Specific Heat of a Normal Fermi Liquid. I. Landau-Theory Approach*

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The contribution of small-momentum-transfer processes to the $T^3 \ln T$ term in the specific heat of a normal Fermi liquid is considered using Landau theory, and is evaluated exactly in terms of Landau parameters. This contribution is shown to be due to a term in the quasiparticle interaction, $f_{\vec{p},\vec{p}+\vec{q}}$, which varies as $(\hat{p}\cdot\hat{q})^2$ when $\vec{q}\to 0$; the physical process responsible for this behavior is the repeated scattering of quasiparticle-quasihole pairs. It is well known that for the two-body problem the interaction energy, while not, in general, equal to the real part of the forward-scattering amplitude, may be expressed in terms of the scattering amplitude. We take over the results for the two-body problem and apply them to a quasiparticle-quasihole pair. This gives an expression for $f_{\vec{p},\vec{p}+\vec{q}}$ in terms of the quasiparticle-quasihole scattering amplitude, which, since $\vec{q}$ is small, may be expressed in terms of Landau parameters. We apply our results to liquid He^3 at low pressure.

I. INTRODUCTION

The quasiparticle concept, introduced by Landau,[1,2] has clarified a whole area of the physics of many-particle systems, and has been applied to a large number of equilibrium and nonequilibrium problems. The basic assumption of Landau theory is that the states of an interacting Fermi system are in one-to-one correspondence with the states of the noninteracting system; thus, one may specify the state of the interacting system by giving the quasiparticle distribution function $n_{\vec{p}}$, which is equal to the distribution of particles in the corresponding state of the noninteracting system. The total energy of the system $E[n_{\vec{p}}]$ is thus a functional only of the distribution function, and the entropy S, which depends purely on combinatorial considerations, has the same form as for a free Fermi gas:

$$S = -k_B \sum_{\vec{p}} [n_{\vec{p}} \ln n_{\vec{p}} + (1-n_{\vec{p}})\ln(1-n_{\vec{p}})] \quad , \qquad (1)$$

where k_B is Boltzmann's constant (and sums over spins are implicit). The equilibrium distribution function $n^0_{\vec{p}}(T)$ at a temperature T is determined by maximizing the entropy for a given total energy; one finds

$$n^0_{\vec{p}}(T) = [\exp(\beta\{\epsilon_{\vec{p}}[n^0_{\vec{p}}(T)] - \mu\}) + 1]^{-1} \quad , \qquad (2)$$

where the quasiparticle energy $\epsilon_{\vec{p}}$ is the functional derivative of the energy with respect to the quasiparticle distribution function,

$$\epsilon_{\vec{p}}[n_{\vec{p}}] = \frac{\delta E[n_{\vec{p}}]}{\delta n_{\vec{p}}} \quad , \qquad (3)$$

μ is the chemical potential, and $\beta \equiv 1/k_B T$. The equilibrium distribution function $n^0_{\vec{p}}(T)$ and quasiparticle energy $\epsilon_{\vec{p}}[n^0_{\vec{p}}(T)] \equiv \epsilon_{\vec{p}}(T)$ are to be determined self-consistently from Eqs. (2) and (3). The entropy of the system in equilibrium is therefore the same as that of a collection of noninteracting fermions whose (temperature dependent) energies are $\epsilon_{\vec{p}}(T)$. Intuitively, one would expect the quasiparticle expression (1) for the entropy to be valid provided the lifetimes of the relevant quasiparticle states are sufficiently long. However, Balian and De Dominicis[3] and Luttinger[4] have shown within the framework of perturbation theory that, in general, it is possible to define "statistical quasiparticle energies" which obey the same equations as the quasiparticle energy introduced by Landau and, in particular, give the correct result for the entropy when inserted into the quasiparticle expression; somewhat surprisingly, this result is true at arbitrary temperatures, since it holds even if the states of the system are short lived.

Work over the last few years on the specific heat of liquid He^3 brought to light two important features of normal Fermi systems. First, the asymptotic expansion for the specific heat at low temperatures contains not only powers of T, but also terms such as $T^3 \ln T$, and second, for calculating the $T^3 \ln T$ term it is incorrect to identify the statistical quasiparticle energy with the "dynamical quasiparticle energy" defined by the poles of the single-particle propagator. When measurements of the heat capacity of liquid He^3 were extended to temperatures below 100 m°K[5] it became clear that the data could not be fitted by a series containing only odd powers of the temperature, as one would expect on the basis of Landau Fermi-liquid theory if the (statistical) quasiparticle energy were analytic at the Fermi surface.[6] Anderson[7] pointed out that the data at that time could be fitted quite well by a $T \ln T$ behavior, and suggested that the nonanalytic behavior might result from the coupling of quasiparticles to collective modes. Following this, Balian and Fredkin[8] explored the coupling of quasiparticles to zero sound and showed that this could lead to a contribution to the specific heat of order $T\ln T$ or $T(\ln T)^{1/2}$. In these calculations the coupling between quasiparticles and zero sound was assumed to be long range, similar to that between electrons and phonons in piezoelectric crystals. Subsequently, Engelsberg and Platzman,[9] among others, showed that this assumption led to inconsistencies and argued that the coupling should be of a deformation-potential type,[10] in which case the nonanalytic contribution to the specific heat would behave as $T^3\ln T$, just as Buckingham and Schafroth[11] and Eliashberg[12] earlier had found for the interacting electron-phonon system. More recent calculations indicated, however, that the $T^3\ln T$ contribution to the specific heat coming from the coupling of quasiparticles to zero sound is too small in magnitude to account for the experimental data.

The next development was the discovery by Doniach and Engelsberg,[13] and by Berk and Schrieffer[14] of the importance of incoherent spin fluctuations. Owing to quasiparticle interactions the magnetic susceptibility of liquid He^3 is very much greater than that of an ideal Fermi gas,[15] and this leads to strong coupling between quasiparticles and low-lying states containing spin fluctuations (quasiparticle-quasihole pairs in triplet spin states). Coupling of quasiparticles to such spin fluctuations was shown to give a $T^3\ln T$ contribution to the specific heat rather than the $T\ln T$ behavior suggested by Anderson.[7] However, the theoretical estimate of the $T^3\ln T$ term was of the right order of magnitude to account for the experimental data, and moreover the $T^3\ln T$ dependence was not inconsistent with the data.

Nonanalytic contributions to the quasiparticle energy, which give rise to the $T^3\ln T$ terms in the specific heat are a very general feature of all normal Fermi liquids, and their existence does not depend in any way on the fact that liquid He^3 is almost ferromagnetic; such contributions were, in fact, obtained in Galitskii's[16] work on the dilute Fermi gas, but their magnitude in the case of liquid He^3 was appreciated only after studies of almost ferromagnetic Fermi liquids. Amit, Kane, and Wagner[17] showed that the nonanalytic contributions to the *dynamical* quasiparticle energy could be calculated in terms of Landau parameters, and they then evaluated the entropy by inserting these energies into the quasiparticle expression (1). Similar calculations were also performed by Brenig and Mikeska[18] and by Emery.[19] Further work by Brenig, Mikeska, and Riedel,[20] by Riedel[21] and by Brinkman and Engelsberg[22] on models of almost ferromagnetic Fermi liquids showed that this way of calculating the entropy, while it does give the term linear in T correctly, does not give the correct result for the $T^3\ln T$ term: it gives a result *three* times larger than the true answer. In these models the physical process taken into account was repeated scattering of a particle-hole pair. Riedel showed that the entropy could be expressed as the usual quasiparticle expression (1) evaluated using dynamical quasiparticle energies, plus an additional "Bose" contribution coming from interacting particle-hole pairs.[21] The Bose contribution has two distinct terms. The first comes from true collective modes (such as zero sound, in liquid He^3) which correspond to isolated poles in the particle-hole propagator; it has the same form as the entropy of a system of noninteracting bosons whose energies are given by the poles of the particle-hole propagator. However, at low temperatures it is of order T^3, and can therefore be neglected here since we are concerned only with $T^3\ln T$ terms.[23] The remaining part of the Bose contribution to the entropy comes from the particle-hole continuum and includes a contribution from incoherent spin fluctuations, which are not true collective modes of the system. Thus in Riedel's calculation the entropy to order $T^3\ln T$ is expressed as a sum of the *dynamical* quasiparticle contribution and the Bose contribution coming from the particle-hole continuum. However, Balian and De Dominicis[3] and Luttinger[4] have established that the entropy is given by the quasiparticle expression evaluated using *statistical* quasiparticle energies. It is therefore clear that Riedel's Bose contribution coming from the particle-hole continuum may be regarded equally well as being due to the difference between dynamical and statistical quasiparticle energies.

Two obvious ways to calculate the $T^3\ln T$ term better are either to extend Riedel's analysis to more general terms in perturbation theory, or to calculate statistical quasiparticle energies from microscopic theory. Here we adopt a simpler approach based on Landau Fermi-liquid theory, rather than microscopic theory. [We note that quasiparticle energies entering the Landau-theory calculations satisfy the relations (3) and are therefore *statistical* quasiparticle energies.] In a companion paper we derive the Landau-theory results from microscopic theory; there we also discuss the difference between statistical and dynamical quasiparticle energies and show how this difference is related to Riedel's Bose contribution to the entropy.[24]

Previous work suggests that the major part of the $T^3\ln T$ contribution to the specific heat of liquid He^3 comes from interactions between quasiparticles whose momenta are almost equal.[17] We shall show that this part of the $T^3\ln T$ contribution may be expressed in terms of Landau parameters, and that the logarithmic behavior is due to the fact that the quasiparticle interaction

$$f_{\vec{p},\vec{p}+\vec{q}} = \frac{\delta\epsilon_{\vec{p}}}{\delta n_{\vec{p}+\vec{q}}} \tag{4}$$

has a contribution which varies as $(\hat{p}\cdot\hat{q})^2$ in the limit $\vec{q}\to 0$.[25] This term in the quasiparticle interaction also leads to $(p-p_F)^3\ln|p-p_F|$ terms in the (statistical) quasiparticle energy (p_F is the Fermi momentum). The physical process responsible for the $(\hat{p}\cdot\hat{q})^2$ terms is the repeated scattering of a quasiparticle-quasihole pair; it is therefore convenient to consider the quasiparticle-quasihole interaction, which is just $-f_{\vec{p},\vec{p}+\vec{q}}$, rather than the quasiparticle-quasiparticle interaction itself. The amplitude for a quasiparticle and quasihole with small total momentum to scatter each other may be expressed in terms of Landau parameters; the essential problem that remains is to relate energies of interaction to scattering amplitudes. It has often been assumed that the energy of interaction and the real part of the forward-scattering amplitude are identical. However, this is not true in general, since it is well known that for two interacting particles the energy of interaction is proportional to the phase shift δ, whereas the real part of the forward-scattering amplitude is proportional to $\sin\delta\cos\delta$.[26] We take over results from the two-body problem and apply them to the case of a quasiparticle-quasihole pair. We find that in calculating the $T^3\ln T$ contribution to the specific heat it is essential to take into account the difference between the energy of interaction and the forward-scattering amplitude. As we shall see, to the order to which we are working the energy of interaction is related to the *statistical* quasiparticle energy in the same way as the real part of the forward-scattering amplitude is related to the *dynamical* quasiparticle energy. With the statistical quasiparticle energies the quasiparticle expression (1) for the entropy is correct to order $T^3\ln T$, while with the dynamical quasiparticle energies it is not.[23]

This paper is organized as follows: In Sec. II we assume that the quasiparticle interaction $f_{\vec{p},\vec{p}+\vec{q}}$ has contributions varying as $(\hat{p}\cdot\hat{q})^2$ in the limit $\vec{q}\to 0$, and show that they give rise to $(p-p_F)^3 \times\ln|p-p_f|$ terms in the quasiparticle energy and $T^3\ln T$ terms in the specific heat. In Sec. III we show that the quasiparticle interaction indeed has the assumed behavior. There we first discuss the difference between forward-scattering amplitudes and energies of interaction, and then apply the results to evaluate the energy of interaction of a quasiparticle with a quasihole. In Sec. IV we discuss the difference between our calculations and previous ones, and in Sec. V we give numerical results for liquid He^3.

II. QUASIPARTICLE ENERGY AND SPECIFIC HEAT

In this section we show that contributions to the quasiparticle interaction $f_{\vec{p},\vec{p}+\vec{q}}$, which vary as $(\hat{p}\cdot\hat{q})^2$ as $\vec{q}\to 0$, lead to $(p-p_F)^3\ln|p-p_F|$ terms in the quasiparticle energy[27] and $T^3\ln T$ contributions to the specific heat. In Sec. III we show that this is the correct form for $f_{\vec{p},\vec{p}+\vec{q}}$ and calculate the coefficient of this term as a function of the Landau parameters.

The quasiparticle interaction $f_{\vec{p}\vec{\sigma},\vec{p}+\vec{q}\vec{\sigma}'}$ depends on the momenta of the quasiparticles $\vec{p}$ and $\vec{p}+\vec{q}$ and on their spins $\vec{\sigma}$ and $\vec{\sigma}'$, and may be expressed in terms of the spin-symmetric and spin-antisymmetric parts f^s and f^a by the usual relation[28]

$$f_{\vec{p}\vec{\sigma},\vec{p}+\vec{q}\vec{\sigma}'} = f^s_{\vec{p},\vec{p}+\vec{q}} + f^a_{\vec{p},\vec{p}+\vec{q}}\vec{\sigma}\cdot\vec{\sigma}' , \tag{5}$$

where $\vec{\sigma}$ and $\vec{\sigma}'$ on the right-hand side of this equation are the Pauli spin matrices for the two quasiparticles. As we shall show in Sec. III, for small q, and p close to the Fermi momentum p_F, $f_{\vec{p},\vec{p}+\vec{q}}$ depends only on $\hat{p}\cdot\hat{q}$, and is an even function of this quantity. For calculating the $T^3\ln T$ contribution to the specific heat we need only $f_{\vec{p},\vec{p}+\vec{q}}$ evaluated for $p=p_F$, since the dependence of f on the magnitude of p gives rise to contributions of higher order in the temperature. As we shall see, the $T^3\ln T$ contribution to the specific heat comes from small values of $\hat{p}\cdot\hat{q}$. We therefore expand f in powers of $\hat{p}\cdot\hat{q}$:

$$f^\lambda_{\vec{p},\vec{p}+\vec{q}} = f^\lambda(0) + b^\lambda(\hat{p}\cdot\hat{q})^2 + O\,[(\hat{p}\cdot\hat{q})^4] , \quad q \ll p_F \tag{6}$$

where the superscript λ is either s, for the spin-symmetric case, or a, for the spin-antisymmetric case, and $f^\lambda(0) \equiv f^\lambda_{\vec{p},\vec{p}+\vec{q}}$ evaluated at $\hat{p}\cdot\hat{q}=0$. Note that when evaluating properties of a Fermi liquid in the extreme low-temperature limit the appropriate quasiparticle interaction to use for small q is $f^\lambda(0)$, since when q is small, $\hat{p}\cdot\hat{q}$ $[=(\epsilon_{\vec{p}+\vec{q}}-\epsilon_{\vec{p}})/v_F q]$ for the quasiparticle states excited appreciably at temperature T is of the order of $k_B T/v_F q$, where v_F is the Fermi velocity. Therefore, for any finite value of q, the typical value of $\hat{p}\cdot\hat{q}$ tends to zero as $T\to 0$.

We now investigate the consequences of expression (6) for the quasiparticle interaction. First, we consider the contribution to the quasiparticle energy which is due to interactions between a quasiparticle of momentum $\vec{p}$ and other quasiparticles whose momenta differ little from $\vec{p}$. The quasiparticle interaction f is the functional derivative of the quasiparticle energy $\epsilon_{\vec{p}\sigma}$, with respect to the quasiparticle distribution function $n_{\vec{p}'\sigma'}$ [see Eq. (4)]:

$$\delta\epsilon_{\vec{p}\sigma} = \sum_{\vec{q}\sigma'} f_{\vec{p}\sigma,\vec{p}+\vec{q}\sigma'}[n_{\vec{p}\sigma}]\delta n_{\vec{p}+\vec{q}\sigma'} \quad . \tag{7}$$

In general, Eq. (7) cannot be integrated simply, since f depends on the quasiparticle distribution $n_{\vec{p}\sigma}$. However, here we are concerned with the nonanalytic contribution to $\epsilon_{\vec{p}}$ due to interactions of the quasiparticle with other quasiparticles whose momenta differ little from $\vec{p}$. To evaluate this contribution we integrate Eq. (7), taking into account values of q less than some cutoff momentum q_c which is small compared with the Fermi momentum p_F; only a small fraction of the total number of quasiparticles are in this region, and therefore in integrating Eq. (7) the dependence of f on $n_{\vec{p}}$ can be neglected. The total contribution to the quasiparticle energy coming from integrating (7) over this region is given by

$$\Delta\epsilon_{\vec{p}\sigma} = \sum_{\vec{q}(q<q_c)\sigma'} f_{\vec{p}\sigma,\vec{p}+\vec{q}\sigma'} n^0_{\vec{p}+\vec{q},\sigma'} \quad , \tag{8}$$

where f stands for $f[n_{\vec{p}\sigma}]$ evaluated using the equilibrium distribution function $n^0_{\vec{p}}$. As one can see from the work in Sec. III, to the order to which we are calculating it is unnecessary to take into account the temperature dependence of f; in Eq. (8) we can therefore use the value of f for the ground state of the system.

For a Fermi system in equilibrium, the $(p-p_F)^3 \times \ln|p-p_F|$ contribution to the quasiparticle energy comes from the $(\hat{p}\cdot\hat{q})^2$ term in the equation for f [Eq. (6)]. We are concerned here with calculating the specific heat of a paramagnetic Fermi liquid in the absence of a magnetic field; therefore the quasiparticle distribution function is independent of spin. Thus the contribution to the quasiparticle energy may be written

$$\Delta\epsilon_{\vec{p}} = 2b^s \sum_{\vec{q}(q<q_c)} (\hat{p}\cdot\hat{q})^2 n^0_{\vec{p}+\vec{q}}(T) \quad , \tag{9}$$

where $n^0_{\vec{p}+\vec{q}}(T)$ is the equilibrium quasiparticle distribution function given in Eq. (2).

Since $q_c \ll p_F$, the curvature of the Fermi surface may be neglected and the sum over $\vec{q}$ in Eq. (9) is easily performed using cylindrical polar coordinates; then $(\hat{p}\cdot\hat{q})^2 = q_\parallel{}^2/(q_\parallel{}^2+q_\perp{}^2)$, where $q_\parallel$ is the component of $\vec{q}$ parallel to $\vec{p}$ (the polar axis), and $q_\perp$ is the component perpendicular to $\vec{p}$. At zero temperature the quasiparticle distribution is just a step function $\theta\,(p_F-p)$; substituting this into Eq. (9), one finds[29]

$$\Delta\epsilon_{\vec{p}}(T=0) = \frac{2b^s}{(2\pi)^3}\int_{-q_c}^{p_F-p} dq_\parallel \int_0^{q_c{}^2-q_\parallel{}^2} \pi d(q_\perp{}^2)\frac{q_\parallel{}^2}{q_\parallel{}^2+q_\perp{}^2} \tag{10}$$

$$= -\frac{b^s}{2\pi^2}\int_{-q_c}^{p_F-p} dq_\parallel\, q_\parallel{}^2 \ln\left|\frac{q_\parallel}{q_c}\right| \tag{11}$$

$$= \frac{b^s}{6\pi^2}(p-p_F)^3 \ln\left|\frac{p-p_F}{q_c}\right| \quad . \tag{12}$$

The integrals here (and in the remainder of this section) have been performed to logarithmic accuracy, and therefore the cutoff in the logarithms is not determined. It is easily verified that terms in f varying as even powers of $\hat{p}\cdot\hat{q}$ higher than $(\hat{p}\cdot\hat{q})^2$ do not give logarithmic contributions to the quasiparticle energy; their leading contribution to the quasiparticle energy varies as $(p-p_F)^3$, which gives rise to T^3 terms in the specific heat.

In calculating the leading contribution to $\Delta\epsilon_{\vec{p}}$ at finite temperatures from Eq. (9), we may neglect the dependence of $n^0_{\vec{p}+\vec{q}}$ on both the temperature-dependent contributions to the quasiparticle energy and also the nonanalytic terms in the quasiparticle energy. That is, we take $n^0_{\vec{p}+\vec{q}}$ to be $(e^{\beta(|\vec{p}+\vec{q}|-p_F)v_F}+1)^{-1}$, where the Fermi velocity v_F is p_F/m^*, m^* being the quasiparticle effective mass. Substituting this expression for n^0 into Eq. (9) and performing the integrals one finds

$$\Delta\epsilon_{\vec{p}}(T) = -\frac{b^s}{2\pi^2}\int_{-\infty}^{\infty} dq_\parallel\, q_\parallel^2 \ln\left|\frac{q_\parallel}{q_c}\right| \frac{1}{e^{\beta(p+q_\parallel-p_F)v_F}+1} \tag{13}$$

$$= \frac{b^s}{6\pi^2 v_F^2}(p-p_F)[(p-p_F)^2 v_F^2 + \pi^2 k_B^2 T^2] \times \ln\left|\frac{\max[(p-p_F)v_F; k_B T]}{k_B T_c}\right| \quad , \tag{14}$$

where $T_c \sim v_F q_c/k_B$. $\Delta\epsilon_{\vec{p}}(T)$ vanishes at the Fermi surface, and therefore there are no $T^2\ln T$ corrections to the chemical potential.

The entropy is given by the usual quasiparticle expression (1); the nonanalytic contributions to the quasiparticle energy $\Delta\epsilon_{\vec{p}}$ give rise to the $T^3\ln T$ contribution to the entropy, which to lowest order in $\Delta\epsilon_{\vec{p}}$ is given by

$$\Delta S = \sum_{\vec{p}\sigma} \Delta\epsilon_{\vec{p}}(T)\frac{\partial n^0_{\vec{p}}}{\partial T} \tag{15}$$

$$= -\tfrac{2}{15}\pi^2 k_B B^s\left(\frac{k_B T}{v_F}\right)^3 \ln\left(\frac{T}{T_c}\right) \tag{16}$$

$$= -\tfrac{1}{20}\pi^4 n k_B B^s\left(\frac{T}{T_F}\right)^3 \ln\left(\frac{T}{T_c}\right) \quad , \tag{17}$$

where

$$T_F = p_F^2/2m^* k_B \tag{18}$$

is the Fermi temperature,

$$n = p_F^3/3\pi^2 \tag{19}$$

is the particle number density,

$$B^s = \nu(0)b^s \quad , \tag{20}$$

and

$$\nu(0) = m^* p_F/\pi^2 \tag{21}$$

is the density of quasiparticle states at the Fermi surface. To the order to which we are working, the temperature dependence of the chemical potential, whose leading term is of order T^2, can be neglected since it does not affect the coefficient of the logarithm. The corresponding contribution to the specific heat per unit volume is

$$\Delta C_v = -\tfrac{3}{20}\,\pi^4 n k_B B^s \left(\frac{T}{T_F}\right)^3 \ln\left(\frac{T}{T_c}\right). \tag{22}$$

We stress that Eq. (22) is an exact expression for the $T^3 \ln T$ contribution to the specific heat coming from the interaction between quasiparticles whose momenta are close to each other. Amit, Kane, and Wagner[17] have shown that there are additional $T^3 \ln T$ contributions to the specific heat coming from interactions between quasiparticles whose momenta differ by $\sim 2p_F$; we shall return to these contributions in Secs. IV and V. We now turn to the evaluation of B^s.

III. QUASIPARTICLE INTERACTION

In this section we show that the quasiparticle interaction for small momentum differences has the form (6), and we evaluate the coefficient B^s which enters the expression for the specific heat, Eq. (22). First, we consider the relationship between energy shifts and forward-scattering amplitudes for two-body potential scattering,[26] and then apply the results to the case of a pair of excitations in a Fermi liquid.

A. Interaction Energies and Scattering Amplitudes

For two particles with relative momentum $\vec{k}$ interacting via a central potential it is well known that the interaction energy, which is the shift of the energy levels of the system produced by the potential, is given by

$$\Delta E_{\vec{k}} = -\frac{2\pi}{mk} \sum_l (2l+1)\delta_l \;, \tag{23}$$

where m is the reduced mass and δ_l is the phase shift for the lth partial wave. On the other hand the forward-scattering amplitude has the form

$$T_{\vec{k}\vec{k}} = -\frac{2\pi}{mk} \sum_l (2l+1)\, e^{i\delta_l} \sin\delta_l \;, \tag{24}$$

and its real part is

$$\mathrm{Re}\, T_{\vec{k}\vec{k}} = -\frac{2\pi}{mk} \sum_l (2l+1) \cos\delta_l \sin\delta_l \;. \tag{25}$$

Only in the limit $\delta_l \to 0$ do the expressions for $\Delta E_{\vec{k}}$ and $\mathrm{Re}\, T_{\vec{k}\vec{k}}$ agree. As we shall see, the difference between $\Delta E_{\vec{k}}$ and $\mathrm{Re}\, T_{\vec{k}\vec{k}}$ arises from terms in perturbation theory which have vanishing energy denominators. It is therefore convenient to express ΔE and T in terms of the reactance matrix, or K matrix, whose magnitude in the lth partial wave is given in terms of the phase shift by the relation

$$K_l = -(2\pi/mk)(2l+1)\tan\delta_l \;. \tag{26}$$

From Eqs. (23), (25), and (26) we obtain the following expressions for $\mathrm{Re}\, T_{\vec{k}\vec{k}}$ and $\Delta E_{\vec{k}}$:

$$\mathrm{Re}\, T_{\vec{k}\vec{k}} = \sum_l \frac{K_l}{1 + [(mk/2\pi)k_l/(2l+1)]^2} \;, \tag{27}$$

$$\Delta E_{\vec{k}} = \frac{2\pi}{mk} \sum_l (2l+1) \arctan\left(\frac{mk}{2\pi}\frac{K_l}{2l+1}\right) \;. \tag{28}$$

To bring out more clearly the importance of vanishing energy denominators, and to facilitate application of the results to the case of a quasiparticle-quasihole pair, it is convenient to express the results in a more general notation. The K matrix is defined by the operator equation

$$K(E) = V + VP(E - H_0)^{-1} K(E) \;, \tag{29}$$

where H_0 is the kinetic energy operator for the two particles, E is the energy, V is the interaction potential, and P denotes a principal-value integral. The T matrix is given formally by the equation

$$T(E) = V + V(E - H_0 + i\eta)^{-1}\, T(E) \,, \qquad \eta \to +0 \tag{30}$$

which, when written in terms of the K matrix becomes

$$T(E) = K(E) - i\pi K(E)\,\delta(E - H_0)\, T(E) \tag{31}$$

$$= K(E)[1 + i\pi\delta(E - H_0)K(E)]^{-1} \;; \tag{32}$$

its real part is

$$\mathrm{Re}\, T(E) = [1 - i\pi K(E)\,\delta(E - H_0)]^{-1} \times K(E)[1 + i\pi\delta(E - H_0)K(E)]^{-1} \tag{33}$$

$$= K(E)\{1 + [\pi\delta(E - H_0)K(E)]^2\}^{-1} \tag{34}$$

$$= K(E)\left\{1 + \sum_{n=1}^{\infty}(-1)^n [\pi\delta(E - H_0)K(E)]^{2n}\right\} \tag{35}$$

$$= K(E) - \pi^2 K(E)\,\delta(E - H_0)K(E)\,\delta(E - H_0)K(E) + \cdots \,. \tag{36}$$

On the other hand, the interaction energy in the state a, ΔE_a, is given in terms of K by the diagonal matrix elements of the operator

$$\Delta E = K(E)\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} [\pi\delta(E - H_0)K(E)]^{2n}\right] \tag{37}$$

$$= K(E) - \tfrac{1}{3}\pi^2 K(E)\,\delta(E - H_0)K(E)\,\delta(E - H_0)K(E) + \cdots \tag{38}$$

evaluated for $E = E_a$, where E_a is the energy of the state a in the absence of the potential. The right-hand side of Eq. (37) is essentially an expansion of the arctangent of the K matrix [cf. Eq. (28)]. To first order in K, ΔE_a [Eqs. (37) and (38)] and $\mathrm{Re}\,T_{aa}(E_a)$ [Eqs. (35) and (36)] are identical, but higher-order terms, which take into account real transitions, are different. This difference is one of the essential features in our work, and the factor-of-3 difference in the K^3 terms is intimately related to the discrepancy of the same factor encountered in model calculations of the $T^3 \ln T$ contribution to the specific heat of an almost ferromagnetic Fermi liquid.[20–22] In Sec. III B we apply the results of this section to the scattering of quasiparticle-quasihole pairs.

B. Energy of Interaction of Two Quasiparticles

As we remarked in Sec. I, the important physical process that leads to the $(\hat{p}\cdot\hat{q})^2$ term in $f_{\vec{p},\vec{p}+\vec{q}}$ in the limit $\vec{q}\to 0$ is the repeated scattering of a quasiparticle-quasihole pair; it is therefore natural to investigate the quasiparticle-quasihole interaction energy $-f_{\vec{p},\vec{p}+\vec{q}}$, rather than the quasiparticle-quasiparticle interaction itself. We first consider scattering of a quasiparticle-quasihole pair and determine the K matrix; then we evaluate the interaction energy of the pair in terms of the K matrix, using the results of Sec. III A.

Consider first the scattering of two *particles*: The T matrix satisfies Eq. (30), which takes in account repeated scattering by the bare potential V; this equation is represented diagrammatically in Fig. 1(a), where the dashed line corresponds to the bare potential, and the shaded circle to the full T matrix. The K matrix is given by Eq. (29), which differs from the equation for the T matrix only in the way in which vanishing energy denominators are treated. The equation for the K matrix may also be represented diagrammatically by Fig. 1(a), provided one evaluates contributions from vanishing energy denominators as principal parts. The K and T matrices for the scattering of quasiparticles by quasiholes are given by a similar diagrammatic equation shown in Fig. 1(b); this takes into account repeated scattering by the *bare* quasiparticle-quasihole interaction, which is represented by the unshaded circle. We denote by $t_{\vec{p}\vec{p}_1}(\vec{q},\omega)$ and $k_{\vec{p}\vec{p}_1}(\vec{q},\omega)$ the T and K matrices for scattering of quasiparticles by quasiholes, where $\vec{p}+\vec{q}$ and $\vec{p}$ are the momenta of the initial quasiparticle and quasihole, and $\vec{p}_1+\vec{q}$ and $\vec{p}_1$ are the momenta of the final quasiparticle and quasihole, respectively; ω is the total energy of the pair. To determine the bare quasiparticle-quasihole interaction we observe that the energy of interaction of two quasiparticles in the states $\vec{p}\vec{\sigma}$ and $\vec{p}_1\vec{\sigma}_1$ is $f_{\vec{p}\vec{\sigma},\vec{p}_1\vec{\sigma}_1} = f^s_{\vec{p}\vec{p}_1} + f^a_{\vec{p}\vec{p}_1}\vec{\sigma}\cdot\vec{\sigma}_1$. The effective Hamiltonian describing interactions between long-wavelength fluctuations in the quasiparticle distribution function is therefore[30]

$$H=\frac{1}{2}\sum_{\vec{p},\vec{p}_1,\vec{q}}\left(f^s_{\vec{p}\vec{p}_1}\,\delta n^\dagger_{\vec{p}\vec{q}}\,\delta n_{\vec{p}_1\vec{q}} + f^a_{\vec{p}\vec{p}_1}\,\delta\vec{\sigma}^\dagger_{\vec{p}\vec{q}}\cdot\delta\vec{\sigma}_{\vec{p}_1\vec{q}}\right), \tag{39}$$

where the density fluctuation operator $\delta n_{\vec{p}\vec{q}}$ and spin fluctuation operator $\delta\vec{\sigma}_{\vec{p}\vec{q}}$ are defined in terms of quasiparticle creation and annihilation operators $a^\dagger$ and a by the relations

$$\delta n_{\vec{p}\vec{q}} = a^\dagger_{\vec{p}\uparrow}a_{\vec{p}+\vec{q}\uparrow} + a^\dagger_{\vec{p}\downarrow}a_{\vec{p}+\vec{q}\downarrow}, \tag{40}$$

$$(\delta\vec{\sigma}_{\vec{p}\vec{q}})_z = a^\dagger_{\vec{p}\uparrow}a_{\vec{p}+\vec{q}\uparrow} - a^\dagger_{\vec{p}\downarrow}a_{\vec{p}+\vec{q}\downarrow}, \tag{41}$$

$$(\delta\vec{\sigma}_{\vec{p}\vec{q}})_x + i(\delta\vec{\sigma}_{\vec{p}\vec{q}})_y = 2a^\dagger_{\vec{p}\uparrow}a_{\vec{p}+\vec{q}\downarrow}, \tag{42}$$

$$(\delta\vec{\sigma}_{\vec{p}\vec{q}})_x - i(\delta\vec{\sigma}_{\vec{p}\vec{q}})_y = 2a^\dagger_{\vec{p}\downarrow}a_{\vec{p}+\vec{q}\uparrow}. \tag{43}$$

$\delta n_{\vec{p}\vec{q}}$ destroys a quasiparticle-quasihole pair in a singlet spin state, whereas $\delta\vec{\sigma}_{\vec{p}\vec{q}}$ destroys pairs in the triplet spin states. $(\delta\vec{\sigma}_{\vec{p}\vec{q}})_x + i(\delta\vec{\sigma}_{\vec{p}\vec{q}})_y$ and $(\delta\vec{\sigma}_{\vec{p}\vec{q}})_x - i(\delta\vec{\sigma}_{\vec{p}\vec{q}})_y$ are just the usual spin-raising and -lowering operators. From Eq. (39) we see that $f^s_{\vec{p}\vec{p}_1}$ and $f^a_{\vec{p}\vec{p}_1}$ play the role of the *bare* quasiparticle-quasihole interaction in the singlet and triplet states, respectively. The algebraic equations for the quasiparticle-quasihole T and K matrices are therefore [cf. Eqs. (30) and (29)]

$$t^\lambda_{\vec{p}\vec{p}_1}(\vec{q},\omega) = f^\lambda_{\vec{p}\vec{p}_1} + 2\sum_{\vec{p}_2} f^\lambda_{\vec{p}\vec{p}_2}\frac{n^0_{\vec{p}_2} - n^0_{\vec{p}_2+\vec{q}}}{\omega - \epsilon_{\vec{p}_2+\vec{q}} + \epsilon_{\vec{p}_2} + i\eta}\,t^\lambda_{\vec{p}_2\vec{p}_1}(\vec{q},\omega), \tag{44}$$

$$k^\lambda_{\vec{p}\vec{p}_1}(\vec{q},\omega) = f^\lambda_{\vec{p}\vec{p}_1} + 2P\sum_{\vec{p}_2} f^\lambda_{\vec{p}\vec{p}_2}\frac{n^0_{\vec{p}_2} - n^0_{\vec{p}_2+\vec{q}}}{\omega - \epsilon_{\vec{p}_2+\vec{q}} + \epsilon_{\vec{p}_2}}\,k^\lambda_{\vec{p}_2\vec{p}_1}(\vec{q},\omega), \tag{45}$$

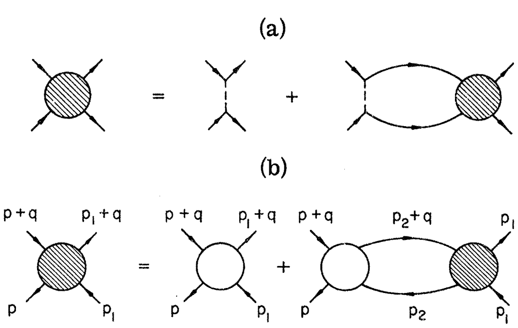

FIG. 1. Diagrammatic equations for the T (or K) matrices. (a) Equation for particle-particle scattering, Eq. (30) [or (29)]; the dashed line represents the bare potential V and the shaded circle represents the full T (or K) matrix. (b) Equation for quasiparticle-quasihole scattering, Eq. (44) [or (45)]; the shaded circle represents the full T (or K) matrix and the unshaded circle represents the bare quasiparticle-quasihole interaction.

where P denotes the principal value. The combination of occupation numbers $n^0_{\vec{p}_2} - n^0_{\vec{p}_2+\vec{q}}$ is the difference of $n^0_{\vec{p}_2}(1 - n^0_{\vec{p}_2+\vec{q}})$ and $n^0_{\vec{p}_2+\vec{q}}(1 - n^0_{\vec{p}_2})$; these two factors correspond to the two possible time directions for the quasiparticle-quasihole bubble in Fig. 1(b). The factor $n^0_{\vec{p}_2}(1 - n^0_{\vec{p}_2+\vec{q}})$ corresponds to the initial pair being scattered to a state with a quasiparticle of momentum $\vec{p}_2 + \vec{q}$, and a quasihole of momentum $\vec{p}_2$, while the second term corresponds to the annihilation of the initial pair by a virtual pair with a quasihole of momentum $\vec{p}_2 + \vec{q}$ and a quasiparticle of momentum $\vec{p}_2$. Equation (44) is the Bethe–Salpeter equation for scattering of a quasiparticle by a quasihole, and was first discussed by Landau.[31] The index λ can be either s (for spin symmetric) or a (for spin antisymmetric); the decomposition of the T and K matrices into spin-symmetric and spin-antisymmetric parts is given by expressions similar to that for f [Eq. (5)]. The T and K matrices for scattering of a quasiparticle-quasihole pair are $2t^s$ and $2k^s$ for the singlet spin state, and $2t^a$ and $2k^a$ for a triplet state.[32]

The energy of interaction of a quasiparticle with a quasihole is given in terms of the K matrix by the quasiparticle-quasihole version of Eqs. (37)–(38):

$$\Delta \omega^\lambda_{\vec{p}\vec{q}} = k^\lambda_{\vec{p}\vec{p}}(\vec{q}, \omega_{\vec{p}\vec{q}}) - \tfrac{1}{3}\pi^2(2)^2 \sum_{\vec{p}_1\vec{p}_2} k^\lambda_{\vec{p}\vec{p}_1}(\vec{q}, \omega_{\vec{p}\vec{q}}) \times [n^0_{\vec{p}_1} - n^0_{\vec{p}_1+\vec{q}}]\, \delta(\omega_{\vec{p}\vec{q}} - \omega_{\vec{p}_1\vec{q}}) k^\lambda_{\vec{p}_1\vec{p}_2}(\vec{q}, \omega_{\vec{p}\vec{q}})[n^0_{\vec{p}_2} - n^0_{\vec{p}_2+\vec{q}}] \times \delta(\omega_{\vec{p}\vec{q}} - \omega_{\vec{p}_2\vec{q}}) k^\lambda_{\vec{p}_2\vec{p}}(\vec{q}, \omega_{\vec{p}\vec{q}}) + \cdots, \quad (46)$$

where

$$\omega_{\vec{p}\vec{q}} = \epsilon_{\vec{p}+\vec{q}} - \epsilon_{\vec{p}} \; . \quad (47)$$

The corresponding equation for the real part of the forward-scattering amplitude is

$$\mathrm{Re}\, t^\lambda_{\vec{p}\vec{p}}(\vec{q}, \omega_{\vec{p}\vec{q}}) = k^\lambda_{\vec{p}\vec{p}}(\vec{q}, \omega_{\vec{p}\vec{q}}) - \pi^2(2)^2 \sum_{\vec{p}_1\vec{p}_2} k^\lambda_{\vec{p}\vec{p}_1}(\vec{q}, \omega_{\vec{p}\vec{q}}) \times [n^0_{\vec{p}_1} - n^0_{\vec{p}_1+\vec{q}}]\, \delta(\omega_{\vec{p}\vec{q}} - \omega_{\vec{p}_1\vec{q}}) k^\lambda_{\vec{p}_1\vec{p}_2}(\vec{q}, \omega_{\vec{p}\vec{q}}) \times [n^0_{\vec{p}_2} - n^0_{\vec{p}_2+\vec{q}}]\, \delta(\omega_{\vec{p}\vec{q}} - \omega_{\vec{p}_2\vec{q}}) k^\lambda_{\vec{p}_2\vec{p}}(\vec{q}, \omega_{\vec{p}\vec{q}}) + \cdots . \quad (48)$$

$2\Delta\omega^s_{\vec{p}\vec{q}}$ and $2\Delta\omega^a_{\vec{p}\vec{q}}$ are the interaction energies of a quasiparticle and a quasihole in the singlet and triplet spin states, respectively. A pair consisting of a quasiparticle ($\vec{p}+\vec{q}\downarrow$) and a quasihole ($\vec{p}\uparrow$) is in a triplet spin state, and therefore the interaction energy is $2\Delta\omega^a_{\vec{p}\vec{q}}$. On the other hand, a pair consisting of a quasiparticle ($\vec{p}+\vec{q}\uparrow$) and a quasihole ($\vec{p}\uparrow$) has equal probability of being a singlet or a triplet; its interaction energy is therefore $\frac{1}{2}(2\Delta\omega^s_{\vec{p}\vec{q}} + 2\Delta\omega^a_{\vec{p}\vec{q}})$. Finally, since the quasiparticle-quasiparticle interaction $f_{\vec{p},\vec{p}+\vec{q}}$ is minus the quasiparticle-quasihole interaction, one finds

$$f_{\vec{p}\uparrow,\vec{p}+\vec{q}\uparrow} = -(\Delta\omega^s_{\vec{p}\vec{q}} + \Delta\omega^a_{\vec{p}\vec{q}}) \; , \quad (49)$$

$$f_{\vec{p}\uparrow,\vec{p}+\vec{q}\downarrow} = -2\omega^a_{\vec{p}\vec{q}} \; . \quad (50)$$

For evaluating the specific heat we need the spin-symmetric part of the interaction

$$f^s_{\vec{p},\vec{p}+\vec{q}} = \tfrac{1}{2}(f_{\vec{p}\uparrow,\vec{p}+\vec{q}\uparrow} + f_{\vec{p}\uparrow,\vec{p}+\vec{q}\downarrow}) = -\tfrac{1}{2}(\Delta\omega^s_{\vec{p}\vec{q}} + 3\Delta\omega^a_{\vec{p}\vec{q}}). \quad (51)$$

Now we consider how the K matrix and the interaction energy $f_{\vec{p},\vec{p}+\vec{q}}$ depend on $\vec{q}$. Since q is small compared to p_F and the temperature is low the distribution functions in the equation for the K matrix [Eq. (45)] constrain p_2 to be close to the Fermi momentum. We may therefore make the replacement

$$\frac{n^0_{\vec{p}_2} - n^0_{\vec{p}_2+\vec{q}}}{\omega - \epsilon_{\vec{p}_2+\vec{q}} + \epsilon_{\vec{p}_2}} = \frac{\hat{p}_2 \cdot \hat{q}}{s - \hat{p}_2 \cdot \hat{q}}\, \delta(\epsilon_{\vec{p}_2} - \mu) \; , \quad (52)$$

where

$$s = \omega / v_F q \; . \quad (53)$$

In the second term on the right-hand side of Eq. (45) we may neglect the dependence of $f_{\vec{p}\vec{p}_2}$ on the magnitudes of $\vec{p}$ and $\vec{p}_2$; in particular the terms $\sim[\hat{p} \cdot (\vec{p} - \vec{p}_2)/|\vec{p} - \vec{p}_2|]^2$ in $f_{\vec{p}\vec{p}_2}$ for $|\vec{p} - \vec{p}_2| \ll p_F$ can be neglected since intermediate states with $|\vec{p} - \vec{p}_2| \ll p_F$ and $\hat{p} \cdot (\vec{p} - \vec{p}_2)/|\vec{p} - \vec{p}_2|$ different from zero give a vanishingly small contribution to the K matrix in the low-temperature limit. Also similar contributions to $f_{\vec{p}\vec{p}_1}$, the first term on the right-hand side of Eq. (45), can also generally be neglected, since in most cases $\vec{p} - \vec{p}_1$ is not small. However, to calculate $\Delta\omega$ [Eq. (46)] we need a diagonal element of the K matrix [Eq. (45)]; the first term is $f_{\vec{p}\vec{p}} = \lim_{\vec{p}' \to \vec{p}} f_{\vec{p}\vec{p}'}$, which is not well defined unless one specifies how the limit is to be evaluated [cf. Eq. (6)]. For our present purposes this is unimportant, since we are interested only in the $\vec{q}$ dependence of $\Delta\omega_{\vec{p}\vec{q}}$. Thus in Eq. (45) we may replace the $f_{\vec{p}\vec{p}'}$ by their values for p and p' equal to p_F,[33] and perform the usual expansion in terms of Legendre polynomials of the angle between $\vec{p}$ and $\vec{p}'$:

$$\nu(0) f^\lambda_{\vec{p}\vec{p}'} = \sum_l F^\lambda_l P_l(\hat{p} \cdot \hat{p}') \; , \quad p = p' = p_F \; . \quad (54)$$

From Eqs. (45), (52), and (54) it is clear that the K matrix $k^\lambda_{\vec{p}\vec{p}_1}(\vec{q}, \omega)$ depends only on $s = \omega/v_F q$ and on the angles between the vectors $\vec{p}$, $\vec{p}_1$, and $\vec{q}$, which we specify by the values of $\hat{p} \cdot \hat{q}$, $\hat{p}_1 \cdot \hat{q}$, and ϕ, the angle between the plane containing $\vec{p}$ and $\vec{q}$, and the plane containing $\vec{p}_1$ and $\vec{q}$. Thus we may write

$$k^\lambda_{\vec{p}\vec{p}_1}(\vec{q}, \omega) \equiv k^\lambda(\hat{p} \cdot \hat{q}, \hat{p}_1 \cdot \hat{q}, \phi; s) \; , \quad p \approx p_1 \approx p_F \; , \quad q \ll p_F \; , \quad \text{and} \quad \omega \ll \mu \; . \quad (55)$$

(We note that for on-energy-shell matrix elements, which are required in evaluating $\Delta\omega$, $\hat{p}\cdot\hat{q}=\hat{p}_1\cdot\hat{q}=s$). The intermediate state sums in Eq. (46) can be converted into integrals in the usual way and the integrals over the magnitudes of the momenta can be performed immediately since the δ functions involve only the angles; the K matrix elements are independent of the magnitudes of the momenta. Using the fact that

$$\int p'^2\,dp'(n^0_{\vec{p}'}-n^0_{\vec{p}'+\vec{q}})\,\delta(\omega_{\vec{p}\vec{q}}-\omega_{\vec{p}'\vec{q}}) = \pi^2\nu(0)\,\hat{p}\cdot\hat{q}\,\delta(\hat{p}\cdot\hat{q}-\hat{p}'\cdot\hat{q}), \quad (56)$$

we may write Eq. (46) as

$$\nu(0)\Delta\omega^\lambda_{\vec{p}\vec{q}}=\mathcal{K}^\lambda(\hat{p}\cdot\hat{q},\phi=0)-\tfrac{1}{3}\pi^2(\tfrac{1}{2}\hat{p}\cdot\hat{q})^2 \times\int_0^{2\pi}\frac{d\phi_1}{2\pi}\int_0^{2\pi}\frac{d\phi_2}{2\pi}\,\mathcal{K}^\lambda(\hat{p}\cdot\hat{q},\phi_1)\mathcal{K}^\lambda(\hat{p}\cdot\hat{q},\phi_2) \times\mathcal{K}^\lambda(\hat{p}\cdot\hat{q},\phi_1+\phi_2)+\cdots, \quad (57)$$

where

$$\mathcal{K}^\lambda(s,\phi)=\nu(0)k^\lambda(s,s,\phi;s) \quad (58)$$

is the on-energy-shell element of the K matrix. We note that the contribution to $\Delta\omega$ of order $(\mathcal{K}^\lambda)^{2n+1}$ has $2n$ intermediate-state summations, and therefore, according to Eq. (56) carries a factor $(\hat{p}\cdot\hat{q})^{2n}$. Since $\mathcal{K}$ has a finite limit as $\hat{p}\cdot\hat{q}\to 0$, and we are calculating only the contribution of order $(\hat{p}\cdot\hat{q})^2$, all terms beyond the $\mathcal{K}^3$ term may be neglected. $\mathcal{K}^\lambda(s,\phi)$ may be expanded in terms of its eigenvalues $\mathcal{K}^\lambda_m(s)$ corresponding to quasiparticle-quasihole pairs whose angular momenta about the direction of $\vec{q}$ have definite values:

$$\mathcal{K}^\lambda(s,\phi)=\sum_{m=-\infty}^{\infty}\mathcal{K}^\lambda_m(s)e^{im\phi}. \quad (59)$$

Substituting (59) into (57), we obtain

$$\nu(0)\Delta\omega^\lambda_{\vec{p}\vec{q}}=\sum_{m=-\infty}^{\infty}\{\mathcal{K}^\lambda_m(\hat{p}\cdot\hat{q}) -\tfrac{1}{3}\pi^2(\tfrac{1}{2}\hat{p}\cdot\hat{q})^2[\mathcal{K}^\lambda_m(\hat{p}\cdot\hat{q})]^3+\cdots\}. \quad (60)$$

Since $\mathcal{K}^\lambda_m(\hat{p}\cdot\hat{q})$ may be expanded in even powers of $\hat{p}\cdot\hat{q}$ it is clear that $\Delta\omega^\lambda_{\vec{p}\vec{q}}$ and $f^s_{\vec{p},\vec{p}+\vec{q}}=-\frac{1}{2}(\Delta\omega^s_{\vec{p}\vec{q}}+3\Delta\omega^a_{\vec{p}\vec{q}})$ [Eq. (51)] contain $(\hat{p}\cdot\hat{q})^2$ contributions, as assumed in Sec. II. The coefficient B^s that enters the $T^3\ln T$ term in the specific heat is proportional to the coefficient of the $(\hat{p}\cdot\hat{q})^2$ term in the expansion of $f^s_{\vec{p},\vec{p}+\vec{q}}$ for small $\hat{p}\cdot\hat{q}$. One finds

$$\nu(0)b^s=B^s=-\frac{1}{2}\sum_{\lambda=s,a}w_\lambda \times\sum_{m=-\infty}^{\infty}\left(\frac{1}{2}\left.\frac{d^2\mathcal{K}^\lambda_m(s)}{ds^2}\right|_{s=0}-\frac{\pi^2}{12}[\mathcal{K}^\lambda_m(0)]^3\right), \quad (61)$$

where the spin multiplicity factors w_λ are $w_s=1$ and $w_a=3$.

In order to illustrate the essential features of the calculation we shall assume here that only F^s_0 and F^a_0 are nonzero. Calculations which include the effects of Landau parameters with $l\leq 2$ are performed in Appendix A. When $f_{\vec{p}_1\vec{p}_2}$ is independent of $\vec{p}_1$ and $\vec{p}_2$, the solution of Eq. (45) is simple; $k(\hat{p}\cdot\hat{q},\,\hat{p}_1\cdot\hat{q},\,s;\phi)$ is independent of the angles and depends only on s. It has the form

$$K^\lambda(s)\equiv\nu(0)k^\lambda_{\vec{p}\vec{p}_1}(s)=F^\lambda_0/[1+F^\lambda_0\,\mathrm{Re}\chi(s)], \quad (62)$$

where

$$\chi(s)=1-\tfrac{1}{2}s\ln[(s+1)/(s-1)] =1-\tfrac{1}{2}s\ln|(s+1)/(s-1)|+i\tfrac{1}{2}\pi s\,\theta(1-|s|). \quad (63)$$

It is obvious from Eqs. (58) and (59) that

$$\mathcal{K}^\lambda(s,\phi)=\mathcal{K}^\lambda_0(s)=K^\lambda(s). \quad (64)$$

Combining Eqs. (61)–(64) we obtain for B^s

$$B^s=-\tfrac{1}{2}[(A^s_0)^2(1-\tfrac{1}{12}\pi^2A^s_0)+3(A^a_0)^2(1-\tfrac{1}{12}\pi^2A^a_0)], \quad (65)$$

where

$$A^\lambda_l=\frac{F^\lambda_l}{1+F^\lambda_l/(2l+1)}. \quad (66)$$

In the above expression for B^s the terms quadratic in A come from the second derivative of $\mathcal{K}_m$ in Eq. (60), whereas the cubic terms, which are proportional to π^2, come from the $\mathcal{K}^3_m$ term in the same equation. Generally, when one includes higher Landau parameters, the $\mathcal{K}^3$ term gives rise to contributions to $B^s\sim\pi^2A^\lambda_{l_1}A^\lambda_{l_2}A^\lambda_{l_3}$, while the second derivative of $\mathcal{K}$ gives rise to quadratic and cubic terms in A^λ_l (without a factor π^2). The expression for B^s taking into account all Landau parameters with $l\leq 2$ is derived in Appendix A.

One deficiency of the Landau-theory calculations described above is that they do not give the correct result when $\hat{p}\cdot\hat{q}=0$. From Eqs. (49), (50), and (57), and the fact that $\mathcal{K}^\lambda(\hat{p}\cdot\hat{q}=0,\ \phi=0)$ is $\sum_{l=0}^{\infty}A^\lambda_l$ [Eq. (A3)], we see that the calculations give in this limit

$$\nu(0)f^s(0)=-\frac{1}{2}\sum_l(A^s_l+3A^a_l), \quad (67)$$

$$\nu(0)f^a(0)=-\frac{1}{2}\sum_l(A^s_l-A^a_l). \quad (68)$$

Using the forward-scattering sum rule[34]

$$\sum_l(A^s_l+A^a_l)=0, \quad (69)$$

we may rewrite Eqs. (67) and (68) as

$$\nu(0)f^\lambda(0)=\sum_l A^\lambda_l. \quad (70)$$

On the other hand $f^\lambda(0)$ may be determined directly from the defining equation for the Landau parameters, Eq. (54). $f^\lambda(0)$ is the energy of interaction of two quasiparticles whose momenta are equal, evaluated when the angle between the momenta tends to zero. It is therefore given by Eq. (54) evaluated for $\hat{p}\cdot\hat{p}'=1$:

$$\nu(0)f^\lambda(0)=\sum_l F_l^\lambda \ . \tag{71}$$

The inconsistency of Eqs. (70) and (71) is due to the contribution $f_{\vec{p}\vec{p}}$ to $\Delta\omega$ being treated incorrectly; in Appendix B we discuss this inconsistency and resolve it on the basis of microscopic theory. However, we stress the fact that the inconsistency in no way affects the calculation of B^s.

IV. COMPARISON WITH PREVIOUS CALCULATIONS

Previous calculations of $T^3\ln T$ contributions to the specific heat fall into two classes. First, there are those that use the quasiparticle expression (1) for the entropy, but insert into it *dynamical* quasiparticle energies[17–19]; we shall refer to these calculations as the dynamical quasiparticle approximation. Second, there are calculations for models of almost ferromagnetic Fermi systems, starting from thermodynamic perturbation theory, which lead to results different from the dynamical quasiparticle approximation.[20–22]

Our calculations, like those using the dynamical-quasiparticle approximation, start from the quasiparticle expression for the entropy; however, we use *statistical*, not *dynamical* quasiparticle energies when evaluating the entropy. By making comparison with the work of Amit, Kane, and Wagner,[17] and Emery[19] one can see that the nonanalytic contributions to the *dynamical* quasiparticle energy due to interactions between quasiparticles with almost equal momenta may be obtained from a relation similar to Eq. (8) for the *statistical* quasiparticle energy by replacing the interaction energy $f_{\vec{p}\sigma,\vec{p}+\vec{q}\sigma'}$ there by the corresponding real part of the forward-scattering amplitude. That is, $f_{\vec{p}\uparrow,\vec{p}+\vec{q}\uparrow}=-(\Delta\omega^s_{\vec{p}\vec{q}}+\Delta\omega^a_{\vec{p}\vec{q}})$ is replaced by $-\mathrm{Re}[t^s_{\vec{p}\vec{p}}(\vec{q},\omega_{\vec{p}\vec{q}})+t^a_{\vec{p}\vec{p}}(\vec{q},\omega_{\vec{p}\vec{q}})]$ and $f_{\vec{p}\uparrow,\vec{p}+\vec{q}\downarrow}=-2\Delta\omega^a_{\vec{p}\vec{q}}$ by $2\mathrm{Re}\,t^a_{\vec{p}\vec{p}}(\vec{q},\omega_{\vec{p}\vec{q}})$, where $\Delta\omega^\lambda_{\vec{p}\vec{q}}$ and $t^\lambda_{\vec{p}\vec{q}}(\vec{q},\omega_{\vec{p}\vec{q}})$ are given in Eqs. (46) and (48). The nonanalytic contribution to the dynamical quasiparticle energy has the same form as the statistical quasiparticle energy [Eq. (14)], but with the coefficient b^s replaced by b^s_D, which is related to the real part of the forward-scattering amplitude $t_{\vec{p}\vec{p}}(\vec{q},\omega_{\vec{p}\vec{q}})$ in the same way that b^s is related to $\Delta\omega_{\vec{p}\vec{q}}$. Explicitly one finds

$$B^s_D=-\frac{1}{2}\sum_{\lambda=s,a}w_\lambda \times\sum_{m=-\infty}^{\infty}\left(\frac{1}{2}\frac{d^2\mathcal{K}^\lambda_m(s)}{ds^2}\bigg|_{s=0}-\tfrac{1}{4}\pi^2[\mathcal{K}^\lambda_m(0)]^3\right) . \tag{72}$$

The only difference between this quantity and B^s itself [Eq. (61)] is that the $\mathcal{K}^3$ term in B^s_D is three times larger; this reflects the same factor of 3 in the K^3 contributions to ΔE [Eq. (38)] and ReT [Eq. (36)]. In Sec. V we shall show that the difference between B^s and B^s_D is very significant for liquid He^3, since the largest contributions come from the $\mathcal{K}^3$ term. This intimate relationship between the scattering amplitude and the self-energy Σ is what one would expect, since both t and Σ are calculated using the scattering boundary condition (in other words, the imaginary parts of energy denominators all have the same sign); since the boundary conditions are the same, real transitions are treated in the same way in the calculations of t and Σ.

In these calculations we have considered interactions only between quasiparticles with almost equal momenta. However, Amit, Kane, and Wagner[17] showed that interactions between quasiparticles whose momenta differ by $\sim 2p_F$ give an additional nonanalytic contribution to the dynamical quasiparticle energy, which affects the $T^3\ln T$ term in the specific heat; this contribution to the dynamical quasiparticle energy can also be calculated in terms of Landau parameters. To evaluate the $T^3\ln T$ term in the specific heat due to these interactions one needs their contribution to the *statistical* quasiparticle energy, which is not known. In Sec. V we shall use their contribution to the *dynamical* quasiparticle energy to estimate their effect on the specific heat.

For an almost ferromagnetic Fermi liquid the Landau parameter F^a_0 approaches -1, and therefore A^a_0 is large and negative. The dominant contribution to B^s is then $\frac{1}{8}\pi^2(A^a_0)^3$, and the corresponding result for the specific heat agrees with the results of the model calculations if one replaces the vertex functions that occur there by fully renormalized ones[20–22]; it is, however, three times larger than the corresponding dynamical quasiparticle result,[17–19] since the dominant term in this limit comes from the $\mathcal{K}^3$ contribution to B^s and B^s_D [Eq. (72)]. In Sec. V we shall see how important the differences between the present calculations and previous ones are by giving numerical results for liquid He^3 at low pressure.

V. APPLICATION TO LIQUID He^3

The $T^3\ln T$ contribution to the specific heat is usually expressed in terms of the coefficient Γ defined by the equation

$$\Delta C_V=nk_B\,\Gamma T^3\ln(T/T_c) \ . \tag{73}$$

Comparing Eqs. (22) and (73) we see that the theoretical value of Γ coming from interactions between quasiparticles with almost equal momenta ($q\to 0$) is given in terms of B^s, the coefficient of

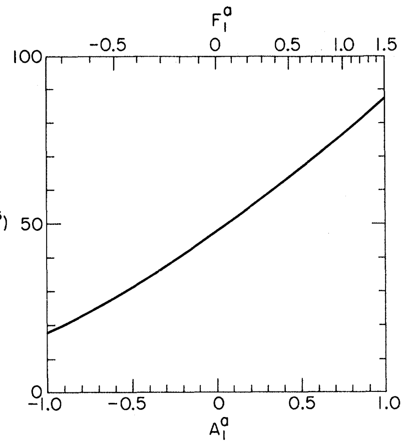

FIG. 2. Theoretical values of $\Gamma_{\rm low}$, Eq. (75), calculated neglecting Landau parameters with $l>1$, as a function of A_1^a (lower scale), and $F_1^a = A_1^a/(1-\frac{1}{3}A_1^a)$ (upper scale). The Landau parameters F_0^s, F_0^a, and F_1^s have the values given in text.

the $(\hat{p}\cdot\hat{q})^2$ term in the quasiparticle interaction, by the relation

$$\Gamma_{\rm low} = -\tfrac{3}{20}\pi^4(B^s/T_F^3)\ . \tag{74}$$

Detailed calculations of B^s are given in Appendix A; if Landau parameters with $l\geq 2$ are neglected the corresponding result for $\Gamma_{\rm low}$ is

$$\Gamma_{\rm low} = \tfrac{3}{40}(\pi^4/T_F^3)\sum_{\lambda=s,a} w_\lambda[(A_0^\lambda)^2(1+A_1^\lambda-\tfrac{1}{12}\pi^2A_0^\lambda) + (A_1^\lambda)^2(1-\tfrac{1}{48}\pi^2A_1^\lambda)-2A_0^\lambda A_1^\lambda], \tag{75}$$

where $w_s=1$ and $w_a=3$. The A_l^λ are defined in Eq. (66).

In Fig. 2 we show theoretical values of $\Gamma_{\rm low}$ for liquid He^3 at low pressure as a function of the parameter A_1^a (F_1^a), which is not well known. The parameters F_0^s, F_1^s, and F_0^a (and the corresponding A's) are well known from measurements of equilibrium properties; we used values $F_0^s=10.76$ ($A_0^s=0.91$), $F_1^s=6.00$ ($A_1^s=2.00$), and $F_0^a=-0.67$ ($A_0^a=-2.00$).[35] For an almost ferromagnetic Fermi liquid, $F_0^a\to-1$ ($A_0^a\to-\infty$), the dominant contribution to $\Gamma_{\rm low}$ is

$$\Gamma_{\rm low}\sim -\tfrac{3}{160}\pi^6[(A_0^a)^3/T_F^3],\quad A_0^a\to-\infty\ . \tag{76}$$

For liquid He^3 at low pressure this has the value 29.2 °K^{-3}, which is only ~60% of the value for $\Gamma_{\rm low}$ evaluated for $F_1^a=0$. Consequently, for liquid He^3 it is a poor approximation to neglect all but the dominant term in the almost ferromagnetic limit. We note that the contribution to $\Gamma_{\rm low}$ coming from spin-symmetric Landau parameters is remarkably small, only 0.77 °K^{-3}, which for most values of F_1^a is no more than a few percent of the total.

Experimentally, Γ is not well known, since the data do not exhibit a well-defined $T^3\ln T$ behavior; but from a forced fit to the data Mota, Platzeck, Rapp, and Wheatley[5] estimate Γ to be 57.4 °K^{-3}. This number would be consistent with the theoretical value if F_1^a were +0.27 ($A_1^a=+0.25$); other estimates of F_1^a are −0.66 ($A_1^a=-0.84$) (from finite-temperature contributions to the thermal conductivity), −0.58 ($A_1^a=-0.72$) (from finite-temperature contributions to the spin diffusion coefficient), and −0.69 ($A_1^a=-0.91$) (from the forward-scattering sum rule)[36]; also recent measurements of spin echoes in high magnetic fields indicate that F_1^a is small, but give no indication of the sign.[37] However, in view of the uncertainties in the experimental value of Γ and in the values of F_1^a obtained from other experiments, the difference between this number and other estimates of F_1^a is not particularly significant. Also one must bear in mind that the above calculation using Eq. (75) neglected the effects of Landau parameters with $l>1$, and also neglected contributions to Γ coming from interaction of quasiparticles whose momenta differ by $\sim 2p_F$.

To estimate the importance of Landau parameters with $l>1$ we have calculated $\Gamma_{\rm low}$ taking into account F_2^a; the expression for B^s is derived in Appendix A, and the results for $\Gamma_{\rm low}$ as a function of A_2^a (F_2^a) and A_1^a (F_1^a) are shown in Fig. 3. For F_0^s, F_1^s, and F_0^a we use the values given earlier.[35] We assumed F_2^s to be zero, since the spin-symmetric Landau parameters contribute little to $\Gamma_{\rm low}$, and experimentally F_2^s is know to be small.[38]

We note that even if $|A_2^a|$ is no bigger than 0.5,

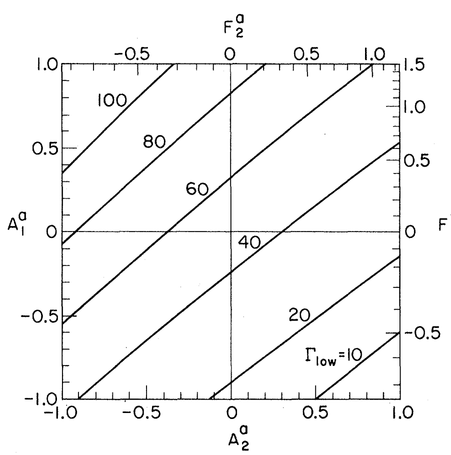

FIG. 3. Contours of constant $\Gamma_{\rm low}$ as a function of A_1^a and A_2^a. The units of $\Gamma_{\rm low}$ are °K^{-3}. The Landau parameters F_0^s, F_0^a, and F_1^s have the values given in the text, and other Landau parameters were neglected. Values of $F_1^a=A_1^a/(1-\frac{1}{3}A_1^a)$ are given on the right-hand vertical scale, and values of $F_2^a=A_2^a/(1-\frac{1}{5}A_2^a)$ are given on the upper horizontal scale.

its contribution to $\Gamma_{\rm low}$ is appreciable and can amount to about 25–50% of the total, depending on the value of A_1^a. The contribution to $\Gamma_{\rm low}$ increases with decreasing A_2^a, and increasing A_1^a. Roughly speaking the contours of constant $\Gamma_{\rm low}$ are lines of constant $A_1^a - A_2^a$. A measurement of Γ may therefore be used to estimate $A_1^a - A_2^a$ (if one assumes higher Landau parameters may be neglected), but it is of little value for estimating A_1^a and A_2^a separately, unless one has some additional information. It is also possible that Landau parameters with $l > 2$ make significant contributions to $\Gamma_{\rm low}$. For example, the $(K_0^a)^3$ contribution to B^s contains terms of the type $(A_0^a)^2 A_{2n}^a$ with a coefficient that falls off rather slowly with increasing n; these may well be important since $|A_0^a|$ is about 2 for liquid He^3.[39]

Some indication of the importance of interactions between quasiparticles whose momenta differ by $\sim 2p_F$ may be obtained by estimating their contribution to the entropy in the *dynamical* quasiparticle approximation; this is given by the difference between the calculation of Amit, Kane, and Wagner[17] and that of Emery,[19] namely,

$$\Gamma_{\rm high}^D = \tfrac{3}{80}\pi^4 T_F^{-3}(A_0^s + \tfrac{1}{18}A_1^s)(A_0^s A_1^s + 3A_0^a A_1^a) . \quad (77)$$

(Here we again neglect all Landau parameters with $l \geq 2$.) Numerical values of $\Gamma_{\rm high}^D$ are given in the third line of Table I, for several values of F_1^a. In Table I we give for comparison the results of the present calculations of $\Gamma_{\rm low}$, and the low-momentum contribution to the entropy in the dynamical quasiparticle approximation. The latter is defined by an equation analogous to Eq. (74) for $\Gamma_{\rm low}$:

$$\Gamma_{\rm low}^D = -\tfrac{3}{20}\pi^4 (B_D^s / T_F^3) , \quad (78)$$

where B_D^s is given in Eq. (72). If all Landau parameters with $l < 2$ are included in the calculation, $\Gamma_{\rm low}^D$ is given by the same expressions as $\Gamma_{\rm low}$ [Eq. (75)] but with the $(A_0^\lambda)^3$ and $(A_1^\lambda)^3$ terms multiplied by a factor of 3. We see that for the values of F_1^a considered, $\Gamma_{\rm high}^D$ is generally small compared with both $\Gamma_{\rm low}$ and $\Gamma_{\rm low}^D$; however, for values of A_1^a between -4.6 and -1.0 ($-1.8 \leq F_1^a \leq -0.75$) $|\Gamma_{\rm high}^D|$ is greater than one-third of $\Gamma_{\rm low}$, and is many times $\Gamma_{\rm low}$ when $A_1^a \approx -2.3$. The lowest estimate of A_1^a is -0.91, obtained from the forward-scattering sum rule; for this value the high-momentum contribution to Γ is probably appreciable.

TABLE I. Theoretical estimates of $\Gamma(°K^{-3})$ as a function of F_1^a.

F_1^a			−0.75	−0.50	0	0.50
A_1^a			−1.0	−0.60	0	0.43
Present calculation		$\Gamma_{\rm low}$	17.7	28.4	48.0	64.1
Dynamical quasiparticle	low momentum	$\Gamma_{\rm low}^D$	71.2	80.0	99.5	115.3
	high momentum	$\Gamma_{\rm high}^D$	−5.9	−4.1	−1.4	0.6

Clearly, it is necessary to evaluate the high-momentum contributions to the *statistical* quasiparticle energy before definitive statements can be made. Another point to notice is that for the range of values of F_1^a considered, the results of the present calculations are smaller than these of the dynamical quasiparticle approximation by a factor which varies from about 2–4. It is also interesting that the contribution to $\Gamma_{\rm low}^D$ coming from spin-symmetric Landau parameters is $-3.97\,°K^{-3}$ which has the opposite sign and is considerably larger in magnitude than the contribution we find ($+0.77\,°K^{-3}$).

To summarize, we have shown that the small-q contributions to the $T^3 \ln T$ term in the specific heat of a normal Fermi liquid may be calculated using the quasiparticle expression (1) provided the quasiparticle energies are determined using Landau's definition, Eq. (3). This statistical quasiparticle energy is not the same as the dynamical quasiparticle energy defined by the poles of the single-particle propagator. In our calculations no Bose contributions of the type found by Riedel[21] appear, although our results are consistent with the earlier calculations. A detailed discussion of Bose contributions is given in a separate paper.[24] One problem still unsolved is how high-momentum processes contribute to the statistical quasiparticle energy.

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APPENDIX A: EVALUATION OF B^s

Here we evaluate B^s [Eq. (61)] in terms of Landau parameters. First we consider the $(\mathcal{K}_m^\lambda)^3$ term. For $s = 0$ the equation for the K matrix reduces to

$$k_{\vec{p}\vec{p}_1}^\lambda = f_{\vec{p}\vec{p}_1}^\lambda + \sum_{\vec{p}_2} f_{\vec{p}\vec{p}_2}^\lambda \frac{\partial n_{\vec{p}_2}^0}{\partial \epsilon_{\vec{p}_2}} k_{\vec{p}_2\vec{p}_1}^\lambda , \quad (A1)$$

which is the same as the equation for the T matrix, since when $s = 0$ no real transitions occur. $k_{\vec{p}\vec{p}_1}$ depends only on the angle between $\vec{p}$ and $\vec{p}_1$, and Eq. (A1) is easily solved by expanding k and f in Legendre polynomials. The result is

$$\nu(0) k^\lambda(\hat{p}\cdot\hat{q},\, \hat{p}_1\cdot\hat{q},\, \phi;\; s=0)$$

$$= K^\lambda(\hat{p}\cdot\hat{q},\, \hat{p}_1\cdot\hat{q},\, \phi;\, s=0) = \sum_l A_l^\lambda P_l(\hat{p}\cdot\hat{p}_1) , \quad (A2)$$

where the A_l^λ are defined in Eq. (66). To obtain the on-energy-shell matrix elements of K [Eq. (58)]

we note that on the energy shell ($s=\hat{p}\cdot\hat{q}=\hat{p}_1\cdot\hat{q}$) and for $s=0$, $\hat{p}\cdot\hat{p}_1$ is simply $\cos\phi$. Therefore from Eqs. (58) and (A2) one finds

$$K^\lambda(0,\,0,\,\phi;\,0)=\mathcal{K}^\lambda(0,\,\phi)=\sum_l A_l^\lambda P_l(\cos\phi)\,. \quad \text{(A3)}$$

The coefficients $\mathcal{K}_m^\lambda(0)$ [Eq. (59)] may be evaluated using the identity[40]

$$P_l(\cos\phi)=\sum_{n=0}^{l} g_n g_{l-n}\cos(l-2n)\phi\,, \quad \text{(A4)}$$

where

$$g_n=\frac{1}{2^{2n}}\frac{(2n)!}{(n!)^2}\,. \quad \text{(A5)}$$

One finds

$$\mathcal{K}_m^\lambda(0)=\sum_{n=0}^{\infty} A^\lambda_{|m|+2n}\, g_{|m|+n}\, g_n\,. \quad \text{(A6)}$$

From this equation the $\mathcal{K}^3$ term in the expression for B^s may be evaluated directly.

For liquid He^3 there are large contributions to B^s coming from the $[\mathcal{K}_0^a(0)]^3$ term in Eq. (61), since $|A_0^a|$ has the large value of approximately 2. According to Eq. (A6) $\mathcal{K}_0^a$ is given by

$$\mathcal{K}_0^a(0)=A_0^a+\tfrac14 A_2^a+\tfrac{9}{64}A_4^a+\cdots\,. \quad \text{(A7)}$$

The coefficient of A_{2n}^a in Eq. (A7) falls off rather slowly with increasing n, and approaches the value $1/\pi n$ asymptotically as $n\to\infty$. Thus, because of the large value of $|A_0^a|$, there may be appreciable contributions to B^s coming from terms involving rather high-order Landau parameters.

To evaluate the term involving the second derivative of $\mathcal{K}_m^\lambda$ in Eq. (61) we solve the K-matrix equation for arbitrary values of s. Expanding K in Legendre polynomials,

$$K^\lambda(\hat{p}\cdot\hat{q},\,\hat{p}_1\cdot\hat{q},\,\phi;\,s)=\nu(0)k^\lambda_{\vec{p}\vec{p}_1}(\vec{q},\,\omega)$$

$$=\sum_{l_1l_2m} K^{\lambda m}_{l_1l_2}(s)\,P_{l_1}^{|m|}(\hat{p}\cdot\hat{q})P_{l_2}^{|m|}(\hat{p}_1\cdot\hat{q})e^{im\phi}\,, \quad \text{(A8)}$$

and substituting into the equation for the K matrix [Eq. (45)] one finds the following equation for the coefficients $K^{\lambda m}_{l_1l_2}$:

$$K^{\lambda m}_{l_1l_2}(s)=\frac{(l_1-|m|)!}{(l_1+|m|)!}F^\lambda_{l_1}\delta_{l_1l_2}-\sum_{l_3}F^\lambda_{l_1}\mathrm{Re}\,\Omega^m_{l_1l_3}(s)K^{\lambda m}_{l_3l_2}(s)\,, \quad \text{(A9)}$$

where

$$\Omega^m_{l_1l_2}(s)=\frac{(l_1-|m|)!}{(l_1+|m|)!}\,\frac12\int_{-1}^{1}d\mu\,P_{l_1}^{|m|}(\mu)\frac{\mu}{\mu-s}P_{l_2}^{|m|}(\mu)\,. \quad \text{(A10)}$$

This equation has the same form as the equation for the T matrix (scattering amplitude), $\nu(0)t^\lambda_{\vec{p}\vec{p}_1}(\vec{q},\,\omega)$, except that in Eq. (A9) one finds $\mathrm{Re}\,\Omega^m_{l_1l_2}$ instead of $\Omega^m_{l_1l_2}$.[41]

Using Eq. (A8) we see that the on-energy-shell matrix elements of the K matrix $\mathcal{K}_m^\lambda(s)$ [Eq. (58)] are given by

$$\mathcal{K}_m^\lambda(s)=\sum_{l_1l_2}K^{\lambda m}_{l_1l_2}(s)P_{l_1}^{|m|}(s)P_{l_2}^{|m|}(s)\,. \quad \text{(A11)}$$

Neglecting all Landau parameters with $l>2$, and solving Eq. (A9) we obtain the following results for $\mathcal{K}_m^\lambda(s)$. For $m=0$ we find

$$\mathcal{K}_0^\lambda(s)=\frac{C^\lambda(s)}{1+D^\lambda(s)+C^\lambda(s)\mathrm{Re}\chi(s)}\,, \quad \text{(A12)}$$

where

$$C^\lambda(s)=(F_0^\lambda+A_1^\lambda s^2)[1-\tfrac12 A_2^\lambda P_2^0(s)]+A_2^\lambda[P_2^0(s)]^2\,, \quad \text{(A13)}$$

$$D^\lambda(s)=\tfrac12 A_2^\lambda[P_2^0(s)-\tfrac12(F_0^\lambda+A_1^\lambda s^2)]\,, \quad \text{(A14)}$$

and $\chi(s)$ is defined in Eq. (63).

For $m=1$, we find

$$\mathcal{K}_1^\lambda(s)=\frac12\,\frac{F_1^\lambda+3A_2^\lambda s^2}{1+(F_1^\lambda+3A_2^\lambda s^2)\mathrm{Re}\,\Omega^1_{11}(s)}[P_1^1(s)]^2\,, \quad \text{(A15)}$$

where

$$\Omega^1_{11}(s)=\tfrac12\{-\tfrac13+[P_1^1(s)]^2\chi(s)\}\,. \quad \text{(A16)}$$

For $m=2$, we find

$$\mathcal{K}_2^\lambda(s)=\frac{1}{4!}\,\frac{F_2^\lambda}{1+F_2^\lambda\mathrm{Re}\,\Omega^2_{22}(s)}[P_2^2(s)]^2\,, \quad \text{(A17)}$$

where

$$\Omega^2_{22}(s)=-\tfrac{1}{20}+(1/4!)P_2^2(s)+(1/4!)[P_2^2(s)]^2\chi(s)\,. \quad \text{(A18)}$$

The $\mathcal{K}_m^\lambda(s)$ for $m>2$ all vanish.

Computing the second derivatives of $\mathcal{K}_m^\lambda$, substituting them into Eq. (61), and adding the $[\mathcal{K}_m^\lambda(0)]^3$ contribution (neglecting Landau parameters with $l>2$), one obtains for B^s the expression

$$B^s=-\frac12\sum_{\lambda=s,a}w_\lambda\big[(A_0^\lambda)^2(1+A_1^\lambda)+\tfrac14(A_2^\lambda)^2+2A_0^\lambda A_2^\lambda-\tfrac{1}{12}\pi^2(A_0^\lambda+\tfrac14A_2^\lambda)^3$$

$$+(A_1^\lambda)^2(1+\tfrac13A_2^\lambda)-2A_1^\lambda A_2^\lambda-\tfrac{1}{48}\pi^2(A_1^\lambda)^3+\tfrac34(A_2^\lambda)^2-\tfrac{9}{1024}\pi^2(A_2^\lambda)^3\big]\,, \quad \text{(A19)}$$

where $w_s=1$ and $w_a=3$. In the above equation the first line comes from the $m=0$ term in Eq. (61), the second line from the $|m|=1$ terms, and the third line from the $|m|=2$ terms. The contributions

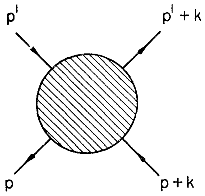

FIG. 4. Two particle vertex $\Gamma(p,\ p';\ k)$.

proportional to π^2 come from the $(\mathcal{K}_m^\lambda)^3$ term in Eq. (61), while the others come from $d^2\mathcal{K}_m^\lambda/ds^2$ in the same equation.

APPENDIX B: QUASIPARTICLE INTERACTION FOR $\hat{p}\cdot\hat{q}=0$

Here we resolve, on the basis of microscopic theory, the inconsistency found in the calculations of $f_{\vec{p},\vec{p}+\vec{q}}$ for small $\vec{q}$ and $\hat{p}\cdot\hat{q}=0$. The reason for the discrepancy is the many-body nature of the problem, which, as we remarked in Sec. III, leads to ambiguities in evaluating the "$f_{\vec{p}\vec{p}}$" term in the expression for the energy of a quasiparticle-quasihole pair. The energy of interaction of two quasiparticles of momenta $\vec{p}$ and $\vec{p}\,'\equiv\vec{p}+\vec{q}$ both of which are on the Fermi surface, is directly proportional to the forward-scattering amplitude, since there are no states available for real transitions, which lead to the differences between energies of interaction and forward-scattering amplitudes discussed in the body of the paper. To be precise, the quasiparticle interaction is then given by the relation[31]

$$f_{\vec{p}\vec{p}}=a^2\Gamma^{s_k=\infty}(\hat{p}\cdot\hat{p}\,'), \tag{B1}$$

where a is the renormalization constant and $\Gamma^{s_k}(\hat{p}\cdot\hat{p}\,')$ is related to the two-particle vertex shown diagrammatically in Fig. 4 by

$$\Gamma^{s_k}(\hat{p}\cdot\hat{p}\,')=\lim_{k\to 0}\left[\Gamma(p,\ p';\ k)\big|_{k^0=s_k|\vec{k}|}\right], \tag{B2}$$

evaluated for $\vec{p}$ and $\vec{p}\,'$ on the Fermi surface and the energies p^0 and p'^0 equal to the chemical potential. Here we use a four-vector notation—for example, $k\equiv(\vec{k},\ k^0)$. (For simplicity we suppress spin indices in most of this Appendix.) As we shall see, it is crucial to notice that Γ is to be evaluated in the limit $k\to 0$, but $s_k\to\infty$; this limit was treated incorrectly in the Landau-theory calculations.

Γ satisfies the equation shown diagrammatically in Fig. 5, which is the analog of Eq. (44) [Fig. 1(b)] in Landau theory. Here we have put $p'=p+q$, since we are interested in the limit $q\to 0$. We assume that all incoherent parts of particle-hole propagators have been removed by renormalizing the vertex functions in the usual way, and therefore all particle lines represent the coherent part of the propagator, and they may be regarded as quasiparticle lines. The unshaded circle in Fig. 5 corresponds to Γ evaluated for $s_q=q^0/v_F|\vec{q}|=\infty$, since the quasiparticle-quasihole propagator in the second term vanishes in this limit. (Note that the energy variable q^0 corresponds to ω in the Landau-theory calculations.)

In evaluating $f_{\vec{p},\vec{p}+\vec{q}}$ using Eq. (B1), one needs the vertex function evaluated for $s_k=\infty$. The only terms which depend on the value of s_k in the limit $k\to 0$ are ones which have an intermediate state containing a single quasiparticle-quasihole pair of total momentum k; graphs representing these terms are shown in Fig. 6. All of these terms occur in the first term on the right-hand side of the integral equation for Γ, Fig. 5; the remaining terms have at least one intermediate state containing a single quasiparticle-quasihole pair with total momentum q, and it is therefore impossible for them to have an intermediate state containing a single pair with total momentum k.[42] To determine $f_{\vec{p},\vec{p}+\vec{q}}$ in the limit $\vec{q}\to 0$, the first term in the equation for the vertex (Fig. 5) must be evaluated in the limits $q\to 0$, $s_q\to\infty$ (as discussed above) and $k\to 0$, $s_k\to\infty$ [from the expression for f in terms of Γ, Eq. (B1)].

In the Landau–theory calculations the first term in the equation for the T or K matrices was replaced by $\lim_{\vec{p}_1\to\vec{p}}f_{\vec{p}\vec{p}_1}$, where both $\vec{p}$ and $\vec{p}_1$ are on

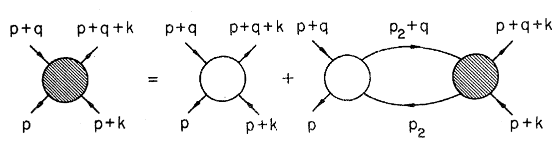

FIG. 5. Equation for $\Gamma(p,\ p+q;\ k)$. The unshaded circle corresponds to Γ evaluated for $s_q=q^0/v_F|\vec{q}|=\infty$, and the shaded circle corresponds to Γ itself.

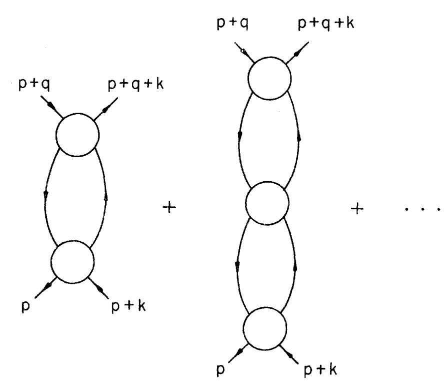

FIG. 6. Graphs which contribute to the difference between the $s_k=0$ and $s_k=\infty$ limits of the vertex function Γ (Fig. 4). The circles in the diagrams represent the part of the vertex function irreducible with respect to a single quasiparticle-quasihole pair of total momentum k, and therefore correspond in the limit of small k to the $s_k=\infty$ limit of the vertex function.

the Fermi surface. To see what this corresponds to in the microscopic calculations we make use of the fact that an expression for $f_{\vec{p},\vec{p}+\vec{k}}$ for $\vec{p}$ and $\vec{p}+\vec{k}$ both on the Fermi surface may be obtained by applying relation (B1) to the crossed channel:

$$f_{\vec{p},\vec{p}+\vec{k}} = \lim_{q\to 0}\left\{ \lim_{s_q = q^0/v_F|\vec{q}|\to\infty} [a^2\Gamma(p,\ p+q;\ k)] \right\} . \quad \text{(B3)}$$

The energies p^0 and p^0+k^0 are to be put equal to the chemical potential. Thus in the Landau-theory calculation the first term corresponds to the vertex function evaluated for $s_k = \hat{p}\cdot\hat{k} = 0$ (since $\vec{p}$ and $\vec{p}+\vec{k}$ are on the Fermi surface) and $s_q \to \infty$ [from the expression (B3) for the Landau parameter]. As we have seen, the vertex should be evaluated for $s_k = \infty$. The difference between the $s_k = 0$ and $s_k = \infty$ values of the vertex function comes from the graphs shown in Fig. 6; the circles in the figure correspond to the part of the vertex irreducible with respect to a single quasiparticle-quasihole pair of momentum k. The contribution from these graphs may be evaluated by applying Landau's equation for quasiparticle-quasihole scattering to the k channel, since k is small. The contribution from these graphs vanishes for $s_k = \infty$, and therefore the Landau-theory calculations overestimated the quasiparticle interaction $\nu(0)f_{\vec{p},\vec{p}+\vec{q}}$ by an amount equal to the value of the contribution for $s_k = 0$, which is

$$-\sum_l \frac{(F_l^\lambda)^2/(2l+1)}{1+F_l^\lambda/(2l+1)} = \sum_l (A_l^\lambda - F_l^\lambda). \quad \text{(B4)}$$

Subtracting this term from the result of the Landau-theory calculations, one recovers the correct result (71). It is important to notice that the $\hat{p}\cdot\hat{q}$ dependence comes from the second term in the equation (Fig. 5) for the vertex, and is therefore not affected by the considerations of this Appendix. We also note that calculations of transport coefficients[36] are also unaffected since processes in which both k and q are small are unimportant there.

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[23]Landau theory in its simplest form neglects the collective modes, which have no analog in the noninteracting Fermi gas. However, the theory can easily be extended by introducing collective-mode distribution functions and a collective-mode contribution to the entropy. The collective-mode distribution functions are to be treated on the same footing as the quasiparticle distribution function $n_{\vec{p}}$. We also note that the perturbation theory calculations of Balian and De Dominicis (Ref. 3) and Luttinger (Ref. 4) do not take into account collective modes; the quasiparticle expression for the entropy evaluated using statistical quasiparticle energies is therefore correct only to order $T^3 \ln T$.

[24]G. M. Carneiro and C. J. Pethick (unpublished).

[25]Calculations which emphasize the analytic properties of the quasiparticle interaction have also been performed by S. Misawa, in *Proceedings of the Twelfth International Conference on Low Temperature Physics*, edited by E. Kanda (Academic Press of Japan, Tokyo, 1971), p. 151.

[26]See, e.g., K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966), Vol. I., Chap. VII. The relation between energies of interaction and forward-scattering amplitudes was considered in some detail in connection with Brueckner-theory calculations by N. Fukuda and R. G. Newton [Phys. Rev. 103, 1558 (1956)] and B. S. De Witt [Phys. Rev. 103, 1565 (1956)]. It is also discussed in the context of many-body theory by Balian and De Dominicis (Ref. 3).

[27]We shall use the phrase "quasiparticle energy" to mean *statistical* quasiparticle energy.

[28]D. Pines and P. Nozières, *The Theory of Quantum Liquids* (Benjamin, New York, 1966), Vol. I., Chap. 1.

[29]Throughout this paper we use a system of units in

which $\hbar = 1$ and put the volume of the system equal to one.

[30]The fact that both the bare quasiparticle-quasihole interaction and the interaction energy [Eqs. (49) and (50)] are related to f is a consequence of the vertex function having two quasiparticle-quasihole channels. Since f is related to the value of the vertex function when the total momentum in a quasiparticle-quasihole channel tends to zero, there are two ways of obtaining f from the vertex function [cf. Eqs. (B1) and (B3) in Appendix B]; this is a consequence of the crossing symmetry of the vertex. In the two-particle problem the bare interaction and the interaction energy are not so simply related, since the vertex has only a single particle-particle channel. As we shall see one has to be rather careful when using this Hamiltonian since $f_{\vec{p}\vec{p}_1}$ does not have a well-defined limit when $\vec{p} \to \vec{p}_1$. However, this ambiguity does not affect our calculations of the $(\hat{p}\cdot\hat{q})^2$ contributions to $f_{\vec{p},\,\vec{p}+\vec{q}}$ in any way.

[31]L. D. Landau, Zh. Eksperim. i Teor. Fiz. 35, 97 (1958) [Sov. Phys. JETP 8, 70 (1959)].

[32]The factor of 2 is a consequence of the definition of spin-symmetric and spin-antisymmetric quantities [Eq. (5)], and may be checked by evaluating matrix elements of the effective Hamiltonian [Eq. (39)]. For example, the normalized state vector for a pair in a singlet spin state is $(1/\sqrt{2})\ (a^\dagger_{\vec{p}+\vec{q}\uparrow}a_{\vec{p}\uparrow}+a^\dagger_{\vec{p}+\vec{q}\downarrow}a_{\vec{p}\downarrow})\,|\,0\rangle = (1\sqrt{2})\ \delta n^\dagger_{\vec{p}\vec{q}}\,|\,0\rangle$, and therefore $\delta n^\dagger_{\vec{p}\vec{q}}$ creates a pair with amplitude $\sqrt{2}$. Thus in the Born approximation the scattering amplitude in the singlet states is *twice* $f^s_{\vec{p}\vec{p}_1}$.

[33]This assumption does treat the $f_{\vec{p}\vec{p}}$ term in the interaction energy incorrectly, and therefore gives an incorrect result for the interaction energy in the limit $\hat{p}\cdot\hat{q}\to 0$. We shall discuss this point further later in this section and in Appendix B.

[34]D. Hone, Phys. Rev. 121, 669 (1961).

[35]These numbers are obtained by using the values of the compressibility and magnetic susceptibility given in Ref. 15 and the value $m^*/m = 3$ taken from Ref. 5.

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[37]L. R. Corruccini, D. D. Osheroff, D. M. Lee, and R. C. Richardson, Phys. Rev. Letters 27, 650 (1971).

[38]C. J. Pethick, Phys. Rev. 185, 384 (1969).

[39]It may also be necessary to take into account Landau parameters with $l > 1$ when estimating finite-temperature contributions to transport coefficients.

[40]See, e.g., *Higher Transcendental Functions, Bateman Manuscript Project* (McGraw-Hill, New York, 1953), Vol. 2, p. 180.

[41]C. J. Pethick, Phys. Rev. 177, 391 (1969). The quantity $A^\lambda(\mu,\ \mu_1,\ \phi;\ s)$ in this reference corresponds to our $\nu(0)\,t^\lambda_{\vec{p}\vec{p}_1}(\vec{q},\ \omega)$.

[42]Problems related to those considered in this Appendix have previously been discussed by N. D. Mermin, Phys. Rev. 159, 161 (1967) and by S. Babu and G. E. Brown (unpublished).