PHYSICAL REVIEW A VOLUME 7, NUMBER 6 JUNE 1973

Kinetic Theory of a Dense Gas: Properties of Collision Kernels of the Bogoliubov Type*

C. D. Boley and Rashmi C. Desai
Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada
(Received 29 January 1973)

We study the collision kernels of two linearized equations for a moderately dense gas. These are the low-density Bogoliubov kinetic equation and the equation at the next order in the density (ternary-collision level), in the form given by Green and Cohen. We reduce the collision kernels of these equations to expressions which are nonlocal in space but local in time, and show that each of them breaks up into static and collisional parts. The former parts agree with the mean-field expressions. We compare the collisional parts with the fully dynamical expressions previously found by Mazenko and us, and find complete agreement only at zero wave vector and frequency. For nonzero wave vectors, the desired symmetry in the momentum variables breaks down. We show that this has no effect on the first-order transport coefficients; it would presumably be significant in the kinetic regime. We also find that the speed of sound predicted by the Bogoliubov equation agrees with the thermodynamic result truncated at first order in the density.

I. INTRODUCTION

Some time ago a classical kinetic equation valid at arbitrary wave vectors and frequencies but limited to first order in the density was derived by Mazenko.[1,2] The present authors later extended this equation to second order to take ternary collisions into account.[3] The collision kernels of these equations have been shown to be consistent with the known general properties, and they can be regarded as wave-vector-frequency generalizations of the linearized Boltzmann collision kernel and its extension to ternary collisions.

In this paper we use these theories to investigate two earlier well-known kinetic equations: the equation at first order in the density proposed by Bogoliubov,[4] and a second-order equation given by Green[5] and Cohen.[6] We consider only the linearized form of these equations, since the development of Refs. 1–3 is restricted to small deviations from thermal equilibrium. The Bogoliubov theory in particular has been very widely used in conceptualizing the kinetic behavior of a gas, and there is much interest in seeing the quantitative limitations of the result. A fundamental assumption in this theory is that the collision term is an instantaneous functional of the single-particle distribution function $F(\vec{r}\vec{p},t)$. Bogoliubov hypothesized that there exists a time scale long compared to the duration of a binary collision during which a synchronization of the higher-order distribution functions occurs, in the sense that they become completely determined by F and its initial condition. The familiar low-density expansion of the collision operator then leads to terms containing successive powers of F:

$$\left(\frac{\partial}{\partial t}+\frac{\vec{p}}{m}\cdot\vec{\nabla}\right)F(\vec{r}\vec{p},t)=J_2[F]+J_3[F]+\cdots \qquad (1)$$

and the quadratic term serves as a spatially nonlocal generalization of the Boltzmann collision integral. The ternary-collision term has been worked out from the Bogoliubov prescription by Choh and Uhlenbeck[7] and, from somewhat different points of view, by Green[5] and Cohen.[6] The form of the ternary-collision operator as given by Choh and Uhlenbeck is different from that given by Green and Cohen, and we deal here with the latter form, which is the one commonly used in calculations of the first density correction to transport coefficients. A demonstration[8] and strong arguments[9] in favor of their equivalence have been proposed, but questions have also been raised.[10]

Despite the success of the kinetic equations of Refs. 4–6 in the calculation of transport coefficients,[11,12] the approximations which lead to the collision terms are still imperfectly understood. One would like an unambiguous way of evaluating the corrections to the functional ansatz or similar factorization hypothesis.[13,14] It is also known that a straightforward expansion beyond the cubic term is invalid, since divergences[15] arise from certain sequences of correlated binary collisions, or ring events, and it is necessary to renormalize[16–18] the collision operator. Since only the binary- and ternary-collision operators are considered here, our results do not have a direct bearing on the divergence problem.

The most concise approach to the problem is to cast the proposed operators into the form of collision kernels and to study their behavior as functions of $\vec{k}$ and the Laplace transform variable z. This reveals the way in which they account for the spatial extent and the duration of collisions. Of course we cannot expect the z dependence to be complete, since the formulations of Refs. 4–7 effectively smooth out time scales comparable to the mean duration τ_d of a binary collision; consequently, the kernels are limited to zero frequency. The k dependence becomes important for wave vectors which are not negligible with respect to the inverse force range. It has been shown[19] that effects due to the k and z dependence of the kernel persist even in the hydrodynamic regime, since terms through second order in $\vec{k}$ and z contribute to the transport coefficients. (The density-independent contributions to the transport coefficients are not affected by the dependence on $\vec{k}, z$.)

We reduce the sum of the kernels of Refs. 4–6 to the form $nI_1(\vec{k};\vec{p}\vec{p}') + n^2I_2(\vec{k};\vec{p}\vec{p}')$, with I_1 accounting for binary collisions and I_2 for ternary collisions (n is the number density). At $k=0$, I_1 is known to reduce to the linearized Boltzmann collision kernel. We show that each kernel breaks up into static and collisional terms:

$$I_l(\vec{k};\vec{p}\vec{p}') = I_l^{(s)}(\vec{k}\vec{p}) + I_l^{(c)}(\vec{k};\vec{p}\vec{p}') \quad (l=1,2), \tag{2}$$

the first term giving the effect of the mean field on one particle owing to the others, and the second term containing the dynamical effects of collisions. The form of $I_l^{(s)}$ is familiar from the early collisionless kinetic equations[20,21] as well as from the more recent weakly coupled[19,22] and low-density[1,3,23] equations. However, to our knowledge it has not previously been recognized that such a term is present in the kinetic theories of Refs. 4–6. Since this term vanishes at $k=0$ it is not seen in any discussion which assumes spatial homogeneity. We cast the low-density collisional term $I_1^{(c)}$ into the form of a zero-frequency limit of a well-defined function of $\vec{k}$ and z. This function summarizes the dynamical information contained in the kernel. Because of difficulties with the long-time streaming operators, we are able to do this for $I_2^{(c)}$ only at $k=0$ but can draw some conclusions about its behavior at $k\neq 0$.

We examine these functions in the light of the fully dynamical kinetic equations[1–3] mentioned earlier. Such equations are formulated for the phase-space density correlation function, since this approach allows one to treat a large class of irreversible phenomena without having to appeal to any kind of functional assumption or factorization hypothesis. Of course the connection between fluctuations and linear response is implicit in such an approach. Evaluated through second order, the kernel takes the form $-in\Sigma_1(\vec{k}z;\vec{p}\vec{p}') - in^2\Sigma_2(\vec{k}z;\vec{p}\vec{p}')$, which is the fully dynamical generalization of the kernel $nI_1 + n^2I_2$. (No account has been taken for the possibility of bound states, and as in Refs. 4–7 we are limited to monotonically repulsive, short-range forces.) Σ_1 and Σ_2 have been shown to divide into static and collisional contributions:

$$\Sigma_l(\vec{k}z;\vec{p}\vec{p}') = \Sigma_l^{(s)}(\vec{k}\vec{p}) + \Sigma_l^{(c)}(\vec{k}z;\vec{p}\vec{p}') \quad (l=1,2), \tag{3}$$

and the static portions agree with their counterparts discussed above. We show, however, that the collisional portions agree with $I_1^{(c)}$ and $I_2^{(c)}$ only in the uniform long-time limit,[24] $k\to 0$ and $z\to i0^+$. For $k\neq 0$, $I_1^{(c)}$ and $I_2^{(c)}$ do not have the correct symmetry in the momentum variables, which is dictated by detailed balance. The symmetry-violating term, which vanishes with $\vec{k}$, is contained in the two- and three-particle propagators and would be expected to have important consequences in the kinetic regime.

Accordingly, the kinetic equations with kernels nI_1 and $nI_1 + n^2I_2$ appear to be completely consistent only at zero wave number and frequency. The first-order analysis is given in Sec. II and the second-order analysis in Sec. III. In Sec. IV we investigate the extent to which this conclusion applies to the most common area in which the equations have been applied, namely, the hydrodynamic regime. On the one hand, the transport coefficients predicted by these equations are known to agree[25,26] through first order in the density with the correlation-function results, and on the other hand, the contributions due to collisional transfer arise from the k dependence of the collision kernel (in this case of $I_1^{(c)}$). We show, however, that the anomalous k dependence does not contribute in the particular long-wavelength low-frequency limit which determines the first-order transport coefficients. Although much work has been done with the transport coefficients, we are not aware of any investiation of the small-k dispersion relation, and hence of the speed of sound, which is predicted by these kinetic equations. It is due to a combination of free-streaming, mean-field, and dynamical effects. Of course, if the collision kernels are taken in the local limit, only the first effect enters and one obtains the familiar $C^2 = 5/3(\beta m)^{-1}$. Using a method developed in the weakly coupled case,[19] we calculate the value of C^2 which is predicted by the Bogoliubov kernel nI_1 and show that it agrees through first order in the density with the usual adiabatic derivative. One would not necessarily expect agreement, since the frequency dependence of the true kernel should contribute to C^2 and is absent in nI_1. However, it turns out that this contribution is supplied by the anomalous k dependence of nI_1.

II. FIRST-ORDER COLLISION KERNEL

We write the low-density collision integral of Bogoliubov[4] in the form

$$J_2[F(1t)] = -\lim_{\tau\to\infty} i\int d2\, L_1(12)\, Z_\tau(12)\, F(1t)\, F(2t), \tag{4}$$

where

$$Z_\tau(12) = e^{-i\tau L'(12)}\, e^{i\tau L_0(12)} \tag{5}$$

and the L's are the usual two-body Liouville operators

$$iL_0(12)=\vec{p}_1\cdot\vec{\nabla}_1+\vec{p}_2\cdot\vec{\nabla}_2,$$
$$iL_1(12)=-\frac{\partial v(12)}{\partial\vec{r}_1}\cdot\left(\frac{\partial}{\partial\vec{p}_1}-\frac{\partial}{\partial\vec{p}_2}\right), \qquad (6)$$

with $L=L_0+L_1$. We use the notation $1=(\vec{r}_1,\vec{p}_1)$ and $v(12)=v(|\vec{r}_1-\vec{r}_2|)$, and set the particle mass to unity. We linearize the distribution function about absolute equilibrium

$$F(1t)=n\phi(\vec{p}_1)+h(\vec{r}_1\vec{p}_1,t), \qquad (7)$$

where

$$\phi(\vec{p})=(\beta/2\pi)^{3/2}e^{-\beta p^2/2} \qquad (8)$$

and $\beta=(k_BT)^{-1}$. The collision integral then reduces to the form

$$J_2[F(1t)]=\int d^3r'\,d^3p'\,nI_1(\vec{r}_1\vec{p}_1;\,\vec{r}'\vec{p}')\,h(\vec{r}'\vec{p}',t)+O(h^2), \qquad (9)$$

where the collision kernel I_1 is

$$I_1(\vec{r}_1\vec{p}_1;\vec{r}'\vec{p}')=-\lim_{\tau\to\infty}i\int d2\,L_1(12)\,Z_\tau(12)\,\phi(\vec{p}_1)\,\phi(\vec{p}_2)\times\sum_{\alpha=1}^{2}\delta(\vec{r}'-\vec{r}_\alpha)\,\delta(\vec{p}'-\vec{p}_\alpha)[1/\phi(\vec{p}_\alpha)]. \qquad (10)$$

The only contribution to the integral arises from configurations for which particles 1 and 2 are within a force range of each other, and the operator $e^{-i\tau L}$ in Z_τ separates the particles far from each other in the distant past. Because the kinetic energy in the infinite past equals the total initial energy $H(12)$ (assuming short-ranged repulsive forces), we have[4] at large τ

$$L_1(12)Z_\tau(12)\,\phi(\vec{p}_1)\,\phi(\vec{p}_2)f(12)=L_1(12)\,\phi(\vec{p}_1)\,\phi(\vec{p}_2)\,e^{-\beta v(12)}Z_\tau(12)f(12),$$

where $f(12)$ is an arbitrary function. Then using the translational invariance of $I_1(\vec{r}_1\vec{p}_1;\vec{r}'\vec{p}')=I_1(\vec{r}_1-\vec{r}';\vec{p}_1\vec{p}')$, we can show that its spatial transform is

$$I_1(\vec{k};\vec{p}\vec{p}')\,\phi(\vec{p}')\equiv\int d^3r\,e^{-i\vec{k}\cdot\vec{r}}I_1(\vec{r};\vec{p}\vec{p}')\,\phi(\vec{p}')=\lim_{\tau\to\infty}i\left(\frac{\beta}{2\pi}\right)^3\int\frac{d1\,d2}{V}\,e^{-\beta H(12)}\,\rho^*_{kp'}(12)\times e^{i\tau L(12)+i\tau\vec{k}\cdot\vec{p}'}L_1(12)\,\rho_{kp}(1), \qquad (11)$$

where

$$\rho_{kp}(1\cdots N)=\sum_{\alpha=1}^{N}\delta(\vec{p}-\vec{p}_\alpha)\,e^{-i\vec{k}\cdot\vec{r}_\alpha}. \qquad (12)$$

It is useful to express the long-time limit in Eq. (11) in frequency language. For a Hermitian operator O such that $\lim_{\tau\to\infty}e^{i\tau O}f$ exists, we have[25]

$$\lim_{\tau\to\infty}e^{iO\tau}f=f-\lim_{z\to i\epsilon}O(z+O)^{-1}f \qquad (13)$$

(ϵ = infinitesimal positive number). Using this in Eq. (11), with $O=L(12)+\vec{k}\cdot\vec{p}'$, we note that the kernel breaks up into two parts, corresponding to the two terms on the right-hand side of Eq. (13):

$$I_1(\vec{k};\vec{p}\vec{p}')=I_1^{(s)}(\vec{k}\vec{p})+I_1^{(c)}(\vec{k};\vec{p}\vec{p}'). \qquad (14)$$

The part which we call the static term reduces to

$$I_1^{(s)}(\vec{k}\vec{p})=if(\vec{k})\,\vec{k}\cdot\vec{p}\,\phi(\vec{p}), \qquad (15)$$

where $f(\vec{k})$ is the transform of the Mayer function $f(r)=e^{-\beta v(r)}-1$. Equation (15) agrees with the expected result for the static kernel, since $f(r)$ is the low-density direct correlation function. More generally,[20,21] one has

$$I^{(s)}(\vec{k}\vec{p})=inc(\vec{k})\,\vec{k}\cdot\vec{p}\,\phi(\vec{p}). \qquad (16)$$

Until now it has not been transparent that the Bogoliubov kernel includes a static (mean-field) term of this kind.

The collisional part of the Bogoliubov kernel can be written

$$I_1^{(c)}(\vec{k};\vec{p}\vec{p}')\,\phi(\vec{p}')=-\lim_{z\to i\epsilon}i\int\frac{d1\,d2}{2V}\,\phi(\vec{p}_1)\,\phi(\vec{p}_2)\times\rho^*_{kp'}(12)M_1'(\vec{k}z,\vec{p}';12)\,L_1(12)\,\rho_{kp}(12), \qquad (17)$$

where

$$M_1'(\vec{k}z,\vec{p}';12)=e^{-\beta v(12)}[\vec{k}\cdot\vec{p}'+L(12)]\times[z+\vec{k}\cdot\vec{p}'+L(12)]^{-1}. \qquad (18)$$

The interesting part of this term is the operator M_1', which is analyzed in more detail later.

Equations (14), (15), and (17) summarize the low-density collision kernel which is deduced from the linearized Bogoliubov theory. We have defined I_1 so that the linear transport equation reads

$$(z-\vec{k}\cdot\vec{p})\,h(\vec{k}\vec{p},z)+h(\vec{k}\vec{p},t=0)=in\int d^3\bar{p}\,I_1(\vec{k};\vec{p}\vec{\bar{p}})\,h(\vec{k}\vec{\bar{p}},z), \qquad (19)$$

in terms of the Laplace transform

$$h(\vec{k}\vec{p},z)=i\int_0^\infty dt\,e^{izt}h(\vec{k}\vec{p},t). \qquad (20)$$

For comparison with the correlation-function approach, we recall the object analogous to h, namely, the thermal average

$$S(\vec{r}-\vec{r}',t-t';\vec{p}\vec{p}')=\langle(f(\vec{r}\vec{p}t)-\langle f(\vec{r}\vec{p}t)\rangle)\times(f(\vec{r}'\vec{p}'t')-\langle f(\vec{r}'\vec{p}'t')\rangle)\rangle, \qquad (21)$$

where f is the local density in phase space

$$f(\vec{r}\vec{p}t)=\sum_\alpha\delta(\vec{r}-\vec{r}_\alpha(t))\,\delta(\vec{p}-\vec{p}_\alpha(t)) \qquad (22)$$

and the sum runs over the particles in the system. S is related in a well-known way to the linear deviation from thermal equilibrium produced by an external force applied before $t=0$, and in particular, $S(\vec{r}t;\vec{p}\vec{p}')$ and $h(\vec{r}\vec{p},t)$ are described by the same kinetic equation in $(\vec{r},\vec{p},t)$. In this context, the variable $\vec{p}'$ enters only into the initial condition and is not involved in the dynamics of collisions. S satisfies a generalized Langevin equation of the form

$$(z-\vec{k}\cdot\vec{p})S(\vec{k}z;\vec{p}\vec{p}')+\tilde{S}(\vec{k};\vec{p}\vec{p}')=\int d^3\bar{p}\,\Sigma(\vec{k}z;\vec{p}\bar{\vec{p}})S(\vec{k}z;\bar{\vec{p}}\vec{p}'), \quad (23)$$

where Σ is the memory function and $\tilde{S}$ the initial condition. Therefore, inI_1 should agree with the low-density expression for Σ. The part of interest here in the low-density expansion of Σ has the form

$$\Sigma\simeq n\Sigma_1+n^2\Sigma_2. \quad (24)$$

The static part of Σ_1 is the expected $iI_1^{(s)}$ and the less trivial collisional part has been shown by Mazenko[1] to be (cf. also Ref. 23)

$$\Sigma_1^{(c)}(\vec{k}z;\vec{p}\vec{p}')\,\phi(\vec{p}')=\int\frac{d1\,d2}{2V}\,\phi(\vec{p}_1)\,\phi(\vec{p}_2)\,\rho^*_{kp'}(12)\times M_1(\vec{k}z,\vec{p}';12)\,L_1(12)\,\rho_{kp}(12), \quad (25)$$

with

$$M_1(\vec{k}z,\vec{p}';12)=e^{-\beta v(12)}[\vec{k}\cdot\vec{p}'+L(12)][z+L(12)]^{-1}, \quad (26)$$

which is to be compared with $iI_1^{(c)}(\vec{k};\vec{p}\vec{p}')\,\phi(\vec{p}')$.

We note that $iI_1^{(c)}$ is the zero-frequency limit of a function which agrees $\Sigma_1^{(c)}$ only at $k=0$, so that

$$iI_1^{(c)}(0;\vec{p}\vec{p}')=\lim_{z\to i\epsilon}\Sigma_1^{(c)}(0z;\vec{p}\vec{p}'). \quad (27)$$

The discrepancy at $k\neq 0$ arises from the term $\vec{k}\cdot\vec{p}'$ in the propagator in M_1', which in turn can be traced to the free-streaming operator in the definition of Z_τ, Eq. (5). The Bogoliubov theory is not improved by elimination of that operator from the definition, since this operator is also responsible for the other term $\vec{k}\cdot\vec{p}'$ in M_1'. One of the important consequences of the discrepancy is that the desired detailed balance symmetry in the momentum indices breaks down for $k\neq 0$. In general, one has[19]

$$\Sigma^{(c)}(\vec{k}z;\vec{p}\vec{p}')\,\phi(\vec{p}')=\Sigma^{(c)}(\vec{k}z;\vec{p}'\vec{p})\,\phi(\vec{p}), \quad (28)$$

which holds[2] for $\Sigma_1^{(c)}$ but which can be seen to fail for $I_1^{(c)}$. As we note in Sec. IV, the asymmetry has no effect on the transport coefficients and the speed of sound through first order in the density, but it would be expected to become more important as k increases.

Finally, we recall that a collision kernel describes a dynamically stable system if[19]

$$\int d^3p\,d^3p'\,g^*(\vec{p})[\mathrm{Im}\Sigma^{(c)}(\vec{k}z;\vec{p}\vec{p}')\,\phi(\vec{p}')]g(\vec{p}')\leq 0, \quad (29)$$

when $\mathrm{Im}z>0$, with g an arbitrary function. (At first order in the density and as $k\to 0$, $z\to i\epsilon$, this states that the eigenvalues of the linearized Boltzmann collision operator are negative or zero.) It can be shown that Eq. (29) still holds for $iI_1^{(c)}$ and so the asymmetry does not affect the stability of the system.

III. SECOND-ORDER COLLISION KERNEL

We write the ternary-collision kernel of Refs. 5 and 6 in the form

$$J_3[F(1t)]=-\lim_{\tau\to\infty} i\int d2\,d3\,L_1(12)\,C_\tau(12;3)\times F(1t)\,F(2t)\,F(3t), \quad (30)$$

where

$$C_\tau(12;3)=Z_\tau(123)-Z_\tau(12)[Z_\tau(13)+Z_\tau(23)-1],\qquad Z_\tau(123)=e^{-i\tau L(123)}e^{i\tau L_0(123)}, \quad (31)$$

and $L(123)$ is the three-particle Liouville operator. When F is linearized and J_3 arranged as a term independent of h plus linear terms, the term independent of h vanishes as in the J_2 calculation. Thus, if we put

$$J_3[F(1t)]=\int d^3r'\,d^3p'\,n^2 I_2(\vec{r}_1\vec{p}_1;\vec{r}'\vec{p}')\times h(\vec{r}'\vec{p}',t)+O(h^2), \quad (32)$$

the kernel again depends spatially only on $\vec{r}_1-\vec{r}'$ and works out to be, after some rearrangements like those in Sec. II,

$$I_2(\vec{k};\vec{p}\vec{p}')\,\phi(\vec{p}')=\lim_{\tau\to\infty} i\int\frac{d1\,d2\,d3}{V}[L_1(12)\,\rho_{kp}(1)]\times\phi(\vec{p}_1)\,\phi(\vec{p}_2)\,\phi(\vec{p}_3)\{e^{-\beta v(123)}Z_\tau(123)-e^{-\beta v(12)}Z_\tau(12)\times[e^{-\beta v(13)}Z_\tau(13)+e^{-\beta v(23)}Z_\tau(23)-1]\}\,\rho^*_{kp'}(123), \quad (33)$$

where $v(123)=v(12)+v(23)+v(13)$.

From the example of Sec. II, it is expected that I_2 can be divided into static and collisional parts, $I_2=I_2^{(s)}+I_2^{(c)}$. The static part is obtained by replacing all the Z_τ's in Eq. (33) with unity, and after a number of elementary manipulations the result is

$$I_2^{(s)}(\vec{k}\vec{p})=ic_1(\vec{k})\,\vec{k}\cdot\vec{p}\,\phi(\vec{p}), \quad (34)$$

where

$$c_1(\vec{r})=f(\vec{r})\int d^3r'\,f(\vec{r}-\vec{r}')\,f(\vec{r}'). \quad (35)$$

The last quantity is the first density correction to the direct correlation function. Therefore, the static term has the expected form, namely, the second-order contribution to Eq. (16).

Now we turn to the remaining portion of I_2, which is the collisional part $I_2^{(c)}$. In order to compare it with the second-order term $\Sigma_2^{(c)}$ obtained by us previously,[3] we first recall that $\Sigma_2^{(c)}$ can be written as the sum of two terms

$$\Sigma_2^{(c)}=\Sigma_{2a}^{(c)}-2\beta_1\Sigma_1^{(c)}, \quad (36)$$

where $\Sigma_{2a}^{(c)}$ involves genuine three-body dynamics and

$$\beta_1=\int d^3r\,f(r). \quad (37)$$

In the second term on the right-hand side of Eq. (36), the presence of the third particle merely modifies the binary-collision rate. The rather lengthy expression for $\Sigma_{2a}^{(c)}$ is

$$\Sigma_{2a}^{(c)}(\vec{k}z;\vec{p}\vec{p}')\,\phi(\vec{p}')=\int\frac{d1\,d2\,d3}{2V}\,\phi(\vec{p}_1)\,\phi(\vec{p}_2)\,\phi(\vec{p}_3)\,\rho_{kp'}^{*}(123)\,M_2(\vec{k}z,\vec{p}';123)\,L_1(12)\,\rho_{kp}(12), \tag{38}$$

where M_2 is the operator

$$M_2(\vec{k}z,\vec{p}';123)=e^{-\beta v(123)}[\vec{k}\cdot\vec{p}'+L(123)][z+L(123)]^{-1}-e^{-\beta v(12)}[\vec{k}\cdot\vec{p}'+L(12)][z+L(12)]^{-1}$$
$$-2e^{-\beta v(13)}[\vec{k}\cdot\vec{p}'+L(13)+L(2)][z+L(13)]^{-1}(z-\vec{k}\cdot\vec{p}_1)\,e^{-\beta v(12)}[z+L(12)]^{-1}. \tag{39}$$

We attempt to break up $I_2^{(c)}$ into a form similar to Eq. (36), and after some labor obtain an expression of the form

$$I_2^{(c)}=I_{2a}^{(c)}+I_{2b}^{(c)}-2\beta_1 I_1^{(c)}, \tag{40}$$

where $I_{2a}^{(c)}$ and $I_{2b}^{(c)}$ involve, respectively, three- and two-body dynamics. We shall find that at $k=0$, $iI_{2a}^{(c)}$ becomes the $z\to i\epsilon$ limit of $\Sigma_{2a}^{(c)}$ and $I_{2b}^{(c)}$ vanishes. These terms are given by

$$I_{2a}^{(c)}(\vec{k};\vec{p}\vec{p}')\,\phi(\vec{p}')=-\lim_{\tau\to\infty} i\int\frac{d1\,d2\,d3}{2V}\,\phi(\vec{p}_1)\,\phi(\vec{p}_2)\,\phi(\vec{p}_3)\,\rho_{kp'}^{*}(123)\,M_2'(\vec{k}\tau,\vec{p}';123)\,L_1(12)\,\rho_{kp}(12), \tag{41}$$

$$I_{2b}^{(c)}(\vec{k};\vec{p}\vec{p}')\,\phi(\vec{p}')=-if(\vec{k})\,\phi(\vec{p}')\left(\frac{\beta}{2\pi}\right)^3\lim_{\tau\to\infty}\int\frac{d1\,d2}{V}\,e^{-\beta H(12)}\,e^{i\vec{k}\cdot\vec{r}_1}\,(e^{i\tau\vec{k}\cdot\vec{p}_1}e^{i\tau L(12)}-1)\,L_1(12)\,\rho_{kp}(12), \tag{42}$$

and M_2' is now the operator

$$M_2'(\vec{k}\tau,\vec{p}';123)=-e^{-\beta v(123)}(e^{i\tau L(123)+i\tau\vec{k}\cdot\vec{p}'}-1)+e^{-\beta v(12)}(e^{i\tau L(12)+i\tau\vec{k}\cdot\vec{p}'}-1)$$
$$+2e^{-\beta v(13)}(e^{i\tau L(13)+i\tau L(2)+i\tau\vec{k}\cdot\vec{p}'}-1)\,e^{-\beta v(12)}e^{i\tau\vec{k}\cdot\vec{p}_1}e^{i\tau L(12)}. \tag{43}$$

The algebra involved in obtaining Eqs. (40)–(43) is given in Appendix A.

The integrands of $I_{2a}^{(c)}$ and $I_{2b}^{(c)}$ have a very complicated τ dependence, owing mainly to the fact that the τ's do not occur in a single simple exponential, e.g.,

$$e^{i\tau\vec{k}\cdot\vec{p}_1}e^{i\tau L(12)}\neq e^{i\tau\vec{k}\cdot\vec{p}_1+i\tau L(12)}. \tag{44}$$

It is not clear how such terms can generally be written as zero-frequency limits. However, at $k=0$ the difficulties can be overcome, and we find

$$I_{2a}^{(c)}(0;\vec{p}\vec{p}')\,\phi(\vec{p}')=-\lim_{z\to i\epsilon} i\int\frac{d1\,d2\,d3}{2V}\,\phi(\vec{p}_1)\,\phi(\vec{p}_2)\,\phi(\vec{p}_3)$$
$$\times\rho_{p'}(123)\,M_2(0z,\vec{p}';123)\,L_1(12)\,\rho_p(12), \tag{45}$$

$$I_{2b}^{(c)}(0;\vec{p}\vec{p}')=0, \tag{46}$$

where $M_2(0z,\vec{p}';123)$ is defined via Eq. (39) and $\rho_p\equiv\rho_{0p}$.

Thus, at $k=0$, $z=i\epsilon$, the collision kernel is in agreement with the second-order memory function:

$$iI_2^{(c)}(0;\vec{p}\vec{p}')=\lim_{z\to i\epsilon}\Sigma_2^{(c)}(0z;\vec{p}\vec{p}'). \tag{47}$$

However, for $k\neq 0$ it is apparent that $I_2^{(c)}$ can no longer agree with the zero-frequency limit of $\Sigma_2^{(c)}$. The reason is essentially the same as that noted in Sec. II: The term $i\vec{k}\cdot\vec{p}'$ in the first two exponential operators of $M_2'(\vec{k}\tau,\vec{p}';123)$ will lead to the undesired term $\vec{k}\cdot\vec{p}'$ in the denominator of the propagators. Of course this point also holds for the final term $-2\beta_1 I_1^{(c)}$ in the expression (40) for $I_2^{(c)}$. Hence $I_2^{(c)}$ will not satisfy the symmetry condition (28). Again, there seems to be no obvious way of changing the definition of the Z_τ's to correct this situation.

IV. HYDRODYNAMIC IMPLICATIONS

In this section we consider the extent to which the conclusions of the previous two sections affect the hydrodynamic behavior. We find that the discrepancies noted earlier do not contribute to the shear viscosity through first order in the density, and the same reasoning applies to the thermal conductivity. (For repulsive forces the bulk viscosity starts at order n^2.) We also calculate the speed of sound appropriate to the kinetic equation with kernel nI_1 and, owing to the cancellation of effects mentioned at the end of Sec. I, we find agreement with the correct first-order expression. To our knowledge this is the first calculation of the speed of sound implied by the Bogoliubov kinetic equation. We have not been able to evaluate all the matrix elements necessary for dealing with the second-order kernel $nI_1+n^2I_2$.

We first recall the essential points pertaining to the first-order shear viscosity which follows from the kinetic equation with memory function $n\Sigma_1+n^2\Sigma_2$. Only a general discussion is necessary, since some details appeared in Ref. 3 and in any event the final answer is well known.[25,26] Writing $\eta=\eta_0+n\eta_1+\cdots$, we have at lowest order the standard Chapman–Enskog result. At next order it is necessary to consider not only the kinetic contribution to the stress tensor but also the low-density potential contribution, so that we have $\eta_1 = \eta_{KK,1} + 2\eta_{KU,1}$, where the subscripts have the usual meaning. The first term, which arises from complete triple collisions and incomplete binary collisions, is determined by the limits of $\Sigma_1^{(c)}$ and $\Sigma_2^{(c)}$ as $k \to 0$, $z \to i\epsilon$. Since in this limit $iI_1^{(c)} = \Sigma_1^{(c)}$ and $iI_2^{(c)} = \Sigma_2^{(c)}$, the expressions for $\eta_{KK,1}$ will also agree.

Our interest focuses on the collisional-transfer term $\eta_{KU,1}$, which depends on $\Sigma_1^{(c)}$ through terms of first order in k, again as $z \to i\epsilon$. Specifically, the contribution of first-order terms in $\Sigma_1^{(c)}$ involves the quantity

$$T_{13,1}(\mathrm{p}') = \lim_{\substack{k\to 0\\ z\to i\epsilon}} \frac{1}{k} \int d^3p\, p_1 \Sigma_1^{(c)}(\vec{\mathrm{k}}z; \vec{\mathrm{p}}\vec{\mathrm{p}}'), \qquad (48)$$

which determines the long-wavelength low-frequency limit of the low-density stress tensor [$\vec{\mathrm{k}}$ is taken in the 3-direction, and the order of the limits in Eq. (48) is immaterial]. From Eq. (25) it can be seen that

$$T_{13,1}(\vec{\mathrm{p}}')\,\phi(\vec{\mathrm{p}}') = -\lim_{z\to i\epsilon} \int \frac{d1\,d2}{2V}\, \phi(\vec{\mathrm{p}}_1)\,\phi(\vec{\mathrm{p}}_2)\,\rho_{p'}(12) \times M_1(0z, \vec{\mathrm{p}}'; 12)\,\lambda_{13}(\vec{\mathrm{r}}_1 - \vec{\mathrm{r}}_2), \qquad (49)$$

with

$$\lambda_{ij}(\vec{\mathrm{r}}) = -2\,\frac{r_i r_j}{r}\,\frac{dv(r)}{dr}. \qquad (50)$$

Thus, the first-order shear viscosity implied by the kinetic equation with kernel $nI_1 + n^2 I_2$ agrees with the above formulation if the corresponding $T_{13,1}$ agrees with Eq. (49). But this is now immediate, since we obtain an expression identical to Eq. (49) except that the operator M_1 is replaced by M_1', and since $M_1 = M_1'$ when $k = 0$. Thus the improper k dependence of nI_1, which occurs in M_1', does not contribute to η_1, and the shear viscosities agree at first order.

The speed of sound C is defined by its appearance in the propagating hydrodynamic mode of the dispersion relation. It has been shown[19] how the modes as well as the density-density correlation function can be calculated directly from the memory function. We cite only the points necessary for our discussion and refer the reader to Ref. 19 for the complete formulation. One deals with the 3×3 matrix $G_{\mu\nu}(\vec{\mathrm{k}}z)$ which gives, apart from a k-dependent factor, the correlations among the density, the longitudinal momentum density, and the kinetic-energy density. It satisfies an equation of the form

$$[z\,\delta_{\mu\sigma} - \Omega_{\mu\sigma}(\vec{\mathrm{k}}z)]\,G_{\sigma\nu}(\vec{\mathrm{k}}z) = \delta_{\mu\nu} \qquad (51)$$

and Ω can be expressed in terms of Σ. If the total energy density rather than the kinetic-energy density had been used, Ω would be essentially the hydrodynamic matrix which has been discussed earlier[27] at small k and z. The dispersion relation is determined by $\det|z - \Omega(\vec{\mathrm{k}}z)| = 0$, and to identify C we need Ω through first order in k, z.

When we specialize to the kinetic equation with the Bogoliubov kernel nI_1, this equation takes the form $\det|z - \Omega^B(\vec{\mathrm{k}})| = 0$, in which Ω^B is independent of z because of the nature of the collision kernel. It is shown in Appendix B that to lowest order in k, Ω^B has the form

$$\Omega^B_{\mu\nu}(\vec{\mathrm{k}}) = (k/\sqrt{\beta})\,\Omega^B_{\mu\nu,1} + \cdots, \qquad (52)$$

$$\Omega^B_{\mu\nu,1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 - nf(0) & 0 & (\tfrac{2}{3})^{1/2}(1 + n\alpha_1) \\ 0 & (\tfrac{2}{3})^{1/2}(1 + n\alpha_1 - n\alpha_1') & 0 \end{pmatrix}, \qquad (53)$$

with α_1 and α_1' as defined by Eqs. (B6) and (B8), respectively. It is important to note that α_1 and α_1' occur in the density expansion of certain thermodynamic quantities. Thus, we have

$$\left(\frac{\partial p}{\partial T}\right)_n = nk_B(1 + n\alpha_1 + \cdots) \qquad (54)$$

and the specific heat at constant volume is

$$c_v = \tfrac{3}{2}k_B(1 + n\alpha_1' + \cdots). \qquad (55)$$

The matrix $\Omega^B_{\mu\nu,1}$ differs in two ways from the prediction of the correct low-density theory.[2] The latter matrix has an element $\Omega_{33,1}(\vec{\mathrm{k}}z) = -zn\alpha_1'\sqrt{\beta}/k$ [which is absent in Eq. (53) since the limit $z \to i\epsilon$ is contained in the Bogoliubov kernel], and it also has

$$\Omega_{23,1} = \Omega_{32,1} = (\tfrac{2}{3})^{1/2}(1 + n\alpha_1). \qquad (56)$$

The anomalous contribution to $\Omega^B_{32,1}$ arises from the part of the Bogoliubov collision kernel which is not symmetric in $\vec{\mathrm{p}} \leftrightarrow \vec{\mathrm{p}}'$.

Thus the Bogoliubov matrix $\Omega^B_{\mu\nu,1}$ agrees with the complete low-density result only when $\alpha_1' = 0$. From Eq. (B8) it follows that the only monotonically repulsive force law for which $\alpha_1' = 0$ is the hard-sphere potential (the specific heat is then $\tfrac{3}{2}k_B$ at all densities). We note that the low-density memory function for this system has been shown to be independent of frequency.[28]

To obtain the dispersion relation implied by $\Omega^B_{\mu\nu,1}$, we set $z^2 = C^2k^2$ and find

$$\beta C^2 = 1 - nf(0) + \tfrac{2}{3}(1 + n\alpha_1 - n\alpha_1')(1 + n\alpha_1) = 1 - nf(0) + \tfrac{2}{3}[1 + n(2\alpha_1 - \alpha_1')] + O(n^2). \qquad (57)$$

The other root $z = 0$ corresponds to heat diffusion. In the conventional hydrodynamic formulation, C is supposed to be the adiabatic speed of sound $C_0 = [(\partial p/\partial n)_S]^{1/2}$, which after some manipulation of thermodynamic derivatives can be written

$$\beta C_0^2 = \beta\left(\frac{\partial p}{\partial n}\right)_T + \frac{k_B}{c_v}\left[\frac{1}{nk_B}\left(\frac{\partial p}{\partial T}\right)_n\right]^2. \qquad (58)$$

At low densities this becomes

$$\beta C_0^2 = 1 - nf(0) + \tfrac{2}{3}(1+n\alpha_1)^2/(1+n\alpha_1'), \qquad (59)$$

where we have used Eqs. (54) and (55) and the compressibility relation $\beta(\partial p/\partial n)_T = 1 - nc(0)$. Through first order in the density, this agrees with the result βC^2 obtained from the Bogoliubov collision kernel. When the Mazenko memory function is used, the result is[2] precisely Eq. (59), with the term $n\alpha_1'$ arising from the frequency dependence in $\Sigma_1^{(c)}$. With the Bogoliubov kernel nI_1 this contribution is dropped but then regained by virtue of the anomalous k dependence. Thus a cancellation of errors occurs in the expression for the speed of sound.

APPENDIX A: DERIVATION OF EQS. (40)–(43)

The collisional part $I_2^{(c)}(\vec{k};\vec{p}\vec{p}')\phi(\vec{p}')$ is defined as the difference between Eq. (33) and this expression with the Z_τ's replaced by unity. It is convenient to interchange particles 1 and 2 within the integrand and then to take half the sum of the two expressions for $I_2^{(c)}$, with the result

$$I_2^{(c)}(\vec{k};\vec{p}\vec{p}')\phi(\vec{p}') = \lim_{\tau\to\infty} i\int\frac{d1\,d2\,d3}{2V}[L_1(12)\rho_{kp}(12)]\phi(\vec{p}_1)\phi(\vec{p}_2)\phi(\vec{p}_3)Q_\tau(123)\rho^*_{kp'}(123), \qquad (A1)$$

where

$$Q_\tau(123) = e^{-\beta v(123)}\tilde{Z}_\tau(123) - 2e^{-\beta v(12)}\tilde{Z}_\tau(12)e^{-\beta v(13)}\tilde{Z}_\tau(13)$$
$$- 2e^{-\beta v(12)}\tilde{Z}_\tau(12)e^{-\beta v(13)} - 2e^{-\beta v(12)-\beta v(13)}\tilde{Z}_\tau(13) + e^{-\beta v(12)}\tilde{Z}_\tau(12) \qquad (A2)$$

and $\tilde{Z}_\tau = Z_\tau - 1$. We separate this into three terms by taking $Q_\tau(123)\rho^*_{kp'}(123) = \rho_a + \rho_b + \rho_c$, with

$$\rho_a = e^{-\beta v(123)}\tilde{Z}_\tau(123)\rho^*_{kp'}(123) - 2e^{-\beta v(12)}\tilde{Z}_\tau(12)\rho^*_{kp'}(1)$$
$$- 2e^{-\beta v(12)}[\tilde{Z}_\tau(12)+1]e^{-\beta v(13)}\tilde{Z}_\tau(13)\rho^*_{kp'}(13), \qquad (A3)$$

$$\rho_b = 2e^{-\beta v(12)}\tilde{Z}_\tau(12)e^{-\beta v(13)}\rho^*_{kp'}(3), \qquad (A4)$$

$$\rho_c = -2e^{-\beta v(12)}\tilde{Z}_\tau(12)e^{-\beta v(13)}\rho^*_{kp'}(2)$$
$$- 2e^{-\beta v(12)}\tilde{Z}_\tau(12)f(13)\rho^*_{kp'}(1)$$
$$+ e^{-\beta v(12)}\tilde{Z}_\tau(12)\rho^*_{kp'}(123), \qquad (A5)$$

and the corresponding contributions to the collision kernel are denoted by $I_2^{(c)} = I_{2a}^{(c)} + I_{2b}^{(c)} + I_{2c}^{(c)}$. Although this separation may appear arbitrary, it leads directly to Eqs. (40)–(43). We can verify without difficulty that $I_{2c}^{(c)} = -2\beta_1 I_1^{(c)}$, and we proceed to $I_{2a}^{(c)}$ and $I_{2b}^{(c)}$.

First, one can write

$$I_{2a}^{(c)}(\vec{k};\vec{p}\vec{p}')\phi(\vec{p}') = \lim_{\tau\to\infty} i\int\frac{d1\,d2\,d3}{2V}[L_1(12)\rho_{kp}(12)]\phi(\vec{p}_1)\phi(\vec{p}_2)\phi(\vec{p}_3)\tilde{\rho}_a, \qquad (A6)$$

with

$$\tilde{\rho}_a \equiv [e^{-\beta v(123)}(e^{-i\tau L(123)+i\tau\vec{k}\cdot\vec{p}'} - 1) - e^{-\beta v(12)}(e^{-i\tau L(12)-i\tau L(3)+i\tau\vec{k}\cdot\vec{p}'} - 1)]\rho^*_{kp'}(123)$$
$$- 2e^{-\beta v(12)}e^{-i\tau L(12)}e^{i\tau L(1)}e^{-\beta v(13)}(e^{-i\tau L(13)+i\tau\vec{k}\cdot\vec{p}'} - 1)\rho^*_{kp'}(13), \qquad (A7)$$

and the operator $e^{i\tau L(1)}$ above can be replaced with $\exp(-\tau\vec{p}_1\cdot\vec{\nabla}_3 + i\tau\vec{k}\cdot\vec{p}_1)$. This last fact is seen by noting that

$$(\vec{\nabla}_1 + \vec{\nabla}_3 - i\vec{k})e^{-\beta v(13)}(e^{-i\tau L(13)+i\tau\vec{k}\cdot\vec{p}'} - 1)\rho^*_{kp'}(13) = 0, \qquad (A8)$$

which holds because the only spatial dependence in (A8) not of the form $\vec{r}_1 - \vec{r}_3$ occurs in $\rho^*_{kp'}(13)$. When $\tilde{\rho}_a$ is eventually integrated over $\vec{r}_3$ and other coordinates to give $I_{2a}^{(c)}$, the operator $\exp(-\tau\vec{p}_1\cdot\vec{\nabla}_3 + i\tau\vec{k}\cdot\vec{p}_1)$ becomes $e^{i\tau\vec{k}\cdot\vec{p}_1}$. Also the last factor in Eq. (A7) can be written

$$(e^{-i\tau L(13)+i\tau\vec{k}\cdot\vec{p}'} - 1)\rho^*_{kp'}(13)$$
$$= (e^{-i\tau L(13)-i\tau L(2)+i\tau\vec{k}\cdot\vec{p}'} - 1)\rho^*_{kp'}(123). \qquad (A9)$$

After putting this information together, we integrate Eq. (A6) by parts with respect to the operators in $\tilde{\rho}_a$ and thereby obtain the expressions in the text, Eqs. (41) and (43).

Similarly, in the expression for $I_{2b}^{(c)}$ we can integrate over the coordinates of the third particle, and we obtain the desired Eq. (42).

APPENDIX B: DERIVATION OF EQ. (53)

In this Appendix, we indicate the calculation of $\Omega_{\mu\nu}$ appropriate to the Bogoliubov collision kernel. To first order in k, the matrix $\Omega_{\mu\nu}$ and the following

$\omega_{\mu\nu}$ are the same:

$$\omega_{\mu\nu}(\vec{k}) \equiv \int d^3p\, d^3p'\, g_\mu(\vec{p}) [\vec{k}\cdot\vec{p}\,\delta(\vec{p}-\vec{p}') + inI_1(\vec{k};\vec{p}\vec{p}')]\,\phi(\vec{p}')g_\nu(\vec{p}'), \quad \text{(B1)}$$

where the functions g_μ are

$$g_1(\vec{p})=1, \quad g_2(\vec{p})=p_3, \quad g_3(\vec{p})=(\beta p^2-3)/\sqrt{6}. \quad \text{(B2)}$$

In order to see the origins of certain terms in the final result, we split $I_1^{(c)}$ into two parts, the first agreeing with the low-frequency limit of the Mazenko memory function, and the second giving the remainder (asymmetric in $\vec{p}\leftrightarrow\vec{p}'$)

$$iI_1^{(c)}(\vec{k};\vec{p}\vec{p}') = \lim_{z\to i\epsilon} \Sigma_1^{(c)}(\vec{k}z;\vec{p}\vec{p}') + iI_1^{(2)}(\vec{k};\vec{p}\vec{p}'), \quad \text{(B3)}$$

with

$$iI_1^{(2)}(\vec{k};\vec{p}\vec{p}')\,\phi(\vec{p}') = -\lim_{z\to i\epsilon} \vec{k}\cdot\vec{p}'\left(\frac{\beta}{2\pi}\right)^3 \int \frac{d1\,d2}{2V}\, e^{-\beta H(12)} \times \rho_{p'}(12)\, L(12)[z+L(12)]^{-2} \times L_1(12)\,\rho_p(12) + O(k^2). \quad \text{(B4)}$$

Similarly, we can write $\omega_{\mu\nu}=\omega_{\mu\nu}^{(1)}+\omega_{\mu\nu}^{(2)}$, corresponding to matrix elements of $\vec{k}\cdot\vec{p}\,\delta(\vec{p}-\vec{p}') + n\Sigma_1(\vec{k}, i\epsilon; \vec{p}\vec{p}')$ and $inI_1^{(2)}(\vec{k};\vec{p}\vec{p}')$, respectively.

The matrix $\omega_{\mu\nu}^{(1)}$ has been calculated to first order in k and z by Mazenko,[2] and taking $z\to i\epsilon$ in his result we obtain

$$\omega_{\mu\nu}^{(1)} = \frac{k}{\sqrt{\beta}} \times \begin{pmatrix} 0 & 1 & 0 \\ 1-nf(0) & 0 & (\frac{2}{3})^{1/2}(1+n\alpha_1) \\ 0 & (\frac{2}{3})^{1/2}(1+n\alpha_1) & 0 \end{pmatrix}, \quad \text{(B5)}$$

where

$$\alpha_1 = -\tfrac{1}{6}\int d^3r\, e^{-u(r)} u(r)\, \vec{r}\cdot\vec{\nabla}u(r) \quad \text{(B6)}$$

and $u(r)=\beta v(r)$. Turning to $\omega_{\mu\nu}^{(2)}$, we can show that all the elements except one vanish because of the symmetry properties of the integrands. The nonzero element works out to be

$$\omega_{32}^{(2)} = -nk\,(2/3\beta)^{1/2}\alpha_1', \quad \text{(B7)}$$

where

$$\alpha_1' = \tfrac{1}{3}\int d^3r\, e^{-u(r)}[u(r)]^2. \quad \text{(B8)}$$

The method of calculation is very similar to an example worked out in Ref. 2. Thus, in the notation of Eq. (52), we finally have

$$(k/\sqrt{\beta})\,\Omega_{\mu\nu,1}^{B} = \omega_{\mu\nu}^{(1)} + \omega_{\mu\nu}^{(2)}. \quad \text{(B9)}$$

---

*Research supported by the National Research Council of Canada.

[1]G. F. Mazenko, Phys. Rev. A **3**, 2121 (1971).

[2]G. F. Mazenko, Phys. Rev. A **5**, 2545 (1972).

[3]C. D. Boley and R. C Desai, Phys. Rev. A 7, 1700 (1973).

[4]N. N. Bogoliubov, *Problems of a Dynamical Theory in Statistical Physics* (State Technical Press, Moscow, 1946) [transl. in *Studies in Statistical Mechanics,* edited by J. de Boer and G. E. Uhlenbeck (North-Holland, Amsterdam, 1962), Vol. 1].

[5]M. S. Green, J. Chem. Phys. **25**, 836 (1956); Physica (Utr.) **24**, 393 (1958).

[6]E. G. D. Cohen, Physica (Utr.) **28**, 1025 (1962); J. Math. Phys. **4**, 183 (1963).

[7]S. T. Choh and G. E. Uhlenbeck, Navy Theoretical Physics Contract No. Nonr 1224 (15) (1958) (unpublished).

[8]E. G. D. Cohen, Physica (Utr.) **28**, 1045 (1962).

[9]R. A. Piccirelli, J. Math. Phys. **7**, 922 (1966).

[10]K. Kawasaki and I. Oppenheim, Phys. Lett. **11**, 124 (1964).

[11]J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, *Molecular Theory of Gases and Liquids* (Wiley, New York, 1964).

[12]E. G. D. Cohen, in *Lectures in Theoretical Physics*, edited by W. E. Brittin (Gordon and Breach, New York, 1967), Vol. IXC; J. V. Sengers, *ibid*.

[13]M. S. Green and R. A. Piccirelli, Phys. Rev. **132**, 1388 (1963).

[14]Other recent work done in this area includes A. H. Kritz, G. V. Ramanathan, and G. Sandri, Phys. Rev. Lett. **25**, 437 (1970); Phys. Rev. A **6**, 1950 (1972); G. V. Ramanathan and G. Sandri, J. Math. Phys. **10**, 1763 (1969); G. V. Ramanathan, A. H. Kritz, and G. Sandri, Phys. Lett. **31A**, 477 (1970). We have not found a relation between these papers and our work, although the results are not contradictory. The above authors criticize the Bogoliubov assumption for the nonequilibrium two-body distribution function F_2, but find the lowest-order kinetic equation itself (involving an integral of F_2) to be reasonably valid. Our work gives no prediction for F_2 but is always concerned with the kinetic equation itself. Their analysis is carried out only for spatially homogeneous systems, while our most important results concern the effects of inhomogeneity.

[15]J. R. Dorfman and E. G. D. Cohen, Phys. Lett. **16**, 124 (1965); K. Kawasaki and I. Oppenheim, Phys. Rev. **139**, A1763 (1965); J. Weinstock, Phys. Rev. **140**, A460 (1965).

[16]Y. Pomeau, Phys. Rev. A **3**, 1174 (1971).

[17]J. R. Dorfman and E. G. D. Cohen, Phys. Rev. A **6**, 776 (1972).

[18]G. F. Mazenko, Phys. Rev. A 7, 209 (1973); Phys. Rev. A 7, 222 (1973).

[19]D. Forster and P. C. Martin, Phys. Rev. A **2**, 1575 (1970).

[20]R. Zwanzig, Phys. Rev. **144**, 170 (1966).

[21]M. Nelkin and S. Ranganathan, Phys. Rev. **164**, 222 (1967).

[22]A. Z. Akcasu and J. J. Duderstadt, Phys. Rev. **188**, 479 (1969).

[23]C. D. Boley, Phys. Rev. A 5, 986 (1972).

[24]The fact that $\Sigma_1^{(c)}$ and $iI_1^{(c)}$ agree at the point $k=0$, $z=i0^+$ follows from Ref. 2, where it is shown that $\Sigma_1^{(c)}$ reduces to the linearized Boltzmann collision kernel.

[25]K. Kawasaki and I. Oppenheim, Phys. Rev. **136**, A1519 (1964).

[26]M. H. Ernst, Physica (Utr.) **31**, 493 (1965).

[27]B. U. Felderhof and I. Oppenheim, Physica (Utr.) **31**, 1441 (1965); P. A. Selwyn and I. Oppenheim, Physica (Utr.) **54**, 161 (1971).

[28]G. F. Mazenko, T. Y. C. Wei, and S. Yip, Phys. Rev. A **6**, 1981 (1972).