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$2N\pi$ Ultrashort Light Pulses

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A method for obtaining analytical expressions for $2N\pi$ pulses in an unbroadened resonant medium is derived by a systematic application of the Bäcklund transformation. The resulting large-area pulses display the expected property of pulse breakup into separate 2π hyperbolic secant pulses.

INTRODUCTION

A distinctive property of ultrashort-light-pulse propagation in resonant media is the conservation of pulse area for multiple 2π pulses. This result, which applies to an ideal attenuator with inhomogeneously broadened line shape, was first stated in an area theorem by McCall and Hahn.[1] A similar result holds for unbroadened ideal attenuators.[2] In this communication, we shall derive analytical expressions for multiple 2π pulses to any order which may propagate in unbroadened media without change of pulse area. As these pulses propagate in the resonant medium, they evolve into separate 2π hyperbolic secant (hs) pulses characteristic of self-induced transparency (SIT). A similar result has been obtained for multiple 2π pulses in broadened media by numerical integration of the equations of motion.[1,3]

EQUATIONS OF MODEL

The equations of motion for ultrashort-pulse propagation in resonant media have been discussed in several papers[1,2,4] and so they will only be reviewed here. The classical field serves as a perturbation to the two-level nondegenerate resonant medium which is treated in the dipole approximation according to the Bloch formalism.[5] The field is assumed to be a linearly polarized plane wave propagating in the z direction and is approximated by a carrier frequency $\phi = \omega_0(t - z/c)$ and a slowly varying amplitude $\epsilon(z, t)$ and phase $\theta(z, t)$:

$$E(z, t) = \epsilon(z, t)\cos[\phi - \theta(z, t)] \ . \tag{1}$$

If we assume that the atomic transition frequency is equal to the field-carrier frequency, the equations of motion reduce[2] to

$$\left(\frac{\partial}{\partial t} + c\,\frac{\partial}{\partial z}\right)\tilde{\epsilon} = \frac{2\pi n_0 \omega_0 p^2}{\hbar}\, S \ , \tag{2}$$

$$C = 0 \ , \tag{3}$$

$$\dot{S} = \tilde{\epsilon}\eta \ , \tag{4}$$

$$\dot{\eta} = -\tilde{\epsilon} S \ , \tag{5}$$

where C and S are defined in terms of the macroscopic polarization $P(z, t)$,

$$P(z, t) = n_0 p[C(z, t)\cos\phi + S(z, t)\sin\phi] \ , \tag{6}$$

and $\eta(z, t)$ is the normalized atomic population inversion

$$\eta(z, t) = |a(t)|^2 - |b(t)|^2 \ .$$

$a(t)$ and $b(t)$ are the coefficients of upper and lower energy states in the atomic wave function, p is the atomic dipole matrix element, n_0 is the density of active atoms, and $\tilde{\epsilon} = (p/\hbar)\epsilon$.

Equations (4) and (5) may be integrated in terms of the partial field area

$$\psi(z, t) = \int_{-\infty}^{t} \tilde{\epsilon}(z, t)\, dt \tag{7}$$

to give

$$S(z, t) = \sin\psi(z, t) \ ,$$
$$\eta(z, t) = -\cos\psi(z, t) \ . \tag{8}$$

Equation (2) may then be expressed as

$$\frac{\partial^2\psi}{\partial\xi\,\partial\tau} = -\sin\psi \tag{9}$$

in terms of the variables $\tau = \alpha(t - z/c)$ and $\xi = \alpha z/c$, where

$$\alpha = (2\pi n_0 \omega_0 p^2/\hbar)^{1/2} \ .$$

We wish to find solutions of Eq. (9) for which $\psi(z, +\infty)$ is an integer multiple of 2π. As may be seen by integrating Eq. (9) from $-\infty$ to $+\infty$, these solutions propagate without change in area. A useful technique for obtaining solutions to Eq. (9) has been pointed out by Lamb[6] and consists of the Bäcklund transformation.[7] This method permits generating a second solution of Eq. (9) from a known one by solving the coupled pair of first-order partial-differential equations

$$\frac{\partial}{\partial\xi}\,\frac{1}{2}\,(\psi_1 - \psi_0) = a\sin\frac{1}{2}\,(\psi_1 + \psi_0) \ ,$$
$$\frac{\partial}{\partial\tau}\,\frac{1}{2}\,(\psi_1 + \psi_0) = -\frac{1}{a}\sin\frac{1}{2}\,(\psi_1 - \psi_0) \ , \tag{10}$$

where a is an arbitrary constant. We denote the solution of Eqs. (10) for ψ_1 in terms of ψ_0 by the

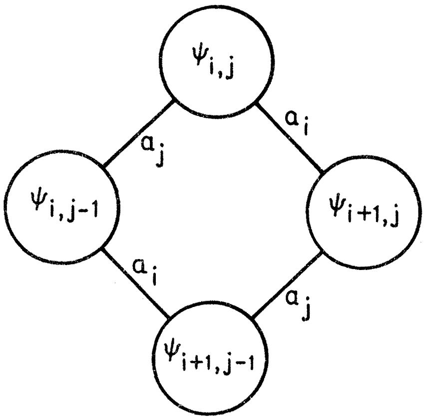

FIG. 1. Notation convention for pulses linked by Bäcklund transformation.

expression $\psi_1 = B_a\psi_0$. If four solutions of Eq. (9) are linked by Bäcklund transformations in terms of two constants, as indicated below,

$$\psi_1 = B_{a_1}\psi_0 \ ,$$
$$\psi_2 = B_{a_2}\psi_0 \ ,$$
$$\psi_3 = B_{a_2}\psi_1 = B_{a_1}\psi_2 \ ,$$

then the four solutions obey a simple trigonometric relationship

$$\tan\left(\frac{\psi_3 - \psi_0}{4}\right) = \frac{a_1 + a_2}{a_1 - a_2}\tan\left(\frac{\psi_1 - \psi_2}{4}\right) \ . \qquad (11)$$

New solutions of the second-order partial-differential equation [Eq. (9)] may thus be expressed as algebraic combinations of known solutions.

$2N\pi$ PULSES

In this section a recursion relation is derived for generating $2N\pi$ pulses, where N is any integer. A diagrammatic representation of the Bäcklund transformation suggested by Lamb[2] will be useful in visualizing the analytical technique (see Fig. 1). Figure 1 also defines the notation convention for the four solutions of Eq. (9), $\psi_{i,j}$, etc., and the two constants a_i and a_j, which are related by the trigonometric form of the Bäcklund transformation [Eq. (11)].

The pulse area $\theta_{i,j} = \psi_{i,j}(z, +\infty)$ is given by

$$\theta_{i,j} = \theta_{i+1,j-1} + \theta_{i,j-1} - \theta_{i+1,j} \ , \qquad (12)$$

if $(a_i + a_j)/(a_i - a_j) > 0$ and if $\theta_{i+1,j-1}$, $\theta_{i,j-1}$, and $\theta_{i+1,j}$ are integer multiples of 2π. This may be demonstrated by following the evolution of

$$\psi_{i,j}(z,t) = \psi_{i+1,j-1}(z,t)$$
$$+ 4\tan^{-1}\left[(a_i + a_j)/(a_i - a_j)\tan\frac{\psi_{i,j-1} - \psi_{i+1,j}}{4}\right] \qquad (13)$$

across the branch points of the tangent function as t goes from $-\infty$ to $+\infty$. The field amplitude is also obtained from this expression by differentiation with respect to time:

$$\tilde{\mathcal{E}}_{i,j} = \tilde{\mathcal{E}}_{i+1,j-1} + K_{ij}\frac{1 + g_{ij}^2}{1 + K_{ij}^2 g_{ij}^2}(\tilde{\mathcal{E}}_{i,j-1} - \tilde{\mathcal{E}}_{i+1,j}) \ , \qquad (14)$$

where

$$K_{ij} = (a_i + a_j)/(a_i - a_j)$$

and

$$g_{ij} = \tan\left[\tfrac{1}{4}(\psi_{i,j-1} - \psi_{i+1,j})\right] \ .$$

The simplest solution of Eq. (9) is the trivial $\psi = 0$. If we let $\psi_{i+1,i} = 0$ for all $i > 0$, then it follows from Eq. (10) that $\psi_{i,i} = 4\tan^{-1}e^{u_i}$, where $u_i = a_i\alpha(t - z/v_i) + \gamma_i$, where γ_i is a constant and $1/v_i = (1/c)(1 + 1/a_i^2)$ for all $i > 0$. The corresponding electric field amplitude is $\mathcal{E}_{i,i} = 2\alpha a_i \operatorname{sech} u_i$. Since $\psi_{i,i}(z, +\infty) = \pm 2\pi$ for $a_i \gtrless 0$, $\psi_{i+1,i}$ and $\psi_{i,i}$ can be used as a basis for building multiple 2π pulses by repeated use of Eq. (13). Figure 2 shows a sequence of transformations that may be extended systematically to obtain analytical expressions for arbitrary large-area (or negative-area) pulses. The pulse areas are indicated in the circles which designate the pulse functions. The pulse area is alternately $+2M\pi$ and $-2M\pi$ across each row, where M is the number of rows from the bottom.

The transformation constants must be constrained by a set of inequalities in order to obtain $2N\pi$ pulses. Since the second row of pulses has alternately $+2\pi$ and -2π area, the constants must be alternately positive and negative.

Thus,

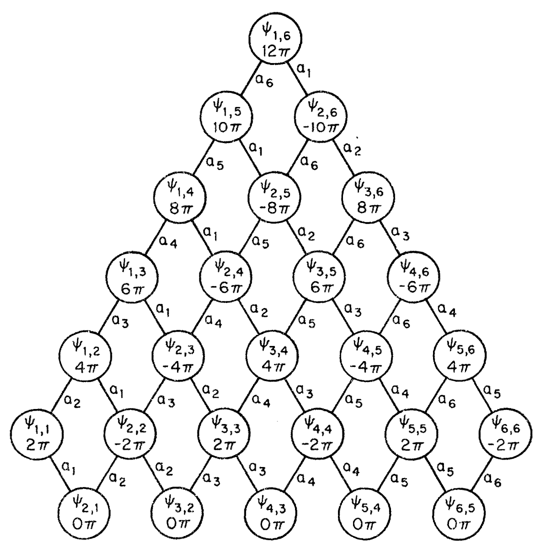

FIG. 2. Sequence of Bäcklund transformations for generation of 12π pulse.

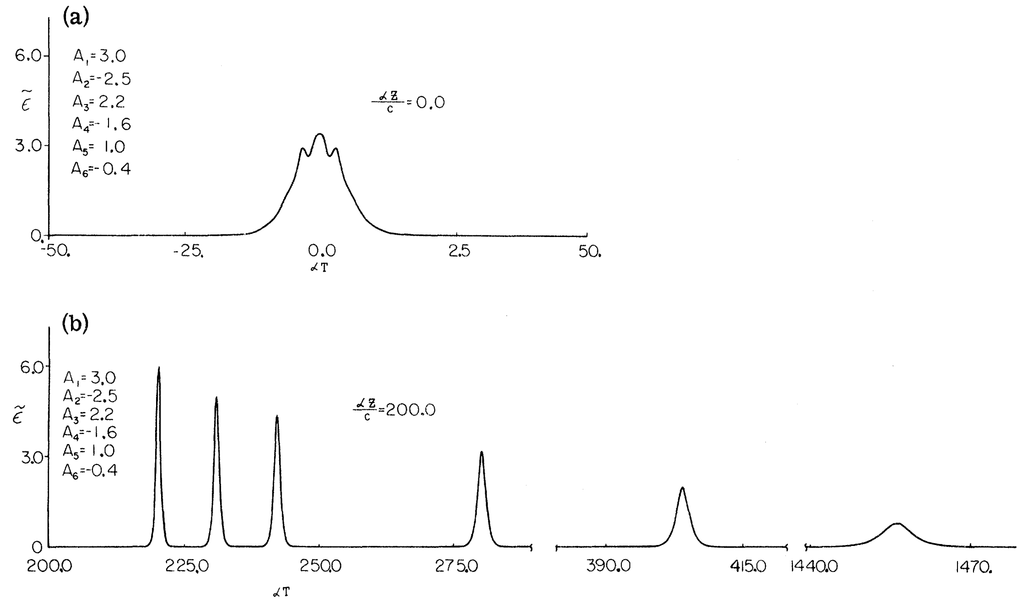

FIG. 3. 12π pulse (a) before and (b) after pulse breakup for the constants $a_1=3.0$, $a_2=-2.5$, $a_3=2.2$, $a_4=-1.6$, $a_5=1.0$, and $a_6=-0.4$.

$$(-1)^i a_i < 0 \; . \tag{15}$$

Also, the inequality

$$(a_i + a_j)/(a_i - a_i) > 0 \tag{16}$$

must be preserved for all $j > i$. It may readily be seen that both Eqs. (15) and (16) are satisfied if

$$(-1)^i a_i < (-1)^j a_j \; , \quad j > i \tag{17}$$

which yields a self-consistent set of inequalities for the transformation constants. The expressions which define the $2N\pi$ pulses are summarized below:

$$\left.\begin{array}{l} \psi_{i+1,i} = 0 \; , \\ \psi_{i,i} = 4\tan^{-1} e^{u_i} \; , \\ (-1)^i a_i < 0 \; , \end{array}\right\} \quad i > 0 \; ;$$

$$\left.\begin{array}{l} \psi_{i,j} = \psi_{i+1,j-1} + 4\tan^{-1}\left[K_{ij}\tan\left(\dfrac{\psi_{i,j-1} - \psi_{i+1,j}}{4}\right)\right] , \\ \tilde{\epsilon}_{i,j} = \tilde{\epsilon}_{i+1,j-1} + K_{ij}\dfrac{1 + g_{ij}^2}{1 + K_{ij}^2 g_{ij}^2}(\tilde{\epsilon}_{i,j-1} - \tilde{\epsilon}_{i+1,j}) \\ \theta_{i,j} = (-1)^{i+1}(j - i + 1)2\pi \; , \\ (-1)^i a_i < (-1)^j a_j \; , \end{array}\right\} \quad j > i \; .$$

In order to generate the analytical expression for one of the higher-order pulses, it is necessary first to obtain the expressions for the constituent pulses. This corresponds in Fig. 2 to finding the expressions for all solutions within the triangle whose apex is the solution in question and whose base is the set of trivial 0π pulses.

DISCUSSION

It has been noted previously[1] that the only stable SIT pulse is the 2π hs pulse and that, consequently, a multiple 2π pulse will break up into separate 2π hs pulses. This fact may be verified by calculating the field envelope according to Eq. (14). The result is also expected from the manner in which higher-order pulses are constructed by means of the Bäcklund transformation.

In order to obtain a $2N\pi$ pulse, one must combine N hs pulses each with a different velocity. For large values of z, only one retarded-time variable $u_i = a_i\alpha(t - z/v_i) + \gamma_i$ will be near zero for a given value of t. Since the pulse is centered about $u_i = 0$ and has significant amplitude only in a limited region around zero, it is as though only one nontrivial pulse were present in the hierarchy of Bäcklund transformations for a given value of z, t. This is equivalent to requiring that all a_j be zero except for one such as a_i, where i depends on z, t. It is clear from Eqs. (13) and (14) that any multiple 2π pulse will consist of separate 2π hs pulses.

A plot of a 12π pulse is shown in Fig. 3 before and after it has evolved into six 2π hs pulses. The pulses are ordered according to their velocities $c[a_i^2/(1 + a_i^2)]$, their widths $1/\alpha a_i$, and their amplitudes αa_i, as would be expected.

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Hartree–Fock Third-Harmonic Coefficient of Neon

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This is a note concerned with a previous publication by R. Klingbeil, V. G. Kaveeshwar, and R. P. Hurst [Phys. Rev. A 4, 1760 (1971)] giving specific results for neon.

Recently Klingbeil, Kaveeshwar, and Hurst[1] (KKH) derived the Hartree–Fock expression for the third-harmonic coefficient (THC) of an atomic system by a method employing Chung's time-dependent perturbation theory.[2] KKH also obtained the Hartree–Fock THC (HFTHC) of helium. It is the purpose of this note to present the results of a calculation of the HFTHC of neon.

It is necessary to obtain zeroth-, first-, and second-order wave functions in order to evaluate the expression for the THC given by Eq. (17) of KKH. The zeroth-order wave function chosen is that of Clementi.[3] First-order wave functions are obtained by the method of Kaveeshwar, Chung, and Hurst,[4] and agreement with their calculation of the linear dynamic polarizability is excellent. Second-order wave functions are obtained by the method of KKH.

The results of the calculation of the HFTHC $\chi_{zzzz}(-3\omega;\omega,\omega,\omega)$ of neon are presented in Table I and plotted in Fig. 1 along with the HFTHC of helium[1] for comparison. At zero frequency the HFTHC of neon is 54.0 a.u. Surprisingly, this result is significantly larger than the Hartree–Fock static hyperpolarizability result of 42 a.u. obtained by Sitter and Hurst.[5] Both results are considerably below the experimental static hyperpolarizability

TABLE I. Hartree–Fock third-harmonic coefficient of neon.

ω (a.u.)	λ(Å)	$\chi_{zzzz}(-3\omega;\omega,\omega,\omega)$ (a.u.)
0.000	∞	54.0[a]
0.025	18 238	54.9
0.050	9119	57.9
0.075	6079	63.2
0.100	4559	72.0
0.125	3648	86.1
0.150	3040	109.6
0.175	2605	152.5
0.200	2280	246.7
0.225	2026	570.6
0.247	1846[b]	100 939.1
0.248	1838	−16 333.3

[a]$\chi_{zzzz}(-3\omega;\omega,\omega,\omega)=54.0$ a.u. at zero frequency.
[b]$\frac{1}{3}$(1846 Å) = 615 Å is the calculated value of the first transition wavelength. The experimental value is 736 Å.

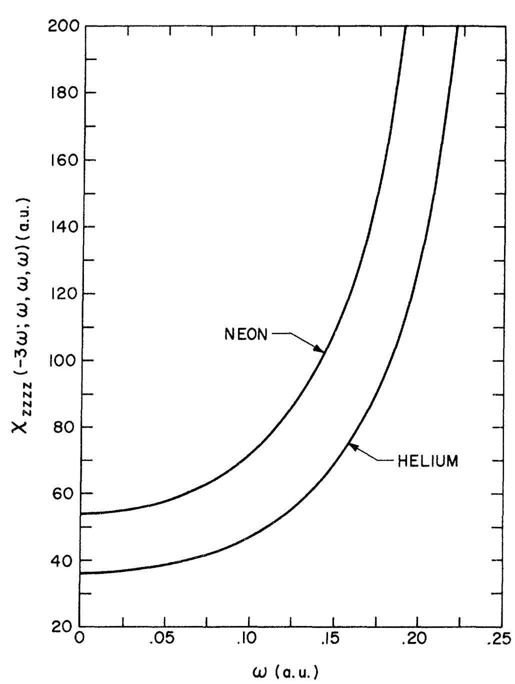

FIG. 1. Hartree–Fock third-harmonic coefficient $\chi_{zzzz}(-3\omega;\omega,\omega,\omega)$ of helium (Ref. 1) and neon (this work).