Since no anomalous term was found in λ and (C18) is consistent with the scaling laws $-\alpha=\lambda\gamma/(2-\eta)$ and $2-\alpha=\nu d$, (C23) implies the breakdown of these scaling laws at these dimensions. On the other hand, we have seen earlier that the $m\to\infty$ limit of the above calculation simply implies that there is no singularity in specific heat at these particular dimensions. It is not impossible that (C22) simply indicates a logarithmic singularity rather than a correction to the exponent. Such a logarithmic singularity may be a result of a singularity in coefficients [as (C20) indicates] as functions of d in the infinite-n limit.[18] This point has to be verified or disproved by further study.

Finally, the reader may ask whether a similar anomaly occurs in $\Gamma_{2s}(0)$ (see Sec. VI), since Fig. 5, which gives $\Gamma_{2s}(0)$, does contain the diagrams in Fig. 4 as a part in Fig. 5(b). The answer is that such an anomaly does occur in the terms in Fig. 5(b) and *also* in Fig. 5(c). However, when we add all diagrams in Fig. 5, the anomaly disappears. The scaling law (6.12) can be explicitly verified.

In the case of long-range interactions of the form k^σ, the above discussion goes through in the same fashion. Abe and Hikami also worked on this case.

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Superfluidity in a Ring*

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The persistent flow of a superfluid in a ring is discussed in terms analogous to those previously used for superconductors. The existence of a phase memory around the ring is shown to be responsible for energy minima with a periodic dependence on the total momentum which is directly related to the quantization of circulation. The general features are illustrated by means of the ideal Bose gas and the model of quasiparticles as examples.

An ideal condensed Bose gas in a rotating container exhibits a series of equilibrium states, characteristic of a superfluid,[1] with the successive entry of vortices responsible for their formation.[2] In the case of an arbitrarily interacting Bose system, one is led to the related but more general conclusion that the angular velocity of the system varies periodically with that of the container.[3] The arguments for this conclusion are analogous to those applied in a general interpretation of the Josephson effect[4] and similar arguments will be used here to discuss persistent flow and the quantization of circulation in a container at rest as a counterpart to the discussion of persistent currents and flux quantization[5,6] in a superconducting ring. Simplifying assumptions will be made wherever possible without loss of the most essential features.

In particular, the consideration will be restricted to the uniform flow around a circular ring with the linear dimensions of the cross section small compared to the radius R. Introducing a coordinate x, measured around the ring (so that x/R represents the angle variable) and omitting the explicit dependence on the two additional coordinates required to locate an atom within the cross section, the system of N atoms is described by a symmetric wave function $\Psi(x_s)$ of the N variables x_s with the conjugate momenta p_s ($s=1, \cdots, N$). The interaction with the containing wall shall be neglected for the time being so that the total momentum

$$P=\sum_s p_s \tag{1}$$

(or the total angular momentum PR) is a constant of motion.[7] A solution of the Schrödinger equation

$$\mathcal{H}\Psi=E\Psi \tag{2}$$

can be written in the form

$$\Psi(x_s)=\exp\left[iP\left(\sum_s x_s\right)/(N\hbar)\right]\chi(x_s-x_{s'}) \ , \tag{3}$$

where χ depends only on the differences $x_s-x_{s'}$ ($s, s'=1, \ldots, N$) and satisfies the condition

$$P\chi=0 \ . \tag{4}$$

Due to the fact that the operators

$$p_s=\frac{\hbar}{i}\frac{\partial}{\partial x_s}$$

appear only in the kinetic energy $(1/2m)\sum_s p_s^2$, and in view of Eqs. (1) and (4), the substitution for Ψ of Eq. (3) into Eq. (2) yields the equation

$$\mathcal{H}\chi=e\chi \tag{5}$$

for χ, with the total energy given by

$$E=P^2/2M+e \ , \tag{6}$$

where $M=Nm$ is the total mass of the system.

So far no more than the customary procedure has been followed to separate the motion of the center of gravity $(\sum_s x_s)/N$, whereby the part e of the energy, owing to the relative motion of the atoms, is normally independent of the total momentum P. In order to recognize the difference arising in superfluid systems it is necessary, in addition, to take into account the general condition that Ψ must be single valued and thus must return to the same value when any atom is brought around the ring to its original position. With the notations used in Eq. (3) it is, therefore, required that χ undergo multiplication by the factor

$$f=e^{-2\pi i(PR/N\hbar)} \tag{7}$$

whenever any one of the variables x_s is augmented by the amount $2\pi R$. This requirement codetermines the permissible solutions and eigenvalues of Eq. (5) and thus may lead to a pronounced dependence of e upon P. It also determines the eigenvalues of P since χ depends only upon the differences of the variables x_s and thus does not change if all N of them are augmented by $2\pi R$. On the other hand, this implies the N-fold multiplication of χ with the factor f of Eq. (7) and thus requires that $f^N=1$. According to Eq. (7), it follows that one has to choose

$$P=\nu(\hbar/R) \ , \tag{8}$$

where ν is an integer. Without hindering a practically continuous variation of P, Eq. (8) merely confirms the eigenvalues of the total angular momentum PR to be integer multiples of $\hbar$.

It is essential, however, to note that f and, hence, the conditions to be imposed upon χ remain unaltered if P is augmented by the amount $N\hbar/R$. This means that every eigenvalue e found for a given value of P is likewise an eigenvalue for $P+N\hbar/R$ or, in abbreviated form, that $e(P)$ is a periodic function of P with period $N\hbar/R$. Moreover, since an inversion of the sense of rotation cannot affect the energy E, the same holds for e so that it must be an even function of P.

More explicitly, a stationary state of the system can be characterized by the total momentum P and an additional set of $N-1$ quantum numbers, to be indicated by the symbol n. The corresponding eigenvalue of the energy, given by Eq. (6), is then of the general form

$$E_n(P)=P^2/2M+e_n(P) \ , \tag{9}$$

where

$$e_n(P+N\hbar/R)=e_n(P) \tag{10}$$

and

$$e_n(-P)=e_n(P) \ . \tag{11}$$

The result for $e(P)$ is analogous to that obtained for the energy of a conducting ring as an even periodic function of the magnetic flux, passing through its opening. Just as in the case of superconductivity, there is no need in the derivation to postulate the particular properties of superfluidity; those of the normal state are included as the special case in which e turns out to be independent of P. In fact, no serious error is normally committed if an atom is considered to be localized well within the available macroscopic dimension $2\pi R$.[8] The condition for Ψ to be single valued is then immaterial and does not affect the energy.[6] A dependence of the energy on the total momentum through the part e of Eq. (6), on the other hand, indicates a phase memory which, irrespective of its origin, extends around the whole ring. A connection between this circumstance and the occurrence of persistent flow remains to be established.

To further simplify the discussion, it will be assumed that one deals with the conditions at the absolute zero of temperature. Besides demanding $P=0$, the ground state of the system shall be characterized by the set of additional quantum numbers, symbolized by letting $n=0$. The same symbol shall be chosen to characterize the lowest one among the functions $E_n(P)$ or $e_n(P)$ for a fixed value of their argument so that

$$E_0(P)=P^2/2M+e_0(P) \tag{12}$$

represents the lowest value of the energy for a given total momentum P and the function $e_0(P)$ has a minimum at $P=0$. In view of Eq. (10), this minimum is periodically repeated at intervals $N\hbar/R$ of P. Assuming that $e_0(P)$ has a finite slope as P approaches zero from either side, this function is schematically represented in Fig. 1. Besides being chosen to conform with the general properties of $e_0(P)$ mentioned above, the plot exhibits in full lines some of the particular features characteristic of the case of phonon excitation in a system of interacting particles while the dashed lines are representative of an ideal Bose gas; both will be explained later. Figure 2 represents the corresponding function $E_0(P)$, given in Eq. (12); it likewise exhibits minima when P is an integer multiple of $N\hbar/R$ with an absolute magnitude below a "critical" value P_c, to be discussed below.

In order to interpret the significance of these minima it is necessary to take into consideration the interaction of the system with the container along with the ensuing equilibrium. Starting from an arbitrary initial state, equilibrium is reached by transitions involving an exchange of momentum and energy. With the container held at zero temperature, the system will necessarily be brought

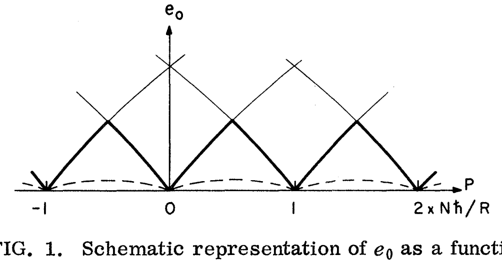

FIG. 1. Schematic representation of e_0 as a function of the total momentum P, assuming finite slope at the periodically repeating minima. The heavy lines are of the character to be expected in the case where the momentum is ascribed to phonons with their prolongations in fine lines indicating the possibility of reaching the same value of P at higher energy by continued phonon excitation. The dashed lines indicate the case of an ideal Bose gas.

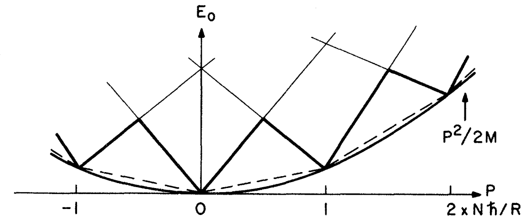

FIG. 2. Schematic representation of E_0 according to Eq. (12) with the part e_0 as shown in Fig. 1. For the heavy lines (phonon case) the slope towards lower values of $|P|$ at successive minima decreases with increasing momentum, thus preventing the occurrence of further minima above a critical value P_c of $|P|$. No persistent (metastable) flow is therefore possible above the critical value $u_c=P_c/M$ of the drift velocity. Referring to an ideal Bose gas, the dashed lines indicate the single trivial minimum at $P=0$ and, hence, the absence of persistent flow in that case.

to the lowest state and hence to a vanishing momentum if, as in the normal case, there are no other minima of E_0. If, however, there are other minima the system can reach one of them through a rapid succession of transitions, each involving an energy loss accompanied by a small transfer of momentum to the container. From then on, only those transitions could occur in which the energy further decreases with a momentum transfer comparable to or larger than the macroscopically large value $N\hbar/R$. Such transitions can be safely considered to be so highly infrequent as to cause a metastability which, in effect, will let the system remain in a state with momentum

$$P_\mu=\mu N\hbar/R\ , \tag{13}$$

where μ is an integer. In view of the general definition $u=P/M$ of the drift velocity and since $M=Nm$, the system therefore exhibits persistent flow with the drift velocity

$$u_\mu=\mu\hbar/mR\ . \tag{14}$$

Furthermore the "circulation" defined by the line integral of u around the ring is given by

$$2\pi R u_\mu=\mu(h/m)\ . \tag{15}$$

Under the circumstances considered here, this confirms its quantization with the "quantum of circulation" h/m, more familiar from superfluid vortex motion. The preceding conclusions remain valid only as long as $|P_\mu|$ does not exceed the critical value P_c mentioned above. In view of Eq. (12), one deals, in fact, no longer with a minimum of E_0 at such large values of P_μ that $P^2/2M$ rises more steeply than e_0, since it allows a decrease of the energy towards the low-momentum side. Expressed in terms of the derivatives and with $e_0'(P_\mu)=e_0'(0)$, persistent flow at the drift velocity

u_μ therefore requires

$$|P_\mu|/M = |u_\mu| < |e_0'(0)|,$$

so that the "critical momentum" is given by $P_c = M\,|e_0'(0)|$ or that $u_c = |e_0'(0)|$ represents a critical velocity.

The preceding considerations demonstrate the possibility to derive some of the salient features of superfluidity from basic principles. While they yield conclusions about the general character of the underlying function $e_0(P)$, more specific assumptions are required to gain additional insights into its behavior. The following remarks shall serve to illustrate this point by means of a few examples.

As a first example, the case of an ideal Bose gas will be considered. In the interval $0 \le P \le N\hbar/R$ the lowest energy for a given value $P = \nu(\hbar/R)$ is obtained by assigning the momentum $p = \hbar/R$ to ν atoms and zero momentum to the remaining $(N-\nu)$ atoms. Thus $E_0(P) = \nu\hbar^2/2mR^2 = P\hbar/2mR$, and with $M = Nm$ from Eq. (12),

$$e_0(P) = \frac{P\hbar}{2mR}\left(1 - \frac{PR}{N\hbar}\right). \tag{16}$$

Extended to other intervals, the corresponding function is indicated by the dashed parabolic arcs in Fig. 1. When added to $P^2/2M$ one finds in the interval $P_\mu \le P \le P_{\mu+1}$

$$E_0(P) = \frac{P_\mu^{\,2}}{2M} + \frac{\Delta P\hbar}{2mR}(2\mu+1), \tag{17}$$

with $\Delta P = P - P_\mu$ and P_μ given by Eq. (13). The straight segments which connect the points on the parabola $P^2/2M$ for $P = P_\mu$ and $P = P_{\mu+1}$ represent this function by the dashed lines in Fig. 2. Since it has no other minimum than that at $P = 0$, one is led to the conclusion that the ideal Bose gas does not exhibit persistent flow in a container at rest. Another way of reaching this conclusion is to note from Eq. (16) that $|e_0'(0)| = \hbar/2mR$. With $u_\mu = \mu\hbar/mR$ from Eq. (14), the requirement $|u_\mu| < |e_0'(0)|$ for the drift velocity imposes the condition $|\mu| < \frac{1}{2}$ upon the integer μ, thus demanding that $\mu = 0$ and excluding any finite stable value of u. The preceding result may seem to be incompatible with the property of the equilibrium states in a rotating container, mentioned in the beginning, to allow stable motion of an ideal Bose gas relative to the container.[9] It is essential, however, that this property derives from a uniform rotation, rather than a translation, of the container so that the invariance of the relative velocity against uniform motion of the system of reference cannot be invoked in concluding upon the case of a container at rest considered here.[10]

Whereas the model of the ideal gas is thus inadequate to explain persistent flow in a container at rest, it is sufficient for our purposes to effectively account for the required interaction of the atoms through their mere replacement by "quasiparticles" of momentum p and energy $\eta(p)$. For $p \ll \hbar/\delta$ where δ is the mean distance between atoms, they can be identified as phonons so that $\eta(p) = c\,|p|$ and where c represents the sound velocity. Phonon excitation from the ground state then leads to a total momentum with the least energy required if it is the resultant of phonon momenta pointing all in the same direction. In the vicinity of the ground state one therefore has $E_0 = c\,|P|$ as indicated by the heavy lines in Fig. 2; continued excitation in the same manner leads to the extension of this function, shown in thin lines, but lower energies are here attainable by excitation from the adjacent minimum. The corresponding function $e_0(P)$ from Eq. (12), schematically represented in Fig. 1, has the initial slope $|e_0'(0)| = c$ so that the critical velocity u_c, mentioned above, is given here by the sound velocity. Depending on the behavior of $\eta(p)$ for larger values of p, this does not necessarily mean, however, that stationary drift velocities can go as high as the sound velocity. Although E_0 still has minima for $P = P_\mu$, it is possible even for $|u_\mu| < c$ that lower energies can be reached by transitions which only require a microscopic momentum transfer to the wall and are hence by no means expected to be infrequent. In particular, this occurs with the customary nonmonotonic form of $\eta(p)$, exhibiting a "roton" minimum for $p \cong \hbar/\delta$, and leads in the usual manner to the somewhat lower value of the critical velocity, required to render the above-mentioned transitions likewise excluded for energetic reasons.

There are no general reasons for appreciable minima of $e_0(P)$ to occur in the interval $0 < P < N\hbar/R$, which would invalidate the previous conclusions. A notable exception occurs, however, if one considers He_3 instead of He_4, thus going over from a boson to a fermion system. In analogy to the fermion system of electrons, responsible for superconductivity, persistent flow requires here a pairing of atoms, effectively taken into account by replacing the total number N of atoms by the number $\frac{1}{2}N$ of pairs, thus reducing the period of $e_0(P)$ to half its former value. For the same reason the mass m of a single atom has to be replaced by $2m$ so that, analogous to the flux quantum for superconductors, the quantum of circulation is likewise reduced to half its former value.

Although generally more intricate, the situation for solutions of He_3 in He_4 stays simple as long as they are sufficiently dilute for He_3 to be in the normal state and thus to remain at rest in the equilibrium reached through momentum transfer to the wall. In fact, there is in this case no essential difference between the effect of the wall and of the atoms of He_3 upon those of He_4, so that the persistent flow of the latter is not affected by the presence of the former.

Further modifications appear if one abandons the assumption of zero temperature but they remain comparatively minor if the temperature T of the wall is assumed to be so low that it causes the system to be found with an appreciable probability only at energies close to a specific minimum of E_0. The essential difference from the case of zero temperature consists of the replacement of $e_0(P)$ by

$$f(P) = -kT \ln \sum_n \exp(-e_n/kT) \ , \qquad (18)$$

where $f(P)$ is likewise an even function of P with period $N\hbar/R$, and in the corresponding replacement, of $E_0(P)$ in Eq. (12) by

$$F(P) = P^2/2M + f(P) \ . \qquad (19)$$

At the assumed low temperature $f(P)$ will not differ appreciably from $e_0(P)$ except that the discontinuous slope for $T=0$ is replaced by a quadratic minimum. Correspondingly, the minima of $F(P)$ appear at values of $|P|$ slightly below those of Eq. (13), to be denoted by

$$\overline{P}_\mu = P_\mu - (\Delta P)_\mu \ , \qquad (20)$$

with $(\Delta P)_\mu$ likewise proportional to μ. It is possible, therefore, to ascribe persistent flow to a "superfluid part" of the liquid with density ρ_s, moving with a finite velocity $u_\mu = P_\mu/M$, while the "normal part" with density $\rho_n = \rho - \rho_s$ remains at rest and to write Eq. (20) in the form

$$\overline{P}_\mu = P_\mu(\rho_s/\rho) \ , \qquad (21)$$

which can be regarded as a definition of the superfluid fraction (ρ_s/ρ), with the property to approach unity as the temperature approaches the absolute zero.[11]

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[7]The symbol P in Eqs. (1) and (4) denotes the operator, but is used elsewhere to denote an eigenvalue of the total momentum.

[8]This is obviously justified in the case of a crystal, but it remains likewise justified, for example, in the case of a nondegenerate gas, since one deals here with a sufficiently smooth and extended momentum distribution to permit the replacement of plane waves by wave packets, extending over far less than macroscopic dimensions.

[9]Contrary to a remark in Ref. 2, one can hardly expect, upon suddenly stopping the container, to observe a continued metastable motion of the gas. Since it is not prevented by conservation of energy, a succession of transitions, owing to interaction with the wall at rest, should be expected to rapidly bring the individual atoms into their state of zero momentum even if they all started from the same finite momentum.

[10]The difference is made obvious by slightly extending the previous considerations to the combined system, obtained by inclusion of the container as a rigidly moving body. With the same notations as before and with the minimum condition of E_0 from Eq. (12) replaced by that of the combined energy for a given combined momentum, one finds the more general relation $u + e_0'(P) = v$, where v is the velocity of the container. It leads to the previous conclusions for $v=0$ and also confirms the drift velocity u to be equal to v for equilibrium in the normal case where e_0 is independent of P but has other consequences if both v and e_0' are finite. In particular, for the ideal Bose gas with e_0 from Eq. (16), one finds that a change of v causes $u - v$ to alter periodically and discontinuously between the values $-(\hbar/2mR)$ and $+(\hbar/2mR)$ as noted in Refs. 1–3. Both, however, vanish in the limit $R \to \infty$ and thus prohibit a finite drift velocity relative to the container, in conformity with the result for a container at rest and the circumstance that rotation and translation are in this limit indistinguishable. On the other hand, for systems which allow persistent flow in a container at rest, a rotation will produce different effects analogous to those of an external flux, discussed in Ref. 4, when its absence does not preclude the existence of persistent currents in a superconductor.

[11]The severely limited temperature range assumed here implies that $(\Delta P)_\mu \ll N\hbar/R$ and $\rho_s/\rho \cong 1$, but it can be extended to higher temperature by means of additional considerations. Adhering to the model of quasiparticles, transitions of states reached by excitation from one minimum of E_0 to states reached from another minimum must then be recognized, even with a microscopic momentum transfer to the wall, as being highly infrequent. With the summation in Eq. (18) thus to be extended over only one such group of states, one deals, in effect, with a manifold of function $f_\mu(P)$ which, although individually not periodic, repeat each other identically upon a shift of P by integer amounts of $N\hbar/R$. For $T=0$ they assume the form schematically indicated by the continuations in thin lines of Fig. 1. Going to higher temperatures, a quadratic minimum not only replaces the discontinuous slope, but is allowed to become much flatter than previously implied. The minima of the corresponding function $F_\mu(P) = P^2/2M + f_\mu(P)$ can then appear at values $\overline{P}_\mu$ well below P_μ but still proportional to μ, thus extending the range of the ratio ρ_s/ρ in Eq. (21) even down to its vanishing point at the onset of the normal state.