}$$

$$m=1,\ 2,\ \ldots$$

$$\tilde Q_0=I\,.$$

By integrating over $t$ in (6.15), we get

$$\tilde Q_m-\tilde Q_{m-1}=\tilde q_m\,,\quad m=1,\ 2,\ \ldots \tag{6.16}$$

where

$$\tilde q_m=(-\gamma)^m\int_m^\infty d\tau_1\,\phi(\tau_1)\int_{m-1}^{\tau_1-1}d\tau_2\,\phi(\tau_2)\cdots\int_1^{\tau_{m-1}-1}\phi(\tau_m)d\tau_m\,. \tag{6.17}$$

Let us now find $f(0)$ from (5.14) by iteration:

$$f(0)=\sum_{m=0}^{\infty}Q_m\,,\quad Q_0=I\,, \tag{6.18}$$

$$Q_m=(-\gamma)^m\int_{-\infty}^0 dx_1\,\phi(x_1)\int_{-\infty}^{x_1-1}dx_2\,\phi(x_2)\times\int_{-\infty}^{x_2-1}dx_3\,\phi(x_3)\cdots\int_{-\infty}^{x_{m-1}-1}dx_m\,\phi(x_m)\cdots,$$

$$m=1,\ 2,\ \ldots. \tag{6.19}$$

Now change variables $x_i=1-\tau_{m+1-i}$. Equation (6.19) becomes

$$Q_m=(-\gamma)^m\int_m^\infty d\tau_1\psi(\tau_1)\int_{m-1}^{\tau_1-1}d\tau_2\,\psi(\tau_2)\cdots\int_1^{\tau_{m-1}-1}d\tau_m\,\psi(\tau_m)\,, \tag{6.20}$$

where

$$\psi(\tau)=\frac{\exp\{\gamma[1/\tau-1/(\tau-1)]\}}{\tau^2}\ .$$

Notice

$$Q_1=(-\gamma)\int_1^\infty d\tau_1\,\frac{\exp\{\gamma[1/\tau_1-1/(\tau_1-1)]\}}{\tau_1^2}=\tilde{Q}_1\ . \tag{6.21}$$

For $m\geq 2$, integrate $d\tau_1$ in (6.20) by parts:

$$Q_m-Q_{m-1}=q_m=(-\gamma)^m\Bigg(\int_m^\infty d\tau_1\,\phi(\tau_1)\int_{m-1}^{\tau_1-1}d\tau_2\,\psi(\tau_2)\int_{m-2}^{\tau_2-1}d\tau_3\psi(\tau_3)$$
$$\times\cdots\int_1^{\tau_{m-1}-1}d\tau_m\psi(\tau_m)+\frac{1}{\gamma}\int_m^\infty d\tau_1\,\frac{\exp\{\gamma[1/\tau_1-1/(\tau_1-2)]\}}{(\tau_1-1)^2}$$
$$\times\int_{m-2}^{\tau_1-2}d\tau_3\,\psi(\tau_3)\cdots\int_1^{\tau_{m-1}-1}d\tau_m\,\psi(\tau_m)\Bigg)\ . \tag{6.22}$$

We have to prove $q_m=\tilde{q}_m$ for $m\geq 2$. First integrate $d\tau_2$ in the first integral of (6.22) by parts. This gives three terms, one of which cancels with the second term in (6.22), to give

$$q_m=(-\gamma)^m\Bigg(\int_m^\infty d\tau_1\,\phi(\tau_1)\int_{m-1}^{\tau_1-1}d\tau_2\,\phi(\tau_2)\int_{m-1}^{\tau_2-1}d\tau_3\,\psi(\tau_3)\cdots$$
$$+\frac{1}{\gamma}\int_m^\infty d\tau_1\,\phi(\tau_1)\int_{m-1}^{\tau_1-1}d\tau_2\,\frac{\exp[\gamma/\tau_2-\gamma/(\tau_2-2)]}{(\tau_2-1)^2}\int_{m-3}^{\tau_2-2}d\tau_4\,\psi(\tau_4)\cdots\Bigg)\ . \tag{6.23}$$

We can now integrate over $d\tau_3$ by parts in the first integral, and so on. Eventually, we obtain

$$q_m=(-\gamma)^m\int_m^\infty d\tau_1\,\phi(\tau_1)\cdots\int_2^{\tau_{m-2}-1}d\tau_{m-1}\,\phi(\tau_{m-1})\int_1^{\tau_{m-1}-1}d\tau_m\,\psi(\tau_m)$$
$$+\frac{(-\gamma)^m}{\gamma}\int_m^\infty d\tau_1\,\phi(\tau_1)\cdots\int_2^{\tau_{m-2}-1}d\tau_{m-1}\,\frac{\exp\{\gamma[1/\tau_{m-1}-1/(\tau_{m-1}-2)]\}}{(\tau_{m-1}-1)^2}\ . \tag{6.24}$$

Integrating over $d\tau_m$ by parts now, we find

$$q_m=(-\gamma)^m\int_m^\infty d\tau_1\,\phi(\tau_1)\int_{m-1}^{\tau_1-1}d\tau_2\,\phi(\tau_2)$$
$$\cdots\int_1^{\tau_{m-1}-1}\phi(\tau_m)d\tau_m$$
$$=\tilde{q}_m\ ,\quad m\geq 2\ . \tag{6.25}$$

Since $Q_0=\tilde{Q}_0$, $Q_1=\tilde{Q}_1$. Using (6.16), (6.22), and mathematical inductions, we conclude that

$$Q_m=\tilde{Q}_m\ ,\quad m=0,\ 1,\ 2,\ \ldots\ .$$

Hence $\tilde{J}=f(0)$ or $W^2=1/J$.

Therefore the result of Frisch and Lloyd (for small $E$) and ours give the same exponential dependence on $E$, with the same coefficient.

## VII. CONCLUSIONS

We have used the assumption of independent big cells in the presence of only small cells in studying the low-lying spectrum of a particle in one-dimensional random repulsive $\delta$ potentials. The resulting density of states agrees with that of Frisch and Lloyd[2] at small energy. The same method can be used for any nonoverlapping finite-ranged repulsive potentials (see Appendix B).

## APPENDIX A: PROBABILITY DISTRIBUTION FUNCTION

The probability distribution function of specific cell lengths being $L^1,\ L^2,\ \ldots,\ L^\nu$ is from (3.8)

$$P(L^1,L^2,\ldots,L^\nu)=\frac{(\nu-1)!}{L^{\nu-1}}\delta(L^1+L^2+\ldots+L^\nu-L)\ . \tag{A1}$$

This is the microcanonical distribution in lengths. For a large system, it can be replaced by the canonical distribution

$$P(L^1,L^2,\ldots,L^\nu)=A'e^{-\rho(L^1+L^2+\cdots+L^\nu)},\quad 0\leq L^i<\infty \tag{A2}$$

$A'=\rho^\nu$ from normalization.

The total length

$$L=\sum_{i=1}^{\nu}L^i$$

now can take on values from 0 to $\infty$. However, we show that the fluctuation of $L$ around its average value is negligible for the distribution (A2).

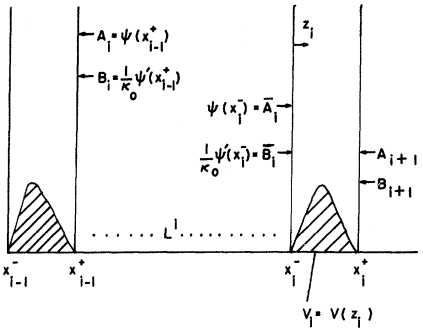


FIG. 4. Potentials for finite-ranged case. The potential is only nonzero between $x_i^-$ and $x_i^+$.

*Proof.*

$$\langle L\rangle = \int_0^\infty L\,\rho^\nu\, e^{-\rho(L^1+L^2+\cdots+L^\nu)}\,dL^1\cdots dL^\nu$$
$$= \nu/\rho\ . \tag{A3}$$

The mean-square fluctuation

$$(\Delta L)^2 = \langle L^2\rangle - \langle L\rangle^2,$$
$$\langle L^2\rangle = \int_0^\infty \left(\sum_i L^i\right)^2 \rho^\nu e^{-\rho(L^1+L^2+\cdots+L^\nu)}dL^1\cdots dL^\nu$$
$$= \int_0^\infty\left(\sum_i (L^i)^2 + \sum\sum_{i\neq j} L^iL^j\right)$$
$$\rho^\nu\, e^{-\rho(L^1+L^2+\cdots+L^\nu)}dL^1\cdots dL^\nu$$
$$= 2\nu/\rho^2 + \nu(\nu-1)/\rho^2\ .$$

Therefore

$$(\Delta L^2) = \nu/\rho^2\ ,$$

or

$$\frac{(\Delta L)}{\langle L\rangle} = \frac{\sqrt{\nu}/\rho}{\nu/\rho} = \frac{1}{\sqrt{\nu}} \to 0\ .$$

Hence (A2) is equivalent to (A1).

## APPENDIX B: GENERALIZATION TO ANY NONOVERLAPPING FINITE-RANGED POTENTIALS

In this appendix, we generalize our result to any nonoverlapping finite-ranged repulsive potentials.

Let the potentials be all the same, and nonzero only inside a range $b$ (Fig. 4):

$$V = \sum_{i=1}^{\nu=1} V_i\ , \tag{B1}$$

$$V_i = V(z_i) \geq 0, \quad b \geq z_i \geq 0$$
$$= 0, \qquad \text{otherwise} \tag{B2}$$

$$z_i = x - x_i^-, \quad b = x_i^+ - x_i\ , \tag{B3}$$

and

$$L^i = x_i^- - x_{i-1}^+ \tag{B4}$$

is the spacing between two potentials. The wave function between the two potentials is still given by that of a free particle:

$$\psi_i(x) = A_i\cos\kappa(x - x_{i-1}^+) + B_i(\kappa_0/\kappa)\sin\kappa(x - x_{i-1}^+),$$
$$x_{i-1}^+ < x < x_i^- \tag{B5}$$

where

$$A_i = \psi(x_{i-1}^+) \text{ and } B_i = (1/\kappa_0)\psi'(x_{i-1}^+). \tag{B6}$$

$\kappa_0$ is a positive number of the dimension of $\kappa$, to be chosen later. It is convenient to define also

$$\bar{A}_i = \psi(x_i^-) = A_i\cos\kappa L^i + B_i\,(\kappa_0/\kappa)\sin\kappa L^i\ ,$$
$$\bar{B}_i = (1/\kappa_0)\psi'(x_i^-) = -A_i(\kappa/\kappa_0)\sin\kappa L^i + B_i\cos\kappa L^i\ . \tag{B7}$$

We again use the assumption of independent big cells, and consider only low-lying energy levels. Call the big cell under consideration $j$, i.e.,

$$\kappa L^i \ll 1, \quad i \neq j$$
$$\kappa/\kappa_0 \ll 1. \tag{B8}$$

Equation (B7) reduces to

$$\bar{A}_i = A_i + B_i(\kappa_0 L^i), \quad \bar{B}_i = B_i\ . \tag{B9}$$

The relationship between $\bar{A}_i$, $\bar{B}_i$, and $A_{i+1}$, $B_{i+1}$ can be found by considering the wave function $\phi$ in $V_i$ (Fig. 4). Since $E \to 0$, this is given by

$$-\frac{\hbar^2}{2m}\frac{d^2}{dz^2}\,\phi(z) + V(z)\,\phi(z) = 0 \tag{B10}$$

or

$$\frac{d^2}{dz^2}\,\phi(z) = u(z)\,\phi(z)\ ,$$

where

$$u(z_i) \equiv 2mV(z)/\hbar^2\ ,$$
$$\phi(z_i) = \psi(z), \quad x_i^- \leq x \leq x_i^+\ .$$

Let $g(z)$ and $h(z)$ be the two independent solutions satisfying the boundary conditions.

$$g(0) = 1, \quad h(0) = 0,$$
$$g'(0) = 0, \quad h'(0) = 1. \tag{B11}$$

$g(z)$ and $h(z)$ are related by the Wronskian

$$g(z)h'(z) - g'(z)h(z) = 1\ . \tag{B12}$$

Define

$$h(z) \equiv \Omega(z)g(z)\ ,$$
$$h' = \Omega' g + g'\Omega\ , \tag{B13}$$
$$\Omega' = \frac{1}{g^2} \text{ or } \Omega = \int_0^z \frac{dz'}{g^2(z')}\ ,$$

i.e.,

$$h(z)=\int_0^z \frac{dz'}{g^2(z')}\, g(z)\ . \tag{B14}$$

The solution in the potential $V_i$, satisfying boundary conditions at $x_i^-$, is thus

$$\phi(z)=\bar{A}_i g(z)+\bar{B}_i \kappa_0 h(z)\ . \tag{B15}$$

Continuity at point $x_i^+$ thus gives

$$\begin{aligned} A_{i+1} &= g(b)\bar{A}_i+\bar{B}_i\kappa_0 h(b)\ , \\ B_{i+1} &= [g'(b)/\kappa_0]\bar{A}_i+\bar{B}_i h'(b)\ . \end{aligned} \tag{B16}$$

Using (B9), we get finally

$$\begin{aligned} A_{i+1} &= g(b)\,A_i+B_i[\kappa_0 L^i g(b)+\kappa_0 h(b)]\ , \\ B_{i+1} &= [g'(b)/\kappa_0]\,A_i+B_i[g'(b)L^i+h'(b)], \\ & \qquad i\neq j \end{aligned} \tag{B17}$$

where

$$h(b)=g(b)\int_0^b \frac{dz'}{g^2(z')}=g\delta\ ,\quad \delta\equiv\int_0^b \frac{dz'}{g^2(z')}\ , \tag{B18}$$

and

$$h'(b)=\frac{1}{g(b)}+g'(b)\int_0^b \frac{dz'}{g^2(z')}=\frac{1}{g}+g'\delta, \tag{B19}$$

in which we do not indicate the dependence on $b$ of $h(b)$, $h'(b)$, $g(b)$, and $g'(b)$ where no confusion may occur. Equation (B17) replaces (3.10) for the $\delta$-function case. Either $A_i$ or $B_i$ can be eliminated from (B17):

$$\begin{aligned} B_{i+2} &= B_{i+1}(g+g'L^{i+1}+h')-B_i\ , \\ A_{i+2} &= A_{i+1}\left(g+\frac{L^{i+1}+\delta}{L^i+\delta}(g'L^i+h')\right) \\ & \qquad -A_i\frac{L^{i+1}+\delta}{L^i+\delta}\ , \\ & \qquad i=1,\,2,\,\ldots,\,j-2\ . \end{aligned} \tag{B20}$$

Notice from Eqs. (B10)–(B14) that $g>1$, $h'>1$, $g'>0$, and $h>0$.

We briefly summarize the proof for localization near cell $j$. Choose $A_1=0$, and $B_1=B^L$; then

$$B_2=B^L(h'+g'L^1),\quad A_2=B^L\kappa_0(gL^1+h). \tag{B21}$$

Therefore, (B20) and mathematical induction give the following theorem.

*Theorem* ($L$):

(i) $B_{i+1}$ and $A_{i+1}$ have the same sign as $B^L$;

(ii) $|B_{i+1}|>|B_i|\quad (g+g'L^i)$;

(iii) $|A_{i+1}|>|A_i|\left(g+\frac{L^i+\delta}{L^{i-1}+\delta}g'L^{i-1}\right)$ ,

$$i=1,\,2,\,\ldots,\,j-1\ .$$

Therefore, the amplitude of the wave function grows exponentially from the left-hand boundary to cell $j$.

Similarly, we can show that the amplitude grows exponentially from the right-hand boundary to the cell $j$. The solution localized in big cell $j$ is thus

$$\kappa L^j = s\pi-\frac{\kappa}{\kappa_0}\frac{A_j}{B_j}-\frac{\kappa}{\kappa_0}\frac{A_j'}{B_j'}\ . \tag{B22}$$

Define

$$\begin{aligned} t_j^L &\equiv \frac{B_j}{A_j}=\frac{g'/\kappa_0+(g'L^{j-1}+h')t_{j-1}^L}{g+\kappa_0(gL^{j-1}+h)t_{j-1}^L}\ , \\ t_{j-1}^L &\equiv B_{j-1}/A_{j-1}\ ,\ \text{etc.} \end{aligned} \tag{B23}$$

Equation (B23) simplifies if we choose

$$\kappa_0\equiv g'/g>0\ , \tag{B24}$$

$$\begin{aligned} t_j^L &= \frac{g+(g'L^{j-1}+h')t_{j-1}^L}{g+(g'L^{j-1}+hg'/g)t_{j-1}^L} \\ &= 1+\frac{1/g}{(g'L^{j-1}+g'\delta)+g/t_{j-1}^L} \\ &= 1+\frac{1/g}{(g'L^{j-1}+g'\delta)+}\,\frac{g}{1+}\,\frac{1/g}{(g'L^{j-2}+g'\delta)+\cdots}\ . \end{aligned} \tag{B25}$$

By the same proof as given in Sec. III, the positive continued fraction in (B25) is convergent. Identical proofs hold for $t_j^R$.

Similar density of states can be obtained:

$$\langle N(E)\rangle=\langle N_\infty(E)\rangle W^2\ , \tag{B26}$$

$$N_\infty(E)=\frac{\nu}{e^{\rho c/\sqrt{E}}-1}\ W=\left\langle \exp\left(\rho\,\frac{g}{g'}\,\frac{1}{1+x_j}\right)\right\rangle\ , \tag{B27}$$

where

$$\rho\equiv\frac{\nu}{L-\nu b}\ , \tag{B28}$$

$$x_j\equiv t_{j-1}^L=\frac{(x_{j-1}+1)}{g[g+(x_{j-1}+1)g'(L^{j-1}+\delta)]}\ , \tag{B29}$$

and the equations satisfied by the distribution function $Q(x_j)$ of $x_j$ are

$$\begin{aligned} Q(x_0) &= \int_0^\infty \frac{Q(x')}{x_0^2 gg'}\,\rho \\ & \qquad \times\exp\left[-\rho\left(\frac{1}{gg'x_0}-\frac{g}{g'(x'+1)}-\delta\right)\right]dx'\ , \\ & \qquad 0\le x_0\le\frac{1}{g(g+g'\delta)} \end{aligned} \tag{B30}$$

$$Q(x_0)=\int_{g/[(gx_0)^{-1}-g'\delta]-1}^\infty \frac{Q(x')}{x_0^2 gg'}\,\rho$$

$$\times \exp\left[-\rho\left(\frac{1}{gg'x_0} - \frac{g}{g'(x'+1)} - \delta\right)\right] dx' ,$$

$$x_0 \geq \frac{1}{g(g+g'\delta)} \quad \text{(B31)}$$

and the normalization condition

$$\int_0^\infty Q(x_0)\,dx_0 = 1 . \quad \text{(B32)}$$

That is,

$$W = \int_0^\infty Q(x_0) \exp\left(\rho \frac{g}{g'(1+x_0)}\right) dx_0 . \quad \text{(B33)}$$

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# Bose–Einstein Condensation in a One-Dimensional Model with Random Impurities*

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We study the thermodynamics of a one-dimensional system of noninteracting bosons in the presence of random fixed finite-range repulsive potentials. It is shown that a Bose–Einstein condensation into the lowest state occurs. The critical temperature increases with the number of impurities. The specific heat exhibits a discontinuity and is infinitely differentiable from both sides of the critical point. For a suitably defined pressure the isothermals exhibit a flat portion below the "transition length."

## I. INTRODUCTION

In this paper we shall study a one-dimensional gas of independent spinless bosons in the presence of random impurities. These impurities are represented by fixed finite-range repulsive potentials. The model is identical to the one studied in the preceding paper,[1] where the structure of the low-lying states and the density of states was investigated. There we found it possible to obtain these levels for an arbitrary finite-range impurity potential. Now we imagine independent bosons occupying these states. In the interest of simplicity the calculations are only carried out for the case where the impurity potential is represented by a $\delta$ function of infinite strength. Considerations of the same nature may be made for the general case[2] (because the structure of the low-lying levels is very similar), but we do not go into these here, as the details prove to be quite complicated.

In Sec. II we show that the Bose–Einstein transition occurs in the thermodynamical limit. In Sec. III the transition temperature and detailed nature of the transition are investigated. Finally, in Sec. IV, it is shown that our result is not in contradiction with the well-known Bogoliubov inequality,[3] which has been used by Hohenberg[4] to show that there is no condensation into the lowest-momentum state in one-dimensional systems of bosons.

## II. EXISTENCE OF BOSE-EINSTEIN TRANSITION

For the case of impurity potentials represented by $\delta$ functions of infinite strength, the energy levels can be given explicitly.[1] Let $L_j$ represent the $j$th-largest distance between $\delta$ functions (i.e., the length of the $j$th-largest "cell"). For $\nu - 1$ impurities, $L_1 \geq L_2 \geq L_3 \cdots \geq L_\nu$.

The levels are

$$E_{j,s} = c^2 s^2 / L_j^2, \quad s = 1, 2, \ldots, \infty \qquad (2.1)$$

$$c^2 \equiv \hbar^2 \pi^2 / 2m .$$