*Research supported in part by the University of Kansas General Research Fund under Grant No. 3235-5038.
†A brief account of some of this work has been published in Phys. Lett. A **37**, 49 (1971).
[1]A. Bijl, Physica (Utr.) **7**, 869 (1940).
[2]R. B. Dingle, Philos. Mag. **40**, 573 (1949).
[3]R. Jastrow, Phys. Rev. **98**, 1479 (1955).
[4]W. E. Massey, Phys. Rev. **151**, 153 (1966).
[5]W. E. Massey and C.-W. Woo, Phys. Rev. **164**, 256 (1967).
[6]W. L. McMillan, Phys. Rev. 138, A442 (1965).
[7]D. Schiff and L. Verlet, Phys. Rev. **160**, 208 (1967).
[8]E. Feenberg, *Theory of Quantum Fluids* (Academic, New York, 1969).
[9]D. K. Lee and E. Feenberg, Phys. Rev. 137, A731 (1965).
[10]D. K. Lee, Phys. Rev. **187**, 326 (1969).
[11]N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).
[12]R. P. Feynman, Phys. Rev. **94**, 262 (1954).
[13]T. B. Davison and E. Feenberg, Ann. Phys. (N.Y.) **53**, 559 (1969).
[14]H. W. Jackson and E. Feenberg, Ann. Phys. (N.Y.) **15**, 266 (1961).
[15]C. E. Campbell and E. Feenberg, Phys. Rev. **188**, 396 (1969).
[16]D. K. Lee, Phys. Rev. A **4**, 1670 (1971).
[17]P. Berdahl and D. K. Lee, Phys. Rev. A 7, 1376 (1973).
[18]K. A. Brueckner, Phys. Rev. **156**, 204 (1967).
[19]N. N. Bogoliubov and D. N. Zubarev, Zh. Eksp. Teor. Fiz. **28**, 129 (1955) [Sov. Phys.-JETP **1**, 83 (1955)].
[20]H.-K. Sim, C.-W. Woo, and J. R. Buchler, Phys. Rev. A **2**, 2024 (1970).
[21]H.-K. Sim and C.-W. Woo, Phys. Rev. A **2**, 2032 (1970).
[22]K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).
[23]T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).
[24]T. T. Wu, Phys. Rev. **115**, 1390 (1959).
[25]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), Chap. 5.
[26]R. B. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966), Chap. 9.
[27]W. Gröbner and N. Hofreiter, *Integraltafel* (Springer-Verlag, Berlin, 1957).
[28]D. K. Lee, Phys. Rev. A **3**, 345 (1971).
[29]D. K. Lee, Phys. Lett. A 42, 1 (1972).

PHYSICAL REVIEW A VOLUME 7, NUMBER 6 JUNE 1973

Critical Exponents above T_c to $O(1/n)$

Shang-keng Ma*
Department of Physics and Institute for Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037

(Received 25 September 1972)

Critical exponents are functions of d, the dimension, and n, the number of components of the order parameter involved in the second-order phase transition. They can be expanded in power series of $1/n$ when n is large. We present details of calculating by perturbation theory the critical exponents above T_c for arbitrary d and to $O(1/n)$ for short-range interacting systems and also for long-range interacting systems.

I. INTRODUCTION

Consider a thermodynamic system which has a critical point at temperature T_c. Let the system be d dimensional and let the order parameter have n equivalent components, e.g., $n=1$, 2, 3 for the Ising model, liquid-helium 4 (scalar Bose system), and the Heisenberg model, respectively. As observed by Wilson,[1] if n is very large, it is possible to calculate critical exponents as expansions in $1/n$ by perturbation methods. In this paper, we discuss in detail the calculation of the critical exponents above T_c to $O(1/n)$ for systems with short-range interactions, for those with long-range interactions, and also briefly for the charged systems (Coulomb interaction). The results of this paper have been reported previously by Ma[2] and by Fisher, Ma, and Nickel.[3] Independent calculations by entirely different methods have been performed by Abe and Hikami[4] and by Suzuki.[5] The screening approximation of Ferrell and Scalapino[6] produced exponents to $O(1/n)$ for $d=3$. Investigations on equations of state and exponents below T_c have been done by Brezin and Wallace[7] and by Suzuki.[5]

The basic idea of our method of calculation is the same as that of the ϵ expansion[8,9] ($\epsilon = 4-d$). Let us illustrate it briefly. Denote the Fourier transform (of wave number k) of the order-parameter correlation function by $G(k)$. Suppose that for small k

$$G(k) \sim k^{-2+\eta}[1+R(k)] \quad , \tag{1.1}$$

where $R(k) \to 0$ as $k \to 0$. Equation (1.1) defines the exponent η. If η can be expanded as a series in $1/n$,

$$\eta = a_1 n^{-1} + a_2 n^{-2} + \cdots \quad , \tag{1.2}$$

then the expansion of $G(k)k^2$ in powers of $1/n$ will include a series in powers of $\ln k$. The $\ln k$ series must exponentiate as implied by (1.1):

$$G(k)k^2 = (\text{const})\left(1 + \eta \ln k + \frac{\eta^2}{2}\ln^2 k + \frac{\eta^3}{3!}\ln^3 k + \cdots + \text{terms vanishing as } k \to 0\right) . \tag{1.3}$$

In view of the universality hypothesis, critical exponents depend only on d and n but not on the details of the interaction (assuming short-range interaction for the moment). Therefore, as far as computing the critical exponent goes, one is free to choose any interaction he pleases as long as the computation can be done expediently. If we choose a point interaction of strength of $O(1/n)$, then it becomes straightforward by the usual perturbation theory to compute $G(k)k^2$ as a series in $1/n$. Furthermore, by choosing the appropriate interaction strength, it turns out that the series in $\ln k$ which exponentiates to give k^η in (1.3) can be singled out order by order in $1/n$, and thereby one determines η as a series in $1/n$. For the calculation to $O(1/n)$ described here, as we shall see, no special choice of the interaction is needed except that it must be of $O(1/n)$. The calculation of η to $O(1/n)$ is accomplished by summing diagrams for $G(k)k^2$ to $O(1/n)$ and computing the coefficient for $\ln k$. Other exponents are computed in the same way.

What is presented in this paper is only the algebra leading from the above idea to the various exponents to $O(1/n)$. No attempt is made here to construct a rigorous formalism to serve as a mathematical foundation. Many points which must be explained for general calculations turn out to be irrelevant for $O(1/n)$ calculation and will not be mentioned here owing to the length of this paper and to our lack of experience in higher-order calculations. One motivation of presenting the "tedious but straightforward algebra," here, is to show explicitly how things turn out so neatly and thereby give hints on the physics of critical phenomena and the structure of simple field theory. This paper should serve as an introduction to some of the technical aspects of the new approaches to critical phenomena. It should also provide some convenient information for higher-order calculations in the future.

The results here are valid for arbitrary d. Therefore, they may be expanded in powers of $\epsilon = 4-d$ and comparison with the known ϵ-expansion results[3,8,9] can be made. Such a comparison is easily made and has shown complete agreement to the orders calculated so far. In the absence of higher-order results, such agreement is very encouraging. When taken together, both the ϵ expansion and the $1/n$ expansion are on firmer ground than either of them alone would be.

The outline of this paper is the following. In Sec. II, we define the models and outline the procedure of calculation. We summarize the results from direct perturbation-theory calculation of four exponents for short-range interacting systems, two exponents for long-range (k^σ-type amplitude-amplitude) interacting system, and also results on systems with Coulomb interaction. Sections III–VI are devoted to the study of short-range interacting systems. The long-range case is considered in Sec. VII, and the charged system is considered in Sec. VIII. Concluding remarks are given in Sec. IX. Appendix A is devoted to the calculation of one diagrammatic element in the case of long-range interaction. Appendix B outlines the calculation of the density-correlation function at T_c to $O(\epsilon^2)$, where $\epsilon = 4-d$. Appendix C is devoted to the calculation of the specific-heat exponent α to illustrate the anomalous terms recently reported by Abe and Hikami.[4]

To aid reading, Table I locates the definitions of frequently occurring symbols in this paper.

II. BASIC FORMALISM AND SUMMARY OF RESULTS

A. Definitions and Models

Let us choose a model that is convenient for our purpose. For counting diagrams, we find it easier to use a model of a *complex* classical field of m components. Such a model is equivalent to an n-component *real* field, i.e., a field of

$$n = 2m \tag{2.1}$$

independent and equivalent components. We shall always use the wave-vector representation and denote the field amplitudes as $a_{j\vec{k}}$, $j = 1, 2, \ldots, m$. The wave vector $\vec{k}$ is limited in magnitude to less than a cutoff value which we arbitrarily have taken as 1. Let us define the "density fluctuation" as

$$\rho_{\vec{k}} = \sum_{\vec{p}} \sum_{j=1}^{m} a^*_{j\vec{p}} a_{j\vec{p}+\vec{k}} \ . \tag{2.2}$$

The Boltzmann factor for statistical averaging has the form $e^{-\mathcal{H}}$ and (a) for a short-range interaction

$$\mathcal{H} = \sum_{\vec{k}j} [(r_0 + k^2) a^*_{j\vec{k}} a_{j\vec{k}} + u \rho_{\vec{k}} \rho^*_{\vec{k}}] \ , \tag{2.3}$$

(b) for a long-range interaction

$$\mathcal{H} = \sum_{\vec{k}j} [(r_0 + k^\sigma) a^*_{j\vec{k}} a_{j\vec{k}} + u \rho_{\vec{k}} \rho^*_{\vec{k}}] \ , \quad \sigma < 2 \tag{2.4}$$

and (c) for a Coulomb interaction (charged system)

$$\mathcal{H} = \sum_{\vec{k} \neq 0, j} [(r_0 + k^2) a^*_{j\vec{k}} a_{j\vec{k}} + e^2 \rho_{\vec{k}} \rho^*_{\vec{k}} / k^2] \ . \tag{2.5}$$

TABLE I. Frequently occurring symbols.

Symbol	Defined by (Eq. No.)	Symbol	Defined by (Eq. No.)
$m = \frac{1}{2}n$	(2.1)	S_d	(2.22)
G	(2.6)	K	(3.7)
χ	(2.7)	$\pi(1)$	(3.8)
λ	(2.9)	J	(3.10)
μ	(2.13)	J_σ	(7.2)
r	(2.15)		

Within a small temperature range near T_c, r_0 effectively measures the temperature. We shall use r_{0c} to denote the value of r_0 at T_c.

The statistical averages of interest are the correlation functions

$$G(k) = \langle a_{j\vec{k}} a^*_{j\vec{k}} \rangle \text{ (independent of } j) \ , \qquad (2.6)$$

$$\chi(k) = \langle \rho_{\vec{k}} \rho^*_{\vec{k}} \rangle \ . \qquad (2.7)$$

The critical exponents are defined by the form of these correlation functions for small k at T_c or for small $r_0 - r_{0c}$ at zero k. We have

$$G(k) \sim k^{-2+\eta} \ , \qquad (2.8)$$

$$\chi(k) \sim (\text{const}) - k^{\lambda} \qquad (2.9)$$

at T_c. These define the exponents η and λ. Above T_c, we have

$$G(0) \sim (r_0 - r_{0c})^{-\gamma} \ , \qquad (2.10)$$

$$\chi(0) \sim \text{const} - (r_0 - r_{0c})^{-\alpha} \ , \qquad (2.11)$$

which define γ and α. The exponent $-\alpha$ also describes the singular part of the specific heat C_p. We shall also be interested in the function

$$\Gamma_{2s}(k) = \langle a_{j\vec{k}/2} a^*_{j\vec{k}/2} \rho^*_{\vec{k}} \rangle / G(\tfrac{1}{2}k)^2 \ . \qquad (2.12)$$

At T_c, we define μ by

$$\Gamma_{2s}(k) \sim k^{\mu} \ . \qquad (2.13)$$

B. Perturbation Theory

We shall calculate the exponents by extracting logarithmic terms in the perturbation expansions of the correlation functions. We are dealing with a classical field and there is no complication of quantum mechanics. The rules for generating diagrams and writing down terms therefrom are very simply derived from the expansion

$$e^{-\mathcal{H}} \equiv e^{-(\mathcal{H}_0 + \mathcal{H}_I)} = e^{-\mathcal{H}_0} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \mathcal{H}_I^n \ , \qquad (2.14)$$

where $\mathcal{H}_0$ is the part of $\mathcal{H}$ which is quadratic in the field amplitudes. We do want to rearrange the perturbation series a bit. Let us define the quantity r and the self-energy Σ by

$$r = G^{-1}(0) \ , \qquad (2.15)$$

$$G^{-1}(k) = r_0 + k^2 + \Sigma(k) \ . \qquad (2.16)$$

The perturbation series generated by (2.14) would have $(r_0 + k^2)^{-1}$ as the free-field propagator [k^2 should be replaced by k^{σ} when (2.4) is used]. We now write [from Eqs. (2.15) and (2.16)]

$$G^{-1}(k) = r + k^2 + \Sigma(k) - \Sigma(0) \qquad (2.17)$$

and use $(r + k^2)^{-1}$ as the free-field propagator. Meanwhile, $\Sigma(0)$ is subtracted out from $\Sigma(k)$ when self-energy insertions are made to propagators. The obvious advantage of this choice of free propagator is that, for our calculation, r is small and becomes zero at T_c, whereas r_0 does not have such a desirable quality.

Drawing diagrams and writing down the terms for correlation functions are easy but the evaluation of the terms is nontrivial and is the main content of the following sections.

C. Summary of Results

Case (a): Short-range interaction. We computed four exponents by perturbation theory:

$$\eta = 4(4/d - 1) S_d n^{-1} + O(n^{-2}) \ , \qquad (2.18)$$

$$\gamma = (\tfrac{1}{2}d - 1)^{-1}(1 - 6 S_d n^{-1}) + O(n^{-2}) \ , \qquad (2.19)$$

$$\lambda = 4 - d - 16(d-2)(d-1) d^{-1} S_d n^{-1} + O(n^{-2}) \ , \qquad (2.20)$$

$$\mu = 4 - d - 4(8/d - 7 + 2d) S_d n^{-1} + O(n^{-2}) \ , \qquad (2.21)$$

where

$$S_d \equiv \frac{\sin\pi(\tfrac{1}{2}d - 1)}{\pi(\tfrac{1}{2}d - 1) B(\tfrac{1}{2}d - 1, \tfrac{1}{2}d - 1)} \ , \qquad (2.22)$$

where B is the β function. These exponents are defined by (2.8), (2.10), (2.9), and (2.13), respectively.

The above values for η, γ, and μ are found to satisfy the scaling law

$$\mu = (2 - \eta)(1 - \gamma^{-1}) \ .$$

We have not computed the exponent α directly from perturbation theory, but it can be obtained from λ if we *assume* the scaling law

$$-\alpha = \lambda\gamma/(2 - \eta) \ . \qquad (2.23)$$

Substituting (2.20) for λ, we find

$$\alpha = -(4-d)(d-2)^{-1}[1 + 8(1-d)(4-d)^{-1} S_d n^{-1}] + O(n^{-2}) . \qquad (2.24)$$

This expression is verified to be consistent with the scaling law

$$2 - \alpha = \nu d \ , \qquad (2.25)$$

where $\nu = \gamma/(2 - \eta)$. In Appendix B, we also give the calculation for λ to $O(\epsilon^2)$ ($\epsilon \equiv 4 - d$). The result is

$$\lambda = \epsilon \frac{n-4}{n+8} + \epsilon^2 \frac{2(n+2)}{(n+8)^2} \left(\frac{13}{2} - \frac{30}{n+8} \right) + O(\epsilon^3) \ , \qquad (2.26)$$

which is exact in n. Calculations of η, γ, and α to $O(\epsilon^2)$ and $O(\epsilon^3)$ and exact in n have been performed by Wilson[8] and by Nickel.[9] All results are so far consistent. However, Abe and Hikami[4] have pointed out recently that for some particular values of d there are additional terms in α, and thereby (2.23) and (2.25) are violated at these dimensions (see Appendix C for details).

Case (b): Long-range interaction k^{σ}, $\sigma < 2$, $\sigma < d < 2\sigma$. Two exponents have been computed by perturbation theory:

$$\eta = 2 - \sigma + O(n^{-2}) \ , \qquad (2.27)$$

$$\gamma = \left(\frac{d}{\sigma} - 1\right)^{-1} \left\{1 + 2K\left(\frac{I(-1,0)}{\Pi(0,1)} - 1\right) \times\left[\sigma\left(\frac{d}{\sigma} - 1\right)\Pi(0,1)\right]^{-1} n^{-1}\right\} + O(n^{-2}) \ , \qquad (2.28)$$

where

$$I(-1,0) = 2^{-d}\pi^{-d/2} \times \frac{\Gamma(\frac{1}{2}d-\sigma)\Gamma(\frac{1}{2}(d-\sigma))\Gamma(-\frac{1}{2}d+\frac{3}{2}\sigma)}{\Gamma(d-\frac{3}{2}\sigma)\Gamma(\frac{1}{2}\sigma)\Gamma(\sigma)} \ , \qquad (2.29)$$

$$\Pi(0,1) = 2^{-d}\pi^{-d/2}B(\tfrac{1}{2}(d-\sigma), \tfrac{1}{2}(d-\sigma)) \times\Gamma(-\tfrac{1}{2}d+\sigma)\Gamma(\tfrac{1}{2}\sigma)^{-2} \ , \qquad (2.30)$$

B is the β function, and $K = 2^{-d+1}\pi^{-d/2}\Gamma(\frac{1}{2}d)^{-1}$. Note that (2.27) and (2.28) do not reduce to (2.18) and (2.19) when σ is set equal to 2. In other words these exponents are not continuous in σ at $\sigma = 2$. Other exponents have not been computed here. Expressions to $O(\epsilon^2)$ and exact in n can be found in Ref. 3. Note that *in this case*, $\epsilon \equiv 2\sigma - d$ (not $4-d$). For $d > 2\sigma$, $\eta = 2 - \sigma$, $\gamma = 1$ for all n (see Ref. 3).

Case (c): Coulomb interaction. The exponents γ and η are found to be the same as those for the case of the short-range interaction. Owing to the infinite range of the Coulomb force, the density fluctuation behaves in a peculiar way, namely,

$$\chi(k) \sim k^2 \ .$$

As has been usually done in discussing uniform charged systems, we define an "irreducible density-correlation function" χ' by

$$\chi = \chi'(1 + e^2\chi'/k^2)^{-1} \ . \qquad (2.31)$$

We find, at T_c

$$\chi'(k) \sim \text{const} - k^{\lambda} \quad \text{if } \lambda > 0 \qquad (2.32)$$

$$\sim -k^{|\lambda|} \quad \text{if } \lambda < 0 \ , \qquad (2.33)$$

and

$$\chi'(0) \sim (\text{const}) - (r_0 - r_{0c})^{-\alpha} \quad \text{if } -\alpha > 0 \qquad (2.34)$$

$$\sim -(r_0 - r_{0c})^{|\alpha|} \quad \text{if } -\alpha < 0 \ , \qquad (2.35)$$

where λ and α are given by (2.20), (2.24), and (2.26).

The results (2.18)–(2.20) and those for the charged system have been previously reported in Ref. 2. The results (2.27)–(2.30) have been reported in Ref. 3. The result (2.21) has not been reported before.

III. EVALUATION OF η

To illustrate the general procedure and technique, we begin with the evaluation of η, which is the simplest.

The Green's function $G(k)$ at T_c behaves like $k^{-2+\eta}$. In terms of the self-energy $\Sigma(k)$, we write

$$G^{-1}(k) = k^2 + \Sigma(k) - \Sigma(0) \ . \qquad (3.1)$$

In this section, $T = T_c$, i.e., $r = 0$, is understood. The leading term in $\Sigma(k) - \Sigma(0)$ is the third diagram in Fig. 1. (The first and the second are momentum independent.) The wavy line represents the geometric sum shown in Fig. 2:

$$-u + (-u)^2 m\Pi + (-u)^3(m\Pi)^2 + \cdots = -u(1 + mu\Pi)^{-1} \ . \qquad (3.2)$$

The summation of these diagrams is necessary and sufficient to $O(1/m)$ because $u = O(1/m)$ and each closed loop implies a factor m. The function Π is the "elementary bubble" making up the strings in Fig. 2, i.e.,

$$\Pi(k) = (2\pi)^{-d}\int d^dp\, p^{-2}(\vec{p}+\vec{k})^{-2} \ . \qquad (3.3)$$

One of the many ways to evaluate this integral is to combine the denominators using the formula

$$A^{-1}B^{-1} = \int_0^1 d\alpha\, [(1-\alpha)A + \alpha B]^{-2} \qquad (3.4)$$

and then to perform the p integral by changing variable to $p' = p + \alpha k$:

$$\Pi(k) = \int_0^1 d\alpha \int d^dp'\,(2\pi)^{-d}[p'^2 + \alpha(1-\alpha)k^2]^{-2} \ . \qquad (3.5)$$

Here we can take the cutoff of the p' integral to infinity and write

$$(2\pi)^{-d}\int d^dp' = K\int_0^\infty p'^{d-1}dp' \ , \qquad (3.6)$$

$$K \equiv 2^{-d+1}\pi^{-d/2}/\Gamma(\tfrac{1}{2}d) \ ; \qquad (3.7)$$

we can evaluate $\Pi(k)$ easily and obtain

$$\Pi(k) = k^{d-4}B(\tfrac{1}{2}d-1, \tfrac{1}{2}d-1)J = k^{d-4}\Pi(1) \ , \qquad (3.8)$$

where B is the β function

$$B(x,y) = \int_0^1 d\alpha\, \alpha^{x-1}(1-\alpha)^{y-1} \qquad (3.9)$$

and

$$J \equiv \tfrac{1}{2}K\pi(\tfrac{1}{2}d-1)\csc\pi(\tfrac{1}{2}d-1) \ . \qquad (3.10)$$

Thus, to $O(1/m)$, we have

$$\Sigma(k) - \Sigma(0) = +(2\pi)^{-d}\int d^dp\, u[1 + mu\Pi(p)]^{-1} \times[(\vec{p}+\vec{k})^{-2} - p^{-2}] \qquad (3.11)$$

following Fig. 1(c). According to (3.8), $\Pi(p)$ blows up as $p \to 0$. For small k, Eq. (3.11) behaves like

$$\sim m^{-1}k^2(\ln k + \text{const}) \ , \qquad (3.12)$$

where the $k^2\ln k$ term comes from the small-p region of the integration. Clearly, for $m \to \infty$, (3.11) goes to zero and $G^{-1} = k^2$ so that $\eta = 0$.

For large but finite m, η is small so that

$$G^{-1} \sim k^{2-\eta} \sim k^2(1 - \eta\ln k + \cdots) \ . \qquad (3.13)$$

According to (3.12) and (3.13), η is therefore determined to $O(1/m)$ by the coefficient of $k^2 \ln k$ term in (3.11).

To determine the $k^2 \ln k$ term, it is sufficient to consider the small-p region of integration in (3.11). Since $\Pi(p)$ is very large, $1+um\Pi(p)$ may be replaced by $um\Pi(p)$. Equation (3.11) may now be written, for our purpose,

$$\Sigma(k)-\Sigma(0)=m^{-1}(2\pi)^{-d}\int d^dp\,\Pi(p)^{-1}[(\vec{p}+\vec{k})^{-2}-p^{-2}]$$
$$=[m\Pi(1)]^{-1}\int d^dp\,p^{4-d}[(\vec{p}+\vec{k})^{-2}-p^{2}] \;, \tag{3.14}$$

where p is restricted to a finite region around the origin. We now write

$$(\vec{p}+\vec{k})^{-2}=(p^2+k^2+2pk\cos\theta)^{-1} \;. \tag{3.15}$$

The angular integral over θ is affected by writing

$$(2\pi)^{-d}\int d^dp=\frac{K_{d-1}}{2\pi}\int P^{d-1}\,dp\int_0^{\pi}\sin^{d-2}\theta\,d\theta \;, \tag{3.16}$$

where K_{d-1} is given by (3.7) with d replaced by $d-1$. Now the integral (3.14) is elementary and the coefficient of the $k^2 \ln k$ term is easily worked out using an elementary integral table.[10] We find

$$\eta=2(4/d-1)m^{-1}S_d+O(m^{-2}) \;, \tag{3.17}$$

where S_d is given by (2.22). This result is meaningful only for $2<d<4$. For $d=4$ or larger, $\Pi(p)$ does not blow up and (3.14) no longer holds. For $d\le 2$, one easily verifies that (3.14) does not converge. In fact, there would be no $\ln k$ term for $d>4$, and therefore $\eta=0$. Nothing definite can be said here for $d\le 2$.

IV. EVALUATION OF γ

We now examine the Green's function at zero momentum (i.e., the static susceptibility) slightly above T_c. For $k=0$, we have

$$G^{-1}=r=r_0+\Sigma(r)=r_0-r_{0c}+\Sigma(r)-\Sigma(0) \;, \tag{4.1}$$

where $k=0$ is understood in the self-energy. The quantity r_{0c} is defined by $r_{0c}+\Sigma(0)=0$, i.e., the value of r_0 at the critical point. Near T_c, we have $r\sim(r_0-r_{0c})^{\gamma}$, which defines the index γ, and thus

$$r+\Sigma(0)-\Sigma(r)=r_0-r_{0c}\sim r^{1/\gamma} \;. \tag{4.2}$$

The lowest-order term in $1/m$ for $\Sigma(r)-\Sigma(0)$ comes from the first diagram in Fig. 1. This

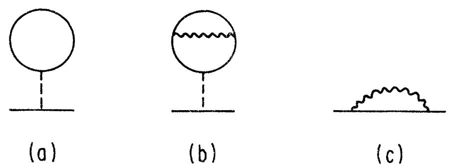

FIG. 1. Self-energy diagrams.

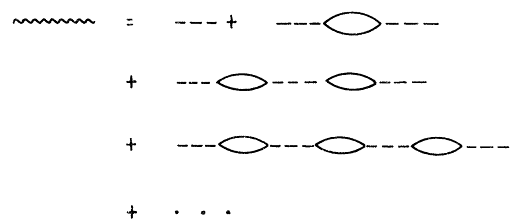

FIG. 2. Definition of a wavy line.

diagram is of the zeroth order; i.e., it does not vanish when $m\to\infty$. It gives

$$\Sigma_a(r)-\Sigma_a(0)=mu(2\pi)^{-d}\int d^dp\,[(r+p^2)^{-1}-p^{-2}]+O(m^{-1})$$
$$=-mur^{d/2-1}J/(\tfrac{1}{2}d-1) \;, \tag{4.3}$$

where J is given by (3.10). The integral is trivial and well defined for $2<d<4$. Comparing (4.2) and (4.3), we have

$$1/\gamma=\tfrac{1}{2}d-1+O(1/m) \;. \tag{4.4}$$

Note that since $\frac{1}{2}d-1<1$, the term r in (4.2) is negligible.

Diagrams 1(b) and 1(c) give $O(1/m)$ contribution to $\Sigma(r)-\Sigma(0)$. Since r is not zero, the elementary bubble Π becomes much more complicated than (3.8). Before evaluating these two diagrams, we shall digress on the structure of Π. For $r\ne 0$, we have

$$\Pi(r,k^2)=(2\pi)^{-d}\int d^dp\,(r+p^2)^{-1}[r+(\vec{p}+\vec{k})^2]^{-1}$$
$$=\Pi(r/k^2,1)k^{d-4} \;. \tag{4.5}$$

Again, let us use (3.4) to combine the denominators and then perform the p integral as we did before. We find

$$\Pi(r,1)=J\int_0^1 d\alpha\,[r+\alpha(1-\alpha)]^{d/2-2} \;, \tag{4.6}$$

where J is given by (3.10). Using the formula

$$\int_{-1}^{1}dx\,(1-zx^2)^{-\nu}=2F(\nu,\tfrac{1}{2},\tfrac{3}{2},z) \tag{4.7}$$

and other identities for hypergeometric functions,[11] we find

$$\Pi(r,1)=Jr^{-(d/2-2)}F(\tfrac{1}{2}d-2,1,\tfrac{3}{2},-1/4r) \;. \tag{4.8}$$

For small r, this expression can be expanded:

$$\Pi(r,1)=\Pi(0,1)\{1-2(3-d)r+2[(4-d)^2-1]r^2+O(r^3)\}$$
$$+[2J/(1-\tfrac{1}{2}d)]r^{d/2-1}[1-(4/d)r+O(r^2)] \;. \tag{4.9}$$

Note that $\Pi(r,1)$ is not a single power series but a sum of two series. Equation (4.9) is all we need to know about Π for our calculation of γ.

Knowing enough about Π, we proceed to evaluate the second and third diagrams of Fig. 1. To $O(1/m)$, we have

$$r^{1/\gamma}\equiv r^{d/2-1-\xi}=r^{d/2-1}(1-\xi\ln r+\cdots) \tag{4.10}$$

for (4.2). Thus, the task is to determine the coefficient ξ of the $r^{d/2-1}\ln r$ term in the self-energy given by these diagrams. We shall show first that the diagram 1(c) does not contribute to such a term, and then evaluate the contribution from diagram 1(b).

Diagram 1(c) gives

$$\Sigma_c(r) = (2\pi)^{-d} \int d^d p\, u[1 + mu\Pi(p)]^{-1}(r+p^2)^{-1} \quad . \tag{4.11}$$

Note that there is no angular integral involved. In view of (4.9) and (4.11), the integral from the large-p region cannot give rise to any logarithmic term. Only the small-p region is of interest, and we may write

$$\Sigma_c(r) - \Sigma_c(0) = m^{-1}K\tfrac{1}{2} \int_0^1 dy\, y^{d/2-1} \times[\Pi(r,y)^{-1}(r+y)^{-1} - \Pi(0,y)^{-1}y^{-1}] \quad , \tag{4.12}$$

where $y \equiv p^2$, and for small r and y, $1 + mu\Pi$ may be replaced by $mu\Pi$. Using the fact that $\Pi(r,y) = y^{d/2-2}\Pi(r/y, 1)$ and (4.9), one easily shows that (4.12) behaves like

$$(\text{const})r\ln r + (\text{const})r^{d/2-1} \tag{4.13}$$

but has no $r^{d/2-1}\ln r$ term.

We proceed to Fig. 1(b). The diagram may be viewed as a correction to Fig. 1(a), i.e., a self-energy Σ_c is inserted into the loop of Fig. 1(a):

$$\Sigma_b(r) = mu(2\pi)^{-d}\int d^d q\,(r+q^2)^{-2}[\Sigma_c(r,q) - \Sigma_c(r,0)] \quad , \tag{4.14}$$

$$\Sigma_c(r,q) = (2\pi)^{-d}\int d^d p\, u[1+mu\Pi(p)]^{-1}[r + (\vec{p}+\vec{q})^2]^{-1} \quad . \tag{4.15}$$

Since $(r+q^2)^{-2} = -(\partial/\partial r)(r+q^2)^{-1}$, we can rewrite (4.14) as

$$\Sigma_b(r) = mu(2\pi)^{-d}\int d^d p\, u[1+mu\Pi(p)]^{-1} \times\left(-\frac{1}{2}\frac{\partial\Pi(p)}{\partial r} - (r+p^2)^{-1}(2\pi)^{-d}\int d^d q\,(r+q^2)^{-2}\right) . \tag{4.16}$$

The q integral in the bracket is elementary:

$$(2\pi)^{-d}\int d^d q\,(r+q^2)^{-2} = \tfrac{1}{2}K\int_0^\infty dy\, y^{d/2-1}(r+y)^{-2} = Jr^{d/2-2} \quad . \tag{4.17}$$

Again, only the small-p region of integration in (4.16) is of interest. We write (4.16) as

$$\Sigma_b(r) = \tfrac{1}{2}uK\int_0^1 dy\, y^{d/2-1}\,\Pi^{-1}(r,y)[-\tfrac{1}{2}\Pi'(r,y) - Jr^{d/2-2}(r+y)^{-1}] , \qquad \Pi' \equiv \frac{\partial\Pi}{\partial r} \tag{4.18}$$

where $y \equiv p^2$. Since Π is known and (4.18) is a one-dimensional integral, extracting the small-r behavior of $\Sigma_b(r)$ is easy. We shall give one way of doing the algebra here for completeness. It follows from (4.5) that

$$\Pi(r,y) = r^{d/2-2}\Pi(1, y/r) \quad . \tag{4.19}$$

We can write the integral in (4.18) in terms of a new variable $z = y/r$:

$$\Sigma_b(r) = \tfrac{1}{2}uKr^{d/2-1}L \quad , \tag{4.20}$$

$$L \equiv \int_0^{1/r} dz\, z^{d/2-1}\Pi^{-1}(1,z) \times[-\tfrac{1}{2}\Pi'(1,z) - J(1+z)^{-1}] \quad . \tag{4.21}$$

We want to extract the $\ln r$ term in L. To this end, we differentiate L to obtain from (4.21)

$$r^{-1}\frac{\partial L}{\partial(r^{-1})} = r^{-1}\frac{-\tfrac{1}{2}r^{-d/2+2}\Pi'(r,1) - J/(1+r)}{\Pi(r,1)} \quad . \tag{4.22}$$

Now substitute (4.9) in (4.22) for Π and collect terms. We get

$$r^{-1}\frac{\partial L}{\partial(r^{-1})} = \frac{-3J}{\Pi(0,1)} + (3-d)r^{1-d/2} + (\text{higher-order terms in } r) \quad . \tag{4.23}$$

Integrating (4.23) to get L and substituting it in (4.20), we obtain

$$(mu)^{-1}\Sigma_b(r) = J/(\tfrac{1}{2}d-1)\xi r^{d/2-1}[\ln r + \text{const}] + \text{const} \quad , \qquad \xi = -3m^{-1}S_d[\tfrac{1}{2}d - 1] \quad . \tag{4.24}$$

Combining (4.24) with (4.3), we have

$$r + \Sigma(0) - \Sigma(r) = muJ(\tfrac{1}{2}d-1)^{-1}r^{d/2-1}(1 - \xi\ln r) + (\text{higher orders in } r) + O(m^{-2}) \sim r^{1/\gamma} \quad . \tag{4.25}$$

It then follows that

$$\gamma = (\tfrac{1}{2}d - 1)^{-1}(1 - 3m^{-1}S_d) + O(m^{-2}) \quad . \tag{4.26}$$

V. EVALUATION OF λ

We proceed to examine the density-correlation function $\chi(k)$ [defined by (2.7)] at T_c. The exponent λ is defined by (2.9), i.e., $\chi(k) \sim (\text{const}) - k^\lambda$. We shall then obtain the exponent α [defined by (2.11)] via a scaling law.

A. Lowest-Order Term

To lowest order in m, the diagrams for χ are shown in Fig. 3. They give just a geometric sum,

$$\chi(k) = m\Pi - u(m\Pi)^2 + u^2(m\Pi)^3 - \cdots = m\Pi(1+mu\Pi)^{-1} = m[(mu)^{-1} - (mu)^{-2}\Pi^{-1} + (mu)^{-3}\Pi^{-2} - \cdots] \quad , \tag{5.1}$$

where we have expanded in powers of $\Pi^{-1} \sim k^{4-d}$ [see Eq. (3.8)]. The first two terms in the last line of (5.1) give

$$\chi(k) \sim (\text{const}) - (\text{const})' k^{4-d} \quad . \tag{5.2}$$

The third term is negligible for small k. By the definition of λ, we have, from (5.2) and (2.9),

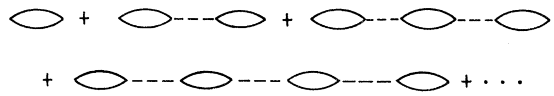

FIG. 3. Lowest-order density-correlation function χ.

$$\lambda = 4 - d + O(1/m) \quad . \tag{5.3}$$

To compute the $O(1/m)$ term of λ, we need to include the next-order diagrams for χ. These diagrams are obtained from those in Fig. 3 by adding one more wavy line in all possible ways. Or, equivalently, they are obtained by adding the correction term Π_1 to an elementary bubble Π making up the diagrams in Fig. 3. Π_1 is the sum of diagrams given in Fig. 4. We write

$$\Pi_1 = \Pi_a + \Pi_b + \Pi_c \tag{5.4}$$

for the three terms shown in Fig. 4(a)–4(c). The contribution to χ is

$$\chi_1 = \Pi_1 (1 + mu\Pi)^{-2} = (mu)^{-2}\Pi^{-2}\Pi_1 + \cdots \quad , \tag{5.5}$$

where we have again expanded in powers of $\Pi^{-1} \sim k^{4-d}$ and kept only the first term. Since $\Pi^{-2} \sim (k^{4-d})^2$ and we are interested in an $m^{-1}k^{4-d}\ln k$ correction to the k^{4-d} in (5.2), we must find the $k^{d-4}\ln k$ term in Π_1 in view of (5.5).

The evaluation of Π_a and Π_b is easy but that of Π_c is much more difficult.

B. Evaluation of Π_a and Π_b

Figure 4(a) represents the $O(m^{-1})$ self-energy correction to one of the propagators in the diagram for Π. We have

$$\Pi_a(k) = -2(2\pi)^{-d} \int d^d p\, (\vec{p} + \vec{k})^{-2} p^{-4} [\Sigma(p) - \Sigma(0)] m \quad . \tag{5.6}$$

As we have shown in Sec. III,

$$\Sigma(p) - \Sigma(0) = -\eta p^2 (\ln p + \text{const}) + (\text{higher order in } p) \quad , \tag{5.7}$$

where η is given by (3.17). Substituting (5.7) in (5.6) and changing the variable p to pk, we obtain

$$\begin{aligned}\Pi_a(k) &= -2k^{d-4}\left[\int d^d p\, (\vec{p} + \vec{1})^{-2} p^{-2}(-\eta \ln k) + \text{const}\right] m \\ &= 2\eta \Pi(1) k^{d-4}(\ln k + \text{const}) m \\ &= 2(4/d - 1) K k^{d-4}(\ln k + \text{const}) \quad ,\end{aligned} \tag{5.8}$$

where we have substituted (3.17) for η and (3.8) for $\Pi(1)$ in the last step. The quantity $\vec{1}$ is the unit vector in the direction of $\vec{k}$. Note that the p integral in (5.8) needs no cutoff; i.e., the large-p contribution is very small.

Figure 4(b) gives

$$\Pi_b(k) = -(2\pi)^{-2d} \int d^d p\, d^d q\, p^{-2} (\vec{p} + \vec{k})^{-2} (\vec{p} + \vec{q})^{-2} \times (\vec{p} + \vec{q} + \vec{k})^{-2} u [1 + mu\Pi(q)]^{-1} \quad . \tag{5.9}$$

The large-q region of the integral does not contribute to the $k^{d-4}\ln k$ term. Again, we drop 1 compared to $mu\Pi(q)$ since only small q is of interest, and then we change the variables p and q to pk and qk. After such a change of variables, the cutoff in the magnitude of p and q becomes $1/k$ instead of 1. We have

$$\Pi_b(k) = -k^{d-4}(2\pi)^{-2d} \int d^d p\, d^d q\, p^{-2} (\vec{p} + \vec{1})^{-2} \times (\vec{p} + \vec{q})^{-2} (\vec{p} + \vec{q} + \vec{1})^{-2} \Pi^{-1}(1) q^{4-d} \; , \quad p, q < 1/k \, . \tag{5.10}$$

After this change of variables, the $\ln k$ term in the integral of (5.10) comes from the upper limit of the q integral. The p integral does not need a cutoff; i.e., it converges when $1/k \to \infty$. Its contribution comes from the regions near $p = 0$ and near $p + q = 0$. Thus, as far as the $\ln(1/k)$ term goes, we may write (5.10) as

$$\Pi_b(k) = -k^{d-4} 2 (2\pi)^{-2d} \int d^d p\, p^{-2} (\vec{p} + \vec{1})^{-2} q^{-d} d^d q\, \Pi^{-1}(1) \quad . \tag{5.11}$$

The p integral gives $\Pi(1)$ and the factor 2 in front counts for the two regions mentioned above. Writing $Kq^{d-1}dq$ for $d^d q\, (2\pi)^{-d}$, we have

$$\Pi_b(k) = -k^{d-4} 2K \int^{1/k} \frac{dq}{q} = 2Kk^{d-4}(\ln k + \text{const}) \quad . \tag{5.12}$$

C. Evaluation of Π_c

Figure 4(c) gives

$$\Pi_c(k) = 2(2\pi)^{-d} \int d^d q\, \Pi^{-1}(q) \Pi^{-1}(\vec{q} + \vec{k}) T^2(\vec{k}, \vec{q}) \quad , \tag{5.13}$$

where $T(k, q)$ is the triangular part:

$$T(k, q) = (2\pi)^{-d} \int d^d p\, (\vec{p} + \vec{k})^{-2} p^{-2} (\vec{p} + \vec{q})^{-2} \quad . \tag{5.14}$$

In (5.13), we have, as before, written $m^{-1}\Pi^{-1}$ for each wavy line, since only small q is of interest. We now change the variable q to qk in (5.13) to obtain

$$\Pi_c(k) = k^{d-4} 2 (2\pi)^{-d} \int d^d q\, \Pi^{-1}(q) \Pi^{-1}(\vec{q} + \vec{1}) T^2(\vec{1}, \vec{q}) \quad , \tag{5.15}$$

where q now has the upper limit $1/k$ instead of 1. The task now is to determine $T(\vec{1}, \vec{q})$ for large q. This task cannot be accomplished easily by brute-force integrations, because the p integral of (5.14) involves two angular integrals owing to the two fixed vectors $\vec{1}$ and $\vec{q}$. To extract the two leading

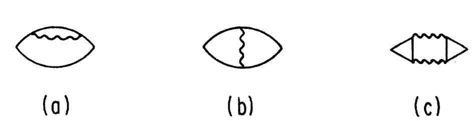

FIG. 4. Diagrams for Π_1.

terms, we can actually avoid one of the angular integrals easily if we use the Mellin-transform method, which is a version of the Laplace transform. The Mellin transform and its inverse are defined by the two formulas

$$g(s)=\int_0^\infty dx\, f(x)x^{s-1}\ ,$$
$$f(x)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} ds\, x^{-s}g(s)\quad . \qquad (5.16)$$

See the Bateman Manuscript Project for details.[12] We write (5.14) as

$$T(\vec{1},\vec{q})=(2\pi)^{-d}\int_0^\infty dp\, p^{d-3}\int d\Omega\,(p^2+1+2p\cos\theta)^{-1}\times(p^2+q^2+2pq\cos\phi)^{-1}\ , \qquad (5.17)$$

where $\int d\Omega$ integrates over the directions of $\vec{p}$. The angles θ and ϕ are defined by

$$\cos\theta=\vec{1}\cdot\vec{p}/p\ ,\qquad \cos\phi=(\vec{q}/q)\cdot(\vec{p}/p)\ . \qquad (5.18)$$

We now take the Mellin transform of (5.17) with respect to q, which appears only in the last factor. Using the formula[13]

$$\int_0^\infty (x^2+2ax\cos\theta+a^2)^{-1}x^{s-1}dx=-\pi\csc\pi s\,\sin(s-1)\theta\,\csc\theta\, a^{s-2}\ , \qquad (5.19)$$

$$0<\mathrm{Re}s<2\ , \qquad (5.20)$$

first in integrating over q and then in the p integral in (5.17), we find

$$T(\vec{1},s)\equiv\int_0^\infty dq\, q^{s-1}T(\vec{1},\vec{q})$$
$$=\pi^2\csc\pi s\,\csc\pi(s+d-4)(2\pi)^{-d}\int d\Omega\times\sin(s-1)\phi\,\csc\phi\,\sin(s+d-5)\theta\,\csc\theta\ , \qquad (5.21)$$

$$0<\mathrm{Re}s<2\ ,\quad 0<\mathrm{Re}(s+d-4)<2\ . \qquad (5.22)$$

The two inequalities in (5.22) imply that

$$4-d<\mathrm{Re}s<2\ . \qquad (5.23)$$

The advantage of the Mellin-transform approach is that the p integral is performed easily with the help of (5.19). To obtain $T(\vec{1},\vec{q})$ for large q, we perform the inverse transform on (5.2):

$$T(\vec{1},\vec{q})=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} ds\, q^{-s}T(\vec{1},s)\ , \qquad (5.24)$$

where $4-d<c<2$ as required by (5.23). For large q, we close the vertical contour to the right and pick up poles of $T(\vec{1},s)$ on the right. According to (5.21), poles are located where s = integer and where $s+d-4$ = integer. Thus, the poles to the right of the vertical contour are at

$$2,\ 3,\ 4,\ \ldots,$$
$$6-d,\ 7-d,\ \ldots\ . \qquad (5.25)$$

There is no pole at $5-d$ owing to the $\sin(s+d-5)\theta$ factor in (5.21); nor is there one at $s=1$. The two poles at 2 and $6-d$ turn out to be the only ones of interest. The residue at 2 gives a q^{-2} term for $T(\vec{1},\vec{q})$. For $s=2$, the angle ϕ drops out of (5.21) and the remaining angular integral is easy. In fact, the residue at 2 can be easily obtained directly from (5.14). For large q, $(\vec{p}+\vec{q})^{-2}\approx q^{-2}$ and

$$T(\vec{1},\vec{q})\to(2\pi)^{-d}\int d^dp\,(\vec{p}+\vec{1})^{-2}p^{-2}q^{-2}=\Pi(1)q^{-2}\ . \qquad (5.26)$$

The residue at the second pole $6-d$ is

$$-q^{-6+d}\pi\csc\pi(6-d)(2\pi)^{-d}\int d\Omega\,\sin(5-d)\phi\,\csc\phi\ . \qquad (5.27)$$

The angle θ has dropped out, and $\int d\Omega$ can easily be performed:

$$(2\pi)^{-d}\int d\Omega\,\sin(5-d)\phi\,\csc\phi=(2\pi)^{-1}K_{d-1}\int_0^\pi d\phi\,\sin^{d-3}\phi\,\sin(5-d)\phi=K_{d-1}2^{2-d}(d-3)\sin\tfrac{1}{2}\pi(5-d)\ . \qquad (5.28)$$

To sum up, we have

$$T(\vec{1},\vec{q})=q^{-2}\Pi(1)-q^{-6+d}\pi\csc\pi(6-d)K_{d-1}2^{2-d}(d-3)\sin\tfrac{1}{2}\pi(5-d)+(\text{higher-order terms in } q^{-1})\ . \qquad (5.29)$$

We now square (5.29) to get $T^2(\vec{1},\vec{q})$ and substitute it in (5.15). The two factors of Π^{-1} give $\sim q^{8-2d}$, and T^2 gives $\sim q^{-4}+q^{-8+d}+\cdots$. The q^{-4} term of T^2 results in a k^{d-4} contribution to the integral, but the q^{-8+d} term gives a $\ln k$ contribution. Copying down the coefficients and simplifying, we obtain

$$\Pi_c(k)=(\text{const})k^{2d-8}+4K(d-3)k^{d-4}(\ln k+\text{const})+(\text{higher orders in } k)\ . \qquad (5.30)$$

Adding (5.8), (5.12), and (5.30), we get Π_1 according to (5.4). We get χ_1 from (5.5):

$$\chi_1=(mu)^{-2}\Pi(1)^{-2}k^{4-d}\times[4(2/d+d-3)K\ln k+\text{const}]+\text{const}, \qquad (5.31)$$

where the last const comes from the k^{2d-8} term of (5.30). Now we add (5.1) and (5.31) to obtain the leading terms in the $1/m$ expansion of $\chi(k)$:

$$\chi(k)=\text{const}-m(mu)^{-2}\Pi^{-1}(1)k^{4-d}[1+\lambda_1(\ln k+\text{const})]\sim\text{const}-k^{4-d+\lambda_1}\ , \qquad (5.32)$$

where

$$\lambda_1\equiv\frac{-m^{-1}4(2/d+d-3)K}{\Pi(1)}\ . \qquad (5.33)$$

Writing out K and $\Pi(1)$ explicitly, we have

$$\lambda=4-d+\lambda_1=4-d-8(2/d+d-3)S_dm^{-1}+O(m^{-2}). \qquad (5.34)$$

D. Exponent α

From the same diagrams we just examined for λ, we can also calculate $\chi(0)$ above T_c and obtain the exponent α. We shall do such a calculation in Appendix C. Here we shall deduce α from λ assuming the validity of the scaling law

$$-\alpha=\lambda\gamma/(2-\eta)\ , \qquad (5.35)$$

which can be obtained in the following way.[14]

Near T_c, we assume the existence of a function $f(x)$ and a constant y such that

$$\chi(k)=f(k\xi)\xi^{y}+\text{const}\,, \qquad (5.36)$$

where ξ is the correlation length and

$$\xi\sim(r_0-r_{0c})^{-\nu}\ , \qquad (5.37)$$

$$\nu=\gamma/(2-\eta)\ . \qquad (5.38)$$

For $k=0$, we have $\chi(0)\sim\text{const}-(r_0-r_{0c})^{-\alpha}$. It follows from (5.36) and (5.37) that

$$y=\alpha/\nu\ , \qquad (5.39)$$

assuming $f(0)$ is a finite constant. At T_c, ξ goes to infinity. We must have $f(\xi k)\to(\xi k)^{-y}$ and $y=-\lambda$ in order that $\chi(k)\sim\text{const}-k^{\lambda}$. By (5.39), we have $-\alpha=\nu\lambda$, which leads to (5.35). Substituting (5.34) in (5.35), we obtain

$$-\alpha=\frac{4-d}{d-2}\left(1+\frac{4(1-d)}{m(4-d)}\,S_d\right)+O(m^{-2})\ . \qquad (5.40)$$

It is easy to check that this expression satisfies the relation $2-\alpha=\nu d$, with ν given by (5.38), and γ and η computed in Secs. III and IV. In Appendix C, we shall discuss possible violations of the scaling laws involving α observed by Abe and Hikami.[4]

VI. EVALUATION OF μ

The functions Γ_{2s} and the exponent μ are defined by (2.12) and (2.13). Figure 5 shows diagrams of the lowest two orders in m^{-1} for Γ_{2s}.

The lowest-order term is given by Fig. 5(a):

$$\Gamma_a=[1+mu\Pi(k)]^{-1}$$
$$=k^{4-d}[mu\Pi(1)]^{-1}+\text{(higher orders in } k)\ . \qquad (6.1)$$

Since $\Gamma_{2s}(k)\sim k^{\mu}$, we have

$$\mu=4-d+O(m^{-1})\ . \qquad (6.2)$$

The rest of Fig. 5 gives the next-order terms. The diagrams in Fig. 5(b) are just trivial modifications of those in Fig. 4. They give

$$\Gamma_b=-u[1+mu\Pi(k)]^{-2}\Pi_1(k)=-u\chi_1(k)\ , \qquad (6.3)$$

where Π_1 and χ are given by (5.4), (5.5), and (5.31). It remains to compute the diagrams 5(c) and 5(d). Diagram 5(c) gives

$$\Gamma_c=-[mu\Pi(1)]^{-1}k^{4-d}(2\pi)^{-d}\int d^dp\,(\vec{p}+\tfrac{1}{2}\vec{k})^{-2}$$
$$\times(\vec{p}-\tfrac{1}{2}\vec{k})^{2}p^{4-d}[mu\Pi(1)]^{-1}\ . \qquad (6.4)$$

From counting the power of p in the integral, it is clear that it diverges logarithmically as $k\to 0$.

Therefore, (6.4) gives

$$\Gamma_c=-u[mu\Pi(1)]^{-2}k^{4-d}K(-\ln k+\text{const})\ . \qquad (6.5)$$

Diagram 5(d) gives

$$\Gamma_d=[mu\Pi(1)]^{-1}k^{4-d}2(2\pi)^{-d}\int d^dq\,[m\Pi(q)]^{-1}$$
$$\times[m\Pi(\vec{q}+\vec{k})]^{-1}(\vec{q}+\tfrac{1}{2}\vec{k})^2 m\,T(\vec{k},\vec{q})\ , \qquad (6.6)$$

where T is the triangular part, which we have evaluated previously. Now we change the variable q to qk. The upper limit of the q integral now becomes $1/k$. Substituting the formula (5.29) in (6.6), we obtain

$$\Gamma_d=u[mu\Pi(1)]^{-2}k^{4-d}2[K(d-3)\ln k+\text{const}]$$
$$+\text{const of } O(m^{-2})\ . \qquad (6.7)$$

The last constant comes from the q^{-2} term in $T(1,q)$. Adding up (6.1), (6.3), (6.5), and (6.7), we have

$$\Gamma_{2s}(k)=\Gamma_a+\Gamma_b+\Gamma_c+\Gamma_d$$
$$=[mu\Pi(1)]^{-1}k^{4-d}\{1+[m\Pi(1)]^{-1}K$$
$$\times[-4(2/d+d-3)+1+2(d-3)](\ln k+\text{const})\}$$
$$+\text{const of } O(m^{-2})\ . \qquad (6.8)$$

It follows that

$$\mu=4-d-2(8/d+2d-7)m^{-1}S_d+O(m^{-2})\ . \qquad (6.9)$$

The scaling law

$$\mu=(2-\eta)(1-\gamma^{-1})\ , \qquad (6.10)$$

with η and γ given by (3.17) and (4.26), can easily be verified. This scaling law can be obtained as follows: Use the identity

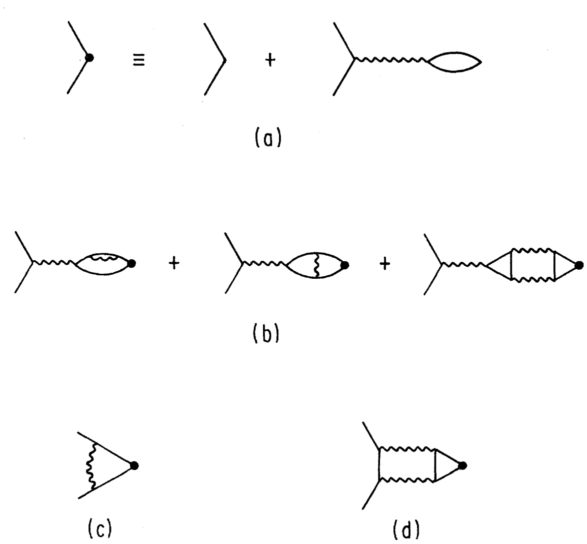

FIG. 5. Diagrams for Γ_{2s}.

$$\Gamma_{2s}(0) = \frac{-\partial G^{-1}(0)}{\partial r_0} \sim (r_0 - r_{0c})^{\gamma-1} \tag{6.11}$$

and then apply the same arguments in Sec. V D. We get

$$\gamma - 1 = \mu\nu = \mu\gamma/(2-\eta) \quad , \tag{6.12}$$

which leads to (6.10).

VII. EVALUATION OF γ AND η FOR LONG-RANGE INTERACTING SYSTEMS

So far we have examined only systems with short-range interactions. In this section we shall evaluate γ and η for systems with k^σ ($\frac{1}{2}d < \sigma < 2$) interaction between field amplitudes. The Hamiltonian is given by (2.4).

The calculation procedure is similar to that for the short-range case. The only difference is that the free propagator is now $(r+p^\sigma)^{-1}$ instead of $(r+p^2)^{-1}$.

A. Evaluation of γ

To evaluate γ, we simply repeat the work of Sec. IV with the new propagator. Our task is to evaluate the self-energy diagrams shown in Fig. 1. The first one is easy. We have

$$\begin{aligned}\Sigma_a(r) - \Sigma_a(0) &= mu \int d^dp\, (2\pi)^{-d}[(r+p^\sigma)^{-1} - p^{-\sigma}] \\ &= muK \int_0^\infty dp\, p^{d-1}[(r+p^\sigma)^{-1} - p^{-\sigma}] \\ &= -mu(d/\sigma - 1)^{-1} J_\sigma r^{d/\sigma - 1} \quad ,\end{aligned} \tag{7.1}$$

where

$$J_\sigma \equiv \frac{(K/\sigma)\pi(d/\sigma - 1)}{\sin\pi(d/\sigma-1)} \quad , \tag{7.2}$$

which reduces to J [see Eq. (3.10)] when $\sigma = 2$. Since $d < 2\sigma$, we have $d/\sigma - 1 < 1$ so that

$$r^{1/\gamma} \sim r + \Sigma(0) - \Sigma(r) \sim r^{d/\sigma - 1} \tag{7.3}$$

to the lowest order in m^{-1}. It follows that

$$1/\gamma = d/\sigma - 1 + O(m^{-1}) \quad . \tag{7.4}$$

Figure 1(b) is more complicated. It gives

$$\Sigma_b(r) = mu(2\pi)^{-d}\int d^dp\, u[1 + mu\Pi(p)]^{-1} \times\left(-\frac{1}{2}\frac{\partial\Pi(p)}{\partial r} - (r+p^\sigma)^{-1}(2\pi)^{-d}\int d^dq\,(r+q^\sigma)^{-2}\right) , \tag{7.5}$$

which has the same structure as (4.16). The q integral is easily evaluated to give

$$(2\pi)^{-d}\int d^dq\,(r+q^\sigma)^{-2} = J_\sigma r^{d/\sigma-2} \quad . \tag{7.6}$$

The function $\Pi(p)$ is given by

$$\Pi(p) \equiv \Pi(r, p^\sigma) = (2\pi)^{-d}\int d^dq\,(r+q^\sigma)^{-1}(r + |\vec{p}+\vec{q}|^\sigma)^{-1} \quad , \tag{7.7}$$

which reduces to (4.5) when $\sigma = 2$. Again the terms of interest come from the small-p region of the integral (7.5). We change variable from p to y,

$$y \equiv p^\sigma \quad , \tag{7.8}$$

and then write (7.5) as

$$\Sigma_b(r) = \frac{K}{\sigma} u \int_0^1 dy\, y^{d/\sigma-1}\Pi^{-1}(r,y) \times[-\tfrac{1}{2}\Pi'(r,y) - J_\sigma r^{d/\sigma-2}(r+y)^{-1}] \quad , \tag{7.9}$$

where $\Pi'(r,y) \equiv \partial\Pi(r,y)/\partial r$. Since

$$\Pi(r,y) = r^{d/\sigma-2}\Pi(1, y/r) \quad , \tag{7.10}$$

we can write the integral of (7.9) in terms of the new variable $z = y/r$:

$$\Sigma_b(r) = (K/\sigma)r^{d/\sigma-1}uL \quad , \tag{7.11}$$

where

$$L \equiv \int_0^{1/r} dz\, z^{d/\sigma-1}\Pi^{-1}(1,z)[-\tfrac{1}{2}\Pi'(1,z) - J_\sigma(1+z)^{-1}] \quad . \tag{7.12}$$

To extract the small-r behavior of L we differentiate L and obtain

$$r^{-1}\frac{\partial L}{\partial(r^{-1})} = r^{-1}\frac{-\frac{1}{2}r^{-d/\sigma+2}\Pi'(r,1) - J_\sigma/(1+r)}{\Pi(r,1)} \quad . \tag{7.13}$$

The function $\Pi(r,1)$ is examined in Appendix A. We obtain from (A19)

$$-\frac{1}{2}\Pi'(r,1)r^{-d/\sigma+2} = J_\sigma - \frac{d}{\sigma}\left(\frac{d}{\sigma}-1\right)^{-1}J_\sigma r + \frac{1}{2}I(-1,0)r^{-d/\sigma+2} + \dots \tag{7.14}$$

and

$$\begin{aligned}r^{-1}\frac{\partial L}{\partial(r^{-1})} &= \frac{J_\sigma}{r}\left[1 - \frac{d}{\sigma}\left(\frac{d}{\sigma}-1\right)^{-1}r - (1-r) + \frac{1}{2}\frac{I(-1,0)r^{-d/\sigma+2}}{J_\sigma} + \cdots\right] \\ &\quad\times\Pi^{-1}(0,1)\left(1 + \frac{2J_\sigma\Pi^{-1}(0,1)r^{d/\sigma-1}}{(d/\sigma-1)} + \cdots\right) \\ &= \frac{J_\sigma\Pi^{-1}(0,1)[I(-1,0)/\Pi(0,1) - 1]}{(d/\sigma-1)} + (\text{const})r^{-d/\sigma+1} + (\text{const})r^{2/\sigma-1} \quad .\end{aligned} \tag{7.15}$$

The quantities $\Pi(0,1)$, $I(-1,0)$ are given explicitly by (A13) and (A15). It follows from (7.11) and (7.15) that

$$\frac{\Sigma_b(r)}{mu} = \frac{J_\sigma}{(d/\sigma-1)}\xi r^{d/\sigma-1}(\ln r + \text{const}) + \text{const} \quad , \tag{7.16}$$

where

$$\xi = \frac{K}{\sigma}\Pi^{-1}(0,1)\left(\frac{I(-1,0)}{\Pi(0,1)} - 1\right)m^{-1} \quad . \tag{7.17}$$

The arguments of (4.11)–(4.13) can be repeated to show that diagram 5(c) contains no $r^{d/\sigma}\ln r$

term.

By the definition of γ, we have

$$r^{1/\gamma} \sim r + \Sigma(0) - \Sigma(r)$$

$$= mu \frac{J_\sigma}{d/\sigma - 1} r^{d/\sigma - 1} [1 - \xi(\ln r + \text{const})] + (\text{higher order in } r) , \quad (7.18)$$

as implied by (7.1) and (7.16). We therefore have

$$1/\gamma = d/\sigma - 1 - \xi + O(m^{-2}) \quad (7.19)$$

or

$$\gamma = \left(\frac{d}{\sigma} - 1\right)^{-1} \left(1 + \frac{\xi}{(d/\sigma - 1)^{-1}}\right) + O(m^{-2}) , \quad (7.20)$$

with ξ given by (7.17).

B. Evaluation of η

At T_c, we have

$$G^{-1}(k) = k^\sigma + \Sigma(k) - \Sigma(0) , \quad (7.21)$$

where $r = 0$ is understood. The lowest-order term contributing to (7.21) is Fig. 1(c), which is

$$\Sigma(k) - \Sigma(0) = (2\pi)^{-d} \int d^d p\, u[1 + mu\Pi(p)]^{-1} \times (|\vec{p} + \vec{k}|^{-\sigma} - p^{-\sigma}) , \quad (7.22)$$

where

$$\Pi(p) = (2\pi)^{-d} \int d^d q\, q^{-\sigma} |\vec{q} + \vec{p}|^{-\sigma} = p^{d-2\sigma} \Pi(0, 1) \quad (7.23)$$

[see Eq. (A13) for $\Pi(0,1)$]. It is just $\Pi(p)$ at $r = 0$, $p = 1$. Only the small-p part of the integral (7.22) is of interest because the large-p part only gives a k^2 contribution. We write (7.22) as

$$\Sigma(k) - \Sigma(0) = m^{-1} \frac{K_{d-1}}{2\pi} \Pi^{-1}(0,1) \times \int_0^1 dp\, p^{2\sigma - 1} \int_0^\pi d\theta \sin^{d-2}\theta \times [(k^2 + p^2 + 2pk\cos\theta)^{-\sigma/2} - p^{-\sigma}] . \quad (7.24)$$

To extract the small-k behavior, we write

$$(k^2 + p^2 + 2pk\cos\theta)^{-\sigma/2} = p^{-\sigma} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \left(\frac{k}{p}\right)^{-s} M(s, \theta) , \quad 0 < c < \sigma \quad (7.25)$$

where $M(s,\theta)$ is given by the expression (A5). Then, we substitute (7.25) in (7.24). The θ integral can be done by using Eq. (8.14.16) of Ref. 11. The p integral is now trivial. We then find

$$\Sigma(k) - \Sigma(0) = (\text{const}) \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{k^{-s}}{s + \sigma} B(s, \sigma - s) \times [\Gamma(\tfrac{1}{2}(d - \sigma + s))\Gamma(\tfrac{1}{2}(d - s)) \times \Gamma(\tfrac{1}{2}(s+1))\Gamma(\tfrac{1}{2}(\sigma - s + 1))]^{-1} . \quad (7.26)$$

To find leading terms in k, we close the contour to the left. Note that the pole at $s = 0$ must be excluded to count for the subtraction $-p^{-\sigma}$ term in (7.24). The leading pole is at $s = -\sigma$. The next ones are at $s = -2, -3, \ldots$, etc. We therefore conclude that

$$\Sigma(k) - \Sigma(0) \sim k^\sigma + O(k^2) . \quad (7.27)$$

There is no logarithmic term since there is no double pole according to (7.26). Note that if $\sigma = 2$, there would be a double pole at $s = -2$ and there would be a logarithmic term as we saw in Sec. III. We then conclude that

$$G^{-1}(k) \sim k^\sigma$$

to two leading orders in m^{-1}. In other words,

$$\eta = 2 - \sigma + O(m^{-2}) . \quad (7.28)$$

Like γ, η is discontinuous going from $\sigma < 2$ to $\sigma = 2$.

VIII. CRITICAL EXPONENTS FOR A CHARGED SYSTEM

A charged system is one which has a Coulomb interaction. A model is provided by the Hamiltonian (2.5). The $k = 0$ term is removed to account for the uniform rigid-charge background, which makes the total charge zero.

The Coulomb interaction is peculiar in that it becomes infinite as $k \to 0$. This makes the density-correlation function $\chi(k)$ behave very differently from that in the case of neutral systems. The repeated isolated (wave vector not integrated) interaction lines like those in Fig. 3 are the cause. We can define χ' by excluding such lines from χ. We then have

$$\chi = \chi'(1 + e^2\chi'/k^2)^{-1} . \quad (8.1)$$

It follows that for small k,

$$\chi \sim k^2 , \quad (8.2)$$

provided $\chi' \neq 0$ for $k \to 0$.

Because the $k = 0$ term in the interaction is removed, diagrams for thermodynamical quantities cannot have isolated interaction lines. In particular, the specific-heat exponent must now be obtained from $\chi'(0)$, not from $\chi(0)$.

Apart from the special features introduced by the isolated Coulomb interaction lines, the charged system is not expected to be different from a neutral system as far as critical behavior goes, because the Coulomb interaction is shielded by charge fluctuations. The shielded interaction is short ranged. The exponents η and γ are the same as those in short-range interacting systems, since $G(k)$ involves no isolated interaction lines. The function $\chi'(k)$ can be expressed in terms of $\chi_n(k)$, which is what one would get for $\chi(k)$ if the shielded Coulomb interaction were used as the interaction. Note that χ_n is not the same as χ' nor χ. Reference 2 has shown in detail how χ' is related to χ_n. We shall not repeat the arguments here. The results (2.32)–(2.35) follow from the fact that χ_n has the same critical behavior as χ in a short-range interacting system.

IX. DISCUSSION

In the above, we have computed the exponents λ and μ for short-range interacting systems in addition to γ and η. In fact this additional work is much more difficult than the γ and η calculation. The reasons for doing it are the following: First, it provides checks on two scaling laws, which were obtained by very different arguments. Second, if one wants to calculate $O(n^{-2})$ terms, the technique developed here, especially in this additional work, will be very helpful.

The calculation to $O(n^{-2})$ is of great interest[15]; however, the mathematical complication is quite serious. For example, the calculation of η, which is believed to be the simplest, would involve the additional diagrams shown in Fig. 6. Also, the diagrams evaluated here must be evaluated to one more order in n^{-1}, the cutoff must be done more elaborately, etc.

In the above calculations, we did not have to choose a special value for the coupling constant u except that it must be of $O(1/n)$. This is in contrast to the ϵ expansion, where one must choose a definite u. The reason is that the critical region in k space (i.e., the region in which the critical exponents describe correlation functions well) for small ϵ is of $O(e^{-1/\epsilon})$ unless coupling constants are properly restricted.[8] In our case here the critical region is of $O(1)$ and no such restriction is needed. This point can be seen from a renormalization-group analysis.[16]

As was mentioned in the Introduction, our results here are for arbitrary d and can be expanded in $\epsilon = 4 - d$ for a comparison with the results of ϵ expansion. This comparison is easy and is left as an exercise. Complete agreement with results in Refs. 3, 8, and 9 has been observed.

There is much being done and to be done to $O(n^{-1})$ besides the work presented here, e.g., investigations below T_c and the study of the renormalization group.[7,16]

ACKNOWLEDGMENTS

It is a great pleasure to thank Professor M. E. Fisher, Dr. B. G. Nickel, and Professor K. G. Wilson for discussions, which were indispensable to this research. I wish to thank Professor J. W. Wilkins and other members of the Laboratory of Atomic and Solid State Physics, Cornell University, for their hospitality during my visit, where most of this work was performed. This work was supported in part by the National Science Foundation under Grant No. GP-31267X.

APPENDIX A

We shall go through the details of the determination of the function

$$\Pi(r,1) = \int d^dp\,(2\pi)^{-d}(r+p^\sigma)^{-1}(r+|\vec{p}+\vec{1}|^\sigma)^{-1}, \tag{A1}$$

where $\vec{1}$ is a unit vector. We shall only evaluate it for small r.

We write the propagator as an inverse Mellin transform [see Ref. 12, Eq. (3), p. 308],

$$(r+p^\sigma)^{-1} = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\, r^{-s} p^{\sigma(s-1)} \pi \csc \pi s \quad , \tag{A2}$$

where $0<c<1$. Substituting (A2) in (A1) for both propagators, we get

$$\Pi(r,1) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \frac{ds'}{2\pi i}\, r^{-s-s'}\pi^2 \times (\csc\pi s)(\csc\pi s')I(s,s') \quad , \tag{A3}$$

where

$$I(s,s') = (2\pi)^{-d}\int d^dp\, p^{\sigma(s-1)}|\vec{p}+\vec{k}|^{\sigma(s'-1)} = \frac{K_{d-1}}{2\pi}\int_0^\infty dp\, p^{d-1+\sigma(s-1)}\int_0^\pi d\theta \sin^{d-2}\theta \times (1+2p\cos\theta+p^2)^{\sigma(s'-1)/2} \quad . \tag{A4}$$

The integral over p can be done using formula (22) (p. 310) of the Bateman Project[12]:

$$\int_0^\infty (1+2x\cos\theta+x^2)^{-\nu}x^{s-1}dx = 2^{\nu-1/2}(\sin\theta)^{1/2-\nu}\Gamma(\nu+\tfrac{1}{2})B(s,2\nu-s)P^{1/2-\nu}_{s-\nu-1/2}(\cos\theta), \quad 0<\mathrm{Re}\,s<2\nu \quad . \tag{A5}$$

Note that in Ref. 12 the power of $\sin\theta$ is in error.[17] Now the integration over θ can be done using the formula 8.14.16 of Abramowitz and Stegun[11] (p. 338). The result is that $I(s,s')$ is a product of quite a few Γ functions. After simplifications using various identities for Γ functions, we get

$$I(s,s') = 2^{-d}\pi^{-d/2}B(\tfrac{1}{2}d+\tfrac{1}{2}\sigma(s-1),\tfrac{1}{2}d+\tfrac{1}{2}\sigma(s'-1)) \times\frac{\Gamma(-\tfrac{1}{2}d-\tfrac{1}{2}\sigma(s+s'-2))}{\Gamma(-\tfrac{1}{2}\sigma(s'-1))\Gamma(-\tfrac{1}{2}\sigma(s-1))} \quad , \tag{A6}$$

$$\mathrm{Re}\,s, s' > 1-d/\sigma \quad , \tag{A7}$$

$$\mathrm{Re}(s+s') < 2-d/\sigma \quad , \tag{A8}$$

$$2\sigma > d > \sigma \quad . \tag{A9}$$

We proceed to substitute (A6) in (A3) and perform the s, s' integrals. Since r is taken to be a very small number, we close the vertical integration contours to the left and pick up poles. Then, the task becomes finding the residues at these poles. The relevant poles come from the β function and csc's. They are at

$$s, s' = 0, -1, -2, \ldots; \quad 1 - d/\sigma, \ 1-(d+2)/\sigma, \ 1-(d+4)/\sigma, \ \ldots \quad . \tag{A10}$$

The factor $r^{-s-s'}$ in (A3) will then give terms proportional to

$$1, r^{d/\sigma-1}, r, r^{d/\sigma}, r^{(d+2)/\sigma-1}, r^2, \ldots \tag{A11}$$

for $\Pi(r, 1)$.

Let us integrate s' around the pole $s'=0$ and get from (A3)

$$\Pi(r,1) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} r^{-s} I(s, 0) \pi \csc \pi s \tag{A12}$$

from $s'=1$.

Now we perform the s integral

(a) around $s=0$ and get

$$2^{-d}\pi^{-d/2} B(\tfrac{1}{2}(d-\sigma), \tfrac{1}{2}(d-\sigma)) \times \Gamma(-\tfrac{1}{2}d+\sigma)\Gamma(\tfrac{1}{2}\sigma)^{-2} = \Pi(0,1) \ ; \tag{A13}$$

(b) around $s = 1 - d/\sigma$ and get

$$\frac{2^{-d}\pi^{-d/2}}{\frac{1}{2}\sigma\Gamma(\frac{1}{2}d)} \pi \csc\pi\left(1-\frac{d}{\sigma}\right) r^{d/\sigma-1} = -\frac{J_\sigma r^{d/\sigma-1}}{d/\sigma-1} \ ; \tag{A14}$$

(c) around $s=-1$ and get

$$-rI(-1,0) = -r 2^{-d}\pi^{-d/2} \times \frac{\Gamma(\frac{1}{2}d-\sigma)\Gamma(\frac{1}{2}(d-\sigma))\Gamma(-\frac{1}{2}d+\frac{3}{2}\sigma)}{\Gamma(d-\frac{3}{2}\sigma)\Gamma(\frac{1}{2}\sigma)\Gamma(\sigma)} \ ; \tag{A15}$$

(d) around $s = 1-(d+2)/\sigma$

$$r^{-1+(d+2)/\sigma} 2^{-d}\pi^{-d/2} \frac{[\frac{1}{2}(d-\sigma)-1]\pi}{\frac{1}{2}d\Gamma(\frac{1}{2}d)\ \sin\pi\ [(d+2)/\sigma-1]} \equiv r^{-1+(d+2)/\sigma}\tilde{J}_\sigma \quad . \tag{A16}$$

We next integrate s' around the pole $s' = 1 - d/\sigma$ and obtain

$$\int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} r^{-s-1+d/\sigma}\pi^2 (\csc\pi s)[\csc\pi(1-d/\sigma)] \times 2^{-d}\pi^{-d/2}\frac{2/\sigma}{\Gamma(\frac{1}{2}d)} \quad . \tag{A17}$$

Integrating over s around the pole $s=-1$ gives

$$-r^{d/\sigma}2^{-d}\pi^{-d/2}\left[\frac{\sigma}{2}\Gamma\left(\frac{d}{2}\right)\right]^{-1}\pi\csc\pi\left(1-\frac{d}{\sigma}\right) = \frac{J_\sigma r^{d/\sigma}}{(d/\sigma-1)} \quad . \tag{A18}$$

Besides (A13)–(A16) and (A18), there are terms with s and s' interchanged. Summing up, we have

$$\Pi(r,1) = \Pi(0,1) - 2r^{d/\sigma-1}\frac{J_\sigma}{d/\sigma-1} - rI(-1,0) + \frac{2r r^{d/\sigma}J_\sigma}{d/\sigma-1} + 2r^{(d+2)/\sigma-1}\tilde{J}_\sigma + O(r^2) \quad . \tag{A19}$$

It should be noted that for $\sigma<2$, the fifth term $r^{(d+2)/\sigma-1}$ is negligible compared to the fourth term $r^{d/\sigma}$ and therefore the fifth term will play no role in γ. However if $\sigma=2$, these two terms are of the same magnitude $r^{d/2}$, and both must be counted. The result then reduces to (4.9). Thus, the exponent γ is discontinuous in change from $\sigma<2$ to $\sigma=2$.

APPENDIX B

We shall sketch the calculation for the exponent λ [defined by (2.9)] for short-range interacting systems to $O(\epsilon^2)$, where $\epsilon = 4-d$. The basic idea behind the method of the ϵ expansion has been discussed and applied,[7,8] and will be taken for granted here.

The interaction u is of $O(\epsilon)$. We therefore need to consider diagrams to $O(\epsilon^2)$. These diagrams are given in Fig. 7. The free propagator has the form

$$k^2(1+k^2)^{-1} = G_0(k) \quad , \tag{B1}$$

where we include the factor $(1+k^2)^{-1}$ to serve as an upper cutoff for k. Unlike the $O(1/n)$ calculation we discussed above, the cutoff has to be included explicitly and precisely in most of the steps.

Figure 7(a) gives

$$\chi_a = m(2\pi)^{-d}\int d^dp\, G_0(\vec{k}+\vec{p})G_0(p) \equiv m\Pi(k) \quad . \tag{B2}$$

Performing the integral and then expanding in ϵ, we get

$$\Pi(k) = (8\pi^2)^{-1}[(1+\epsilon-\epsilon J')(-\ln k + \tfrac{1}{2}\epsilon\ln^2 k) - \tfrac{1}{6}\epsilon^2\ln^3 k + O(\epsilon^3) + O(k^2)] \ , \tag{B3}$$

where

$$J' = -\ln(2\pi^{1/2}) + \tfrac{1}{2}(0.577\,215\,7) \quad . \tag{B4}$$

Figures 7(a) and 7(b) are simply powers of $\Pi(k)$:

$$\chi_b = -um(m+1)\Pi^2(k) \quad , \tag{B5}$$

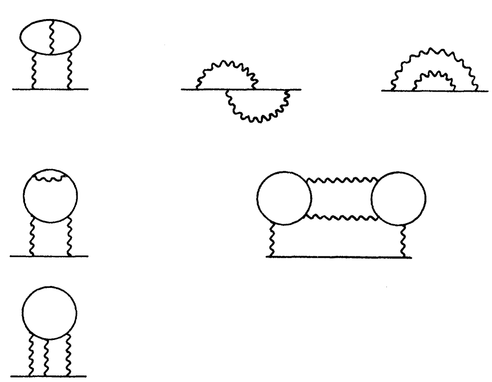

FIG. 6. Additional diagrams for calculating η to $O(n^{-2})$.

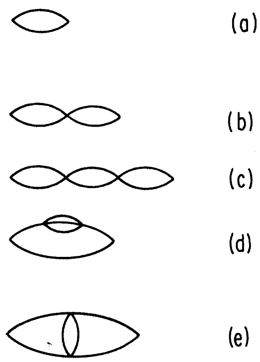

FIG. 7. Diagrams for $\chi(k)$ up to $O(\epsilon^2)$.

$$\chi_c = u^2 m(m+1)^2 \Pi^3(k) \quad . \tag{B6}$$

The quantity u to $O(\epsilon^2)$ is given by

$$u(8\pi^2)^{-1} = \frac{\epsilon}{m+4}\left[1+\epsilon\left(\frac{9m+21}{2(m+4)^2} - 1 + J'\right)\right] + O(\epsilon^3) \quad , \tag{B7}$$

which has to be obtained by calculating some other quantities. Note that our u is not identical to the $u_0(\epsilon)$ in Ref. 7, owing to the difference in the form of the cutoff. Nor is it the same as that in Ref. 8, owing to the difference in the definition of the magnitude of the interaction.

Figure 7(d) is easily calculated. The answer is

$$\chi_d = -m(8\pi^2)^{-1}\eta \ln^2 k + (\text{const})\epsilon^2 \ln k + (\text{const})\epsilon^2 \quad . \tag{B8}$$

The calculation for the diagram 7(e) is not trivial and we shall outline the steps. We view it as a product of two triangular parts:

$$\chi_e = 3m(m+1)u^2(2\pi)^{-4} \int d^4q \, T(\vec{k}, \vec{q})^2 \quad . \tag{B9}$$

For our purpose, $T(\vec{k}, \vec{q})$ is just (5.14) with $d=4$, and the cutoff for q is set to 1. We write T as an inverse Mellin transform using (5.21):

$$T(\vec{k}, \vec{q}) = q^2 T\left(\frac{\vec{k}}{q}, \frac{\vec{q}}{q}\right) = q^{-2}\int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\left(\frac{k}{q}\right)^{-s} \pi^2 \csc^2 \pi s \int d\Omega \, A(s, \Omega) \quad , \tag{B10}$$

where

$$A(s, \Omega) \equiv \sin(s-1)\phi \csc\phi \sin(s-1)\theta \csc\theta \quad . \tag{B11}$$

The angles θ, ϕ are defined by (5.18).

Substituting the square of (B10) into (B9) and integrating over q, we have

$$\chi_e = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\frac{ds'}{2\pi i} \pi^2 (\csc^2 \pi s)(\csc^2 \pi s')(s+s')^{-1} k^{-s-s'} \times \int d\Omega \, d\Omega' A(s, \Omega) A(s', \Omega') 3m(m+1)u^2(2\pi)^{-4} \quad . \tag{B12}$$

For leading terms in small k, we close the contours in s, s' planes to the left. The logarithmic terms of interest come from the fourth-order pole ($\ln^3 k$) and third-order pole ($\ln^2 k$) at the origin. The work remaining to obtain (2.26) is straight labor and no further trick is needed.

APPENDIX C

Recently, Abe and Hikami[4] observed that there are anomalous terms for the exponent α to $O(1/n)$ when

$$d = 2 + 2/l \, , \quad l = 2, 3, 4, \ldots \tag{C1}$$

in addition to those given by (2.24). This would imply that the scaling law $2 - \alpha = \nu d$ breaks down at these dimensions. It also implies that the scaling law $-\alpha = \lambda\gamma/(2-\eta)$ [see Eqs. (2.23) and (5.35)–(5.39)] also breaks down at these dimensions since we found no anomalous term in λ. In this appendix, we shall calculate α to $O(1/n)$ explicitly, and demonstrate how these anomalous terms appear. However, it seems that further study is necessary to firmly establish these important conclusions.

To lowest order in n^{-1}, χ is given by the diagrams in Fig. 3. We simply copy down (5.1). The only change is that now $k=0$ and $r \neq 0$:

$$\chi(0) = m\Pi(r, 0)[1 + mu\Pi(r, 0)]^{-1} \quad . \tag{C2}$$

From (4.5), we get

$$\Pi(r, 0) = J r^{d/2-2} \quad . \tag{C3}$$

Thus, (C2) gives

$$\chi(0) = u^{-1} - m^{-1}u^{-2}J^{-1}r^{2-d/2} + (\text{higher order in } r) \quad . \tag{C4}$$

Since $r \sim (r_0 - r_{0c})^\gamma$ and $\gamma = (\frac{1}{2}d - 1)^{-1} + O(m^{-1})$, we have

$$-\alpha = \gamma(2 - \tfrac{1}{2}d) + O(m^{-1}) = (4-d)/(d-2) + O(m^{-1}) \quad . \tag{C5}$$

Before proceeding to computing α to $O(m^{-1})$, let us see what (C5) gives if d satisfies (C1). We have, substituting (C1) in (C5),

$$-\alpha = l - 1, \quad l = 2, 3, 4, \ldots \quad . \tag{C6}$$

Since $l - 1$ is a positive integer, our conclusion is that $\chi(0)$ is not singular at these dimensions when $m \to \infty$. Since we expect that there are other nonsingular terms contributing to the specific heat, (C6) contains useful information only for $d = 3$ ($l = 2$).

The $O(m^{-1})$ term of α can be obtained by computing the next-order correction to $\chi(0)$ given by (C4). This correction is given by the same formula as (5.5):

$$\chi_1 = \Pi_1(1 + mu\Pi)^{-2} = (mu)^{-2}\Pi^{-2}\Pi_1 + \cdots \quad , \tag{C7}$$

where Π_1 is the sum of diagrams in Fig. 4. Taking advantage of the fact that

$$(r+q^2)^{-2} = -\frac{\partial}{\partial r}(r+q^2)^{-1} \ , \qquad \text{(C7a)}$$

we arrive at

$$\Pi_1(r,0) = \frac{\partial}{\partial r}\left\{m(2\pi)^{-d} \times \int d^dp\,(r+p^2)^{-2}[\Sigma_c(r,p^2)-\Sigma_c(r,0)]\right\} + m(2\pi)^{-d}\int d^dp\,(r+p^2)^{-2}\frac{\partial}{\partial r}\Sigma_c(r,0) \ , \qquad \text{(C8)}$$

where Σ_c is the self-energy given by Fig. 1(c). The first term of (C8) is just the derivative of the diagram in Fig. 1(b) with the $-u$ (dashed line) removed. The differentiation $-\partial/\partial r$ simply adds another density vertex in all possible ways and thus generates diagrams in Fig. 4. However, while the subtraction $-\Sigma_c(r,0)$ is needed in Fig. 4(a), it is not present in obtaining Figs. 4(b) and 4(c). The last term in (C8) is to take this fact into account.

The integral in the first term of (C8) has already been evaluated when we calculated Fig. 1(b) for γ. Apart from a factor $-u$, it is given by $\Sigma_b(r,0)$. Differentiating (4.24), we obtain (leaving out $-u$)

$$-mJ\xi r^{d/2-2}(\ln r + \text{const}) = 3JS_d(\tfrac{1}{2}d-1)(\ln r + \text{const}) \ . \qquad \text{(C9)}$$

The second term in (C8) contains the anomalous term of Abe and Hikami. Let us evaluate $\Sigma_c(r,0)$:

$$\Sigma_c(r,0) = m^{-1}(2\pi)^{-d}\int d^dp\,(r+p^2)^{-1}\Pi(r,p^2)^{-1} = m^{-1}\tfrac{1}{2}KrL \ , \qquad \text{(C10)}$$

$$L \equiv \int_0^{1/r} dz\, z^{d/2-1}(1+z)^{-1}\Pi(1,z)^{-1} \ . \qquad \text{(C11)}$$

To find the small-r behavior of L, we differentiate it:

$$r^{-1}\frac{\partial L}{\partial r^{-1}} = r^{-1}(1+r)^{-1}\Pi(r,1)^{-1} \ . \qquad \text{(C12)}$$

Substituting (4.9) for $\Pi(r,1)$ in (C12), we obtain

$$r^{-1}\frac{\partial L}{\partial r^{-1}} = r^{-1}(1+r^{-1})\Pi(0,1)^{-1}\{1-2(3-d)r+O(r) - A_d r^{d/2-1}[1+O(r)]\}^{-1} \ , \qquad \text{(C13)}$$

where

$$A_d \equiv 2J(\tfrac{1}{2}d-1)^{-1}\Pi(0,1)^{-1} \qquad \text{(C14)}$$

and $O(r)$ contains only integral powers of r. Expanding (C13), we get

$$r^{-1}\frac{\partial L}{\partial r^{-1}} = \Pi(0,1)^{-1}r^{-1}\left(1+(5-2d)r+O(r^2) + A_d r^{d/2-1}[1+O(r)] + \sum_{j=2}^{\infty} A_d^j r^{(d/2-1)j}[1+O(r)]\right) \ . \qquad \text{(C15)}$$

We then integrate (C15) to obtain L and substitute it in (C10) to obtain $\Sigma_c(r,0)$. Let us ignore the last sum in (C15) for the moment and get

$$\Sigma_c(r,0) = (\text{const}) + (\text{const})r^{d/2-1} - (K/2m)\Pi(0,1)^{-1}(5-2d)r(\ln r + \text{const}) \ . \qquad \text{(C16)}$$

Substituting (C9) and (C16) in (C8), we get Π_1. From (C7), we then get χ_1. Combining the result with (C4), we obtain

$$\chi(0) \sim (\text{const}) - r^{2-d/2}[1+m^{-1}S_d(8-\tfrac{7}{2}d)(\ln r + \text{const})] + (\text{higher orders of } r) + O(m^{-2}) \ . \qquad \text{(C17)}$$

The exponent α is thus

$$-\alpha = \gamma\left[2-\frac{d}{2}+m^{-1}S_d\left(8-\frac{7}{2}d\right)\right] + O(m^{-2}) = \left(\frac{4-d}{d-2}\right)\left(1+\frac{4(1-d)}{m(4-d)}S_d + O(m^{-2})\right) \ , \qquad \text{(C18)}$$

where (4.26) is used for γ. Equation (C18) is just (5.40) and thereby confirms the scaling law [Eq. (5.35)] provided the last sum in (C15) may be ignored. What happens if we do not ignore it? If none of the terms in the sum is proportional to r, i.e., $(\frac{1}{2}d-1)j \neq 1$ for all j, we get a contribution to $\chi(0)$ proportional to

$$r^{2-d/2}\sum_{j=2}^{\infty}\frac{A_d^j}{(\frac{1}{2}d-1)j-1}r^{(d/2-1)j-1}[1+O(r)] \ . \qquad \text{(C19)}$$

Since $r \propto (r_0 - r_{0c})^\gamma$, $\gamma = (\frac{1}{2}d-1)^{-1}+O(m^{-1})$, (C19) is

$$\sum_{j=2}^{\infty}[(\tfrac{1}{2}d-1)j-1]^{-1}A_d^j(r_0-r_{0c})^{j-1}[1+O(r)] \ , \qquad \text{(C20)}$$

which is just a power series in $r_0 - r_{0c}$ apart from the last factor, and the result (C18) is not affected.

If $(\frac{1}{2}d-1)l = 1$ for some integer $l \geq 2$, i.e., for

$$d = 3, \tfrac{8}{3}, \tfrac{5}{2}, \tfrac{12}{5}, \ldots, \qquad \text{(C21)}$$

the term $j = l$ in the sum would contribute to $\chi(0)$ the term

$$\chi_A \equiv -JS_d A_d^l(\ln r + \text{const})r^{2-d/2} \propto (r_0-r_{0c})^{l-1}(\ln(r_0-r_{0c})+\text{const}) \ , \qquad \text{(C22)}$$

plus integral powers of $(r_0 - r_{0c})$. If this logarithmic term is taken as the manifestation of an $O(1/m)$ correction to the exponent $l-1$, then an anomalous term in α appears. In addition to (C18), we have

$$(-\alpha)_{\text{anomalous}} = (S_d/m)A_d^l = \frac{\sin(\pi/l)(2l)^l}{m(\pi/l)B(1/l,1/l)^{l+1}} \ , \quad d = 2(1+1/l) \ . \qquad \text{(C23)}$$

This is the content of Eq. (6) of Abe and Hikami.[4]

Since no anomalous term was found in λ and (C18) is consistent with the scaling laws $-\alpha=\lambda\gamma/(2-\eta)$ and $2-\alpha=\nu d$, (C23) implies the breakdown of these scaling laws at these dimensions. On the other hand, we have seen earlier that the $m\to\infty$ limit of the above calculation simply implies that there is no singularity in specific heat at these particular dimensions. It is not impossible that (C22) simply indicates a logarithmic singularity rather than a correction to the exponent. Such a logarithmic singularity may be a result of a singularity in coefficients [as (C20) indicates] as functions of d in the infinite-n limit.[18] This point has to be verified or disproved by further study.

Finally, the reader may ask whether a similar anomaly occurs in $\Gamma_{2s}(0)$ (see Sec. VI), since Fig. 5, which gives $\Gamma_{2s}(0)$, does contain the diagrams in Fig. 4 as a part in Fig. 5(b). The answer is that such an anomaly does occur in the terms in Fig. 5(b) and *also* in Fig. 5(c). However, when we add all diagrams in Fig. 5, the anomaly disappears. The scaling law (6.12) can be explicitly verified.

In the case of long-range interactions of the form k^σ, the above discussion goes through in the same fashion. Abe and Hikami also worked on this case.

---

*Alfred P. Sloan Foundation Fellow.

[1]I learned of this idea from K. G. Wilson.

[2]S. Ma, Phys. Rev. Lett. **29**, 1311 (1972).

[3]M. E. Fisher, S. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).

[4]R. Abe, Prog. Theor. Phys. **48**, 1414 (1972); Prog. Theor. Phys. **49**, No. 1 (1973); S. Hikami, Prog. Theor. Phys. (to be published); R. Abe and S. Hikami, Prog. Theor. Phys. (to be published); Phys. Lett. (to be published). I thank Professor Abe for sending me preprints.

[5]M. Suzuki, Phys. Lett. **42A**, 5 (1972); Prog. Theor. Phys. (to be published); M. Suzuki and G. Igarashi, Prog. Theor. Phys. (to be published). I thank Professor Suzuki for sending me preprints.

[6]R. A. Ferrell and D. J. Scalapino, Phys. Rev. Lett. **29**, 413 (1972); and University of California at Santa Barbara, 1972 (report of work prior to publication).

[7]E. Brezin and D. Wallace, Princeton University (report of work prior to publication).

[8]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972); K. G. Wilson, Phys. Rev. Lett. **28**, 548 (1972); E. Brezin, D. Wallace, and K. G. Wilson, Phys. Rev. Lett. **29**, 591 (1972); and unpublished ; K. G. Wilson and J. Kogut, Phys. Report (to be published).

[9]B. G. Nickel, Cornell University (report of work prior to publication).

[10]H. B. Dwight, *Tables of Integrals and Other Mathematical Data*, , 4th ed. (MacMillan, New York, 1967), pp. 36 and 215.

[11]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 15.

[12]A. Erdélyi *et al.*, *Table of Integral Transforms*, Vol. I (McGraw-Hill, New York, 1954), Chap. VI.

[13]See Ref. 12, formula 12, p. 309.

[14]B. I. Halperin and P. C. Hohenberg, Phys. Rev. **177**, 952 (1969).

[15]R. Abe (unpublished) has calculated η to $O(n^{-2})$ for $d=3$.

[16]S. Ma (unpublished).

[17]I thank Dr. B. G. Nickel for pointing out this error in Ref. 12.

[18]K. G. Wilson (private communication). F. J. Wegner pointed out that, when $2-\alpha=$ integer, a logarithmic term in $T-T_c$ can appear. See Phys. Rev. B 5, 4529 (1972), Sec. IV. I thank Professor Wilson and Professor Wegner for brief but informative communications on this subject.

PHYSICAL REVIEW A VOLUME 7, NUMBER 6 JUNE 1973

Superfluidity in a Ring*

F. Bloch

Department of Physics, Stanford University, Stanford, California 94305

(Received 5 February 1973)

The persistent flow of a superfluid in a ring is discussed in terms analogous to those previously used for superconductors. The existence of a phase memory around the ring is shown to be responsible for energy minima with a periodic dependence on the total momentum which is directly related to the quantization of circulation. The general features are illustrated by means of the ideal Bose gas and the model of quasiparticles as examples.

An ideal condensed Bose gas in a rotating container exhibits a series of equilibrium states, characteristic of a superfluid,[1] with the successive entry of vortices responsible for their formation.[2] In the case of an arbitrarily interacting Bose system, one is led to the related but more general conclusion that the angular velocity of the system varies periodically with that of the container.[3] The arguments for this conclusion are analogous to those applied in a general interpretation of the Josephson effect[4] and similar arguments will be used here to discuss persistent flow and the quantization of circulation in a container at rest as a counterpart to the discussion of persistent currents and flux quantization[5,6] in a superconducting ring. Simplifying assumptions will be made wherever