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Dissipative Coefficients of Superfluid Helium†

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The linearized hydrodynamic equations for superfluid He^4 are derived starting from the kinetic equations of Khalatnikov. Only the lowest-order dissipative terms are considered. The Boltzmann equation for the phonons and rotons is solved formally by expanding the distribution function in terms of the eigenfunctions of the linearized collision operator. The resulting hydrodynamic equations contain dissipative terms of the form proposed by Landau and Khalatnikov. Formal expressions are given for the viscosities and the thermal kinetic coefficient. Finally, relations which may hold between the dissipative coefficients when the collision operator has special properties are discussed.

I. INTRODUCTION

The first satisfactory explanation of the superfluid properties of liquid He^4 below the lambda point was provided by Landau[1] in a classic paper in 1941. By introducing the concept of elementary excitations, he was able to give a microscopic derivation of the two-fluid model and to clarify the physical meaning of the model. The equations of motion for the two-fluid model are

$$\vec{j}=\rho_n\vec{v}_n+\rho_s\vec{v}_s \;, \tag{1}$$

$$\rho=\rho_n+\rho_s \;, \tag{2}$$

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\vec{j}=0 \;, \tag{3}$$

$$\frac{\partial\vec{j}}{\partial t}+\operatorname{grad}P=0 \;, \tag{4}$$

$$\frac{\partial S}{\partial t}+S\operatorname{div}\vec{v}_n=0 \;, \tag{5}$$

$$\frac{\partial\vec{v}_s}{\partial t}+\operatorname{grad}G=0 \;, \tag{6}$$

where ρ_n, ρ_s, $\vec{v}_n$, and $\vec{v}_s$ are respectively the densities and velocities of the normal and superfluid components of the liquid, $\vec{j}$ is the momentum density, P is the pressure, G is the Gibbs free energy per unit mass, and S is the entropy per unit mass. These equations have been linearized by neglecting terms of second or higher order in $\vec{v}_n$ or $\vec{v}_s$. Equations (3) and (4) follow from conservation of mass and from Newton's second law, respectively. Equation (5) is equivalent to the statement that only the normal fluid carries entropy. Equations (4) and (6) together provide a means for calculating the relative acceleration of the two fluids. The two-fluid theory is completed by giving expressions relating the thermodynamic functions to the energy spectrum of the elementary excitations. These expressions are[2,3]

$$\rho_n=\frac{\beta}{3}\int n_p^0(n_p^0+1)p^2\,d\tau_p \;, \tag{7}$$

$$G=G_0+\int n_p^0\,\frac{\partial\epsilon}{\partial\rho}\,d\tau_p \;, \tag{8}$$

$$F=F_0-\frac{1}{3\rho}\int n_p^0\,\frac{\partial\epsilon}{\partial p}\,p\,d\tau_p \;, \tag{9}$$

$$S=\frac{\beta}{3T\rho}\int n_p^0(n_p^0+1)\epsilon\,\frac{\partial\epsilon}{\partial p}\,p\,d\tau_p \;, \tag{10}$$

$$P=\rho^2\left(\frac{\partial F}{\partial\rho}\right)_T \;, \tag{11}$$

where $\beta=1/k_BT$, ϵ is the energy of an excitation of momentum $\vec{p}$, F is the Helmholtz free energy per unit mass, F_0 and G_0 are the values of F and G, respectively, at zero temperature and the same density, and n_p^0 is the equilibrium distribution function of excitations at temperature T:

$$n_p^0=(e^{\epsilon/k_BT}-1)^{-1} \;. \tag{12}$$

The integrals are over all of momentum space. The thermodynamic quantities may be calculated as functions of ρ and T using Eqs. (7)–(11) together with the dispersion relation for the elementary excitations as measured by neutron scattering. At temperatures less than about 1.7 °K the excitations are weakly interacting and the results are in excellent agreement with experiment.[3] The two-fluid model also explains such phenomena as the existence of two types of sound and the thermomechanical or "fountain" effect.

The two-fluid equations are derived under the assumption that at each point in the liquid the distribution of excitations is in local statistical equilibrium. This assumption is valid only if the field quantities such as T, $\vec{v}_n$, $\vec{v}_s$, and ρ are very slowly varying in space over distances of the order of the mean free path of the excitations and slowly varying in time on the scale of the mean free time of the excitations. For more rapidly varying disturbances the distribution of excitations no longer corresponds to exact local equilibrium and extra terms appear in the two-fluid equations.[2-12] Landau and Khalatnikov[2,4-10] have shown that to lowest order in the space and time variation of the field quantities Eqs. (4)–(6) are modified as follows:

$$\frac{\partial j_\alpha}{\partial t}+\frac{\partial P}{\partial X_\alpha}=\frac{\partial}{\partial X_\beta}\left[\eta\left(\frac{\partial v_{n\alpha}}{\partial X_\beta}+\frac{\partial v_{n\beta}}{\partial X_\alpha}-\tfrac{2}{3}\delta_{\alpha\beta}\frac{\partial v_{n\gamma}}{\partial X_\gamma}\right)+\delta_{\alpha\beta}\zeta_1\operatorname{div}(\vec{j}-\rho\vec{v}_n)+\delta_{\alpha\beta}\zeta_2\operatorname{div}\vec{v}_n\right] , \tag{13}$$

$$\frac{\partial S}{\partial t}+S\operatorname{div}\vec{v}_n=\frac{\kappa}{T}\operatorname{div}\operatorname{grad}T , \tag{14}$$

$$\frac{\partial \vec{v}_s}{\partial t}+\operatorname{grad}G=\operatorname{grad}[\zeta_3\operatorname{div}(\vec{j}-\rho\vec{v}_n)+\zeta_4\operatorname{div}\vec{v}_n] . \tag{15}$$

In these equations η is the normal fluid viscosity, ζ_1, ζ_2, ζ_3, and ζ_4 are second viscosities,[13] and κ is the thermal kinetic coefficient. We use the convention that repeated Greek subscripts are summed over. ζ_1 and ζ_4 are equal by an Onsager relation.[2,6] It can be shown that these dissipative coefficients satisfy the conditions[2]

$$\kappa > 0 , \tag{16}$$

$$\eta > 0 , \tag{17}$$

$$\zeta_2 > 0 , \tag{18}$$

$$\zeta_3 > 0 , \tag{19}$$

$$\zeta_2\zeta_3 \geq \zeta_1^2 . \tag{20}$$

Landau and Khalatnikov have made detailed calculations of these coefficients. They consider the various possible collision processes amongst the excitations and derive approximate expressions for the lifetime of the different excitations. Further approximations are used in solving the resulting Boltzmann equation for the excitations. They obtained reasonably good agreement with experiment. It is *extremely* hard to judge the correctness of the intermediate steps in their theory, particularly as there are a number of adjustable parameters involved.[14] Specific details of their calculations have been criticized by a number of authors.[15–20] It has been proposed[21,22] that the hydrodynamic equations should be modified by adding extra dissipative terms on the right-hand sides of Eqs. (13) and (15). These terms are proportional to

$$\operatorname{grad}\Gamma ,$$

where Γ is the rate at which normal fluid is converted into superfluid, and is given by

$$\frac{\partial \rho_n}{\partial t}+\rho_n\operatorname{div}\vec{v}_n=\Gamma , \tag{21}$$

$$\frac{\partial \rho_s}{\partial t}+\rho_s\operatorname{div}\vec{v}_s=-\Gamma . \tag{22}$$

On the other hand, it has also been proposed[23] that there may be extra relations between the dissipative coefficients, in addition to the Onsager relation between ζ_1 and ζ_4. These relations would only hold if one makes simplifying assumptions about the collision rates between the excitations.

In this paper we investigate the possibility that there are either additional dissipative coefficients or relations between the Landau–Khalatnikov coefficients. We proceed by expanding the distribution function for the excitations in terms of the eigenfunctions of the collision operator (Sec. II). In the present context this approach has the great advantage of leading to formally exact results for the dissipative coefficients—thus enabling any relations between them to become readily apparent. Any additional terms in the two-fluid equations should also be exposed by this procedure (Sec. III). Finally, we may easily find relations which should hold between the dissipative coefficients, when the collision processes amongst the excitations satisfy certain special conditions. Examples of this procedure are given in Sec. IV.

II. KINETIC EQUATIONS

Khalatnikov has derived equations of motion for a superfluid containing a distribution n_p of excitations.[2] The equations are

$$\vec{j}=\rho\vec{v}_s+\int n_p\vec{p}\,d\tau_p , \tag{23}$$

$$\frac{\partial \rho}{\partial t}+\operatorname{div}\vec{j}=0 , \tag{24}$$

$$\frac{\partial \vec{v}_s}{\partial t}+\operatorname{grad}\left(G_0+\int n_p\frac{\partial \epsilon}{\partial \rho}\,d\tau_p\right)=0 . \tag{25}$$

Equation (23) expresses the fact that the total momentum density is the momentum density arising from the motion of the superfluid plus the momentum of the excitations. Equation (24) arises from conservation of mass. Equation (25) may be regarded as a natural generalization of Eq. (6), to situations where the excitations are not necessarily described by an equilibrium distribution function. The rate of change of the distribution function with time is given by the Boltzmann equation[2]:

$$\frac{\partial n_p}{\partial t}=\left(\frac{\partial n_p}{\partial t}\right)_{\text{coll}}+\frac{\partial n_p}{\partial \vec{p}}\cdot\frac{\partial H_p}{\partial \vec{X}}-\frac{\partial H_p}{\partial \vec{p}}\cdot\frac{\partial n_p}{\partial \vec{X}} . \tag{26}$$

The first term on the right-hand side is the contribution to the rate of change of n_p arising from collision between excitations. H_p is the Hamiltonian for an excitation with momentum $\vec{p}$ and is given by

$$H_p=\epsilon+\vec{p}\cdot\vec{v}_s . \tag{27}$$

Note that H_p depends upon position $\vec{X}$, not only because of the spatial variation of $\vec{v}_s$, but also because ϵ depends on the density. We may linearize the Boltzmann equation by setting

$$n_p=\{e^{(\epsilon+\vec{p}\cdot\vec{v}_s)/k_B\bar{T}}-1\}^{-1}+\Delta n_p , \tag{28}$$

where $\bar{T}$ is a constant temperature chosen so that Δn_p is a small quantity everywhere in the fluid. We also introduce a density $\bar{\rho}$ such that $\Delta\rho$, defined by

$$\Delta\rho = \rho - \bar{\rho} \quad , \tag{29}$$

is always small. Let the energy of an excitation of momentum $\vec{p}$ be $\bar{\epsilon}$ when the density is $\bar{\rho}$. We will denote the value that a function has when the density is $\bar{\rho}$ and the temperature is $\bar{T}$ by adding a bar to it. Thus,

$$\bar{n}_p \equiv (e^{\bar{\epsilon}/k_B\bar{T}} - 1)^{-1} \quad . \tag{30}$$

Then, to first order in $\Delta\rho$ and $\vec{v}_s$,

$$n_p = \bar{n}_p - \bar{\beta}\bar{n}_p(\bar{n}_p + 1)\left(\frac{\partial\bar{\epsilon}}{\partial\bar{\rho}}\,\Delta\rho + \vec{p}\cdot\vec{v}_s\right) + \Delta n_p \quad . \tag{31}$$

Inserting this result into the Boltzmann equation gives

$$\frac{\partial\Delta n_p}{\partial t} = \frac{\partial n_p}{\partial t} + \bar{\beta}\bar{n}_p(\bar{n}_p + 1)\left[\frac{\partial\bar{\epsilon}}{\partial\bar{\rho}}\,\frac{\partial\rho}{\partial t} + p_\alpha\,\frac{\partial v_{s\alpha}}{\partial t}\right] - v_{p\alpha}\,\frac{\partial\Delta n_p}{\partial X_\alpha} \quad , \tag{32}$$

where $v_{p\alpha}$ is the α component of the group velocity for a particle with momentum $\vec{p}$. In a linear theory the collision term may be written

$$\left(\frac{\partial n_p}{\partial t}\right)_{\text{coll}} = \int C(\vec{p}, \vec{p}\,')\Delta n_{p'}\,d\tau_{p'} \quad . \tag{33}$$

As defined this way, the kernel of the collision integral is not symmetric. It can be shown, however, that the function[24]

$$\tilde{C}(\vec{p}, \vec{p}\,') \equiv C(\vec{p}, \vec{p}\,')[\bar{n}_{p'}(\bar{n}_{p'} + 1)/\bar{n}_p(\bar{n}_p + 1)]^{1/2} \tag{34}$$

is symmetric in $\vec{p}$ and $\vec{p}\,'$. We may then define eigenfunctions $\psi_i(\vec{p})$ and eigenvalues λ_i by

$$\int \tilde{C}(\vec{p}, \vec{p}\,')\psi_i(\vec{p}\,')\,d\tau_{p'} = -\lambda_i\psi_i(\vec{p}) \quad . \tag{35}$$

Since $\tilde{C}(\vec{p}, \vec{p}\,')$ is real and symmetric, the eigenfunctions can be chosen to be orthonormal:

$$\int \psi_i(\vec{p})\psi_j(\vec{p})\,d\tau_p = \delta_{ij} \quad . \tag{36}$$

The total energy and the total momentum of the excitations cannot change as a result of collisions. Hence, for example, we have

$$\int\left(\frac{\partial n_p}{\partial t}\right)_{\text{coll}}\bar{\epsilon}\,d\tau_p = 0 \quad .$$

Using Eqs. (33) and (34), this becomes

$$\int\int \tilde{C}(\vec{p}, \vec{p}\,')[\bar{n}_p(\bar{n}_p + 1)/\bar{n}_{p'}(\bar{n}_{p'} + 1)]^{1/2} \times \bar{\epsilon}\Delta n_{p'}\,d\tau_p\,d\tau_{p'} = 0 \quad .$$

This must be true for all possible $\Delta n_{p'}$. Hence

$$\int \tilde{C}(\vec{p}, \vec{p}\,')\bar{n}_p^{1/2}(\bar{n}_p + 1)^{1/2}\bar{\epsilon}\,d\tau_p = 0 \quad .$$

Thus, $\bar{\epsilon}\bar{n}_p^{1/2}(\bar{n}_p + 1)^{1/2}$ is an eigenfunction of the collision operator with zero eigenvalue. We denote this eigenfunction by $\psi_0(\vec{p})$. It is straightforward to show that the correctly normalized form is

$$\psi_0(\vec{p}) = (\bar{\beta}/\bar{C}\bar{\rho}\,\bar{T})^{1/2}\,\bar{\epsilon}\bar{n}_p^{1/2}(\bar{n}_p + 1)^{1/2} \quad . \tag{37}$$

$\bar{C}$ is the specific heat per unit mass at temperature $\bar{T}$ and density $\bar{\rho}$. Since momentum is conserved, there are three more zero-eigenvalue eigenfunctions. The correctly normalized form for these is

$$\psi_\alpha(\vec{p}) = (\bar{\beta}/\bar{\rho}_n)^{1/2}p_\alpha\bar{n}_p^{1/2}(\bar{n}_p + 1)^{1/2} \quad , \tag{38}$$

where α may be 1, 2, or 3, corresponding to the three components of momentum. We now expand Δn_p as

$$\Delta n_p = \bar{n}_p^{1/2}(\bar{n}_p + 1)^{1/2}\sum_i A_i\psi_i(\vec{p}) \quad , \tag{39}$$

where $\{A_i\}$ are coefficients varying in space and time. If we insert this into the Boltzmann equation, multiply through by $\psi_j(\vec{p})$, and integrate over momentum space, we find

$$\dot{A}_j = -\lambda_j A_j + \left(\frac{\bar{C}\bar{T}\bar{\beta}}{\bar{\rho}}\right)^{1/2}\langle j|u_p|0\rangle\,\frac{\partial\rho}{\partial t} + (\bar{\rho}_n\bar{\beta})^{1/2}\delta_{\alpha j}\,\frac{\partial v_{s\alpha}}{\partial t} - \sum_i\langle j|v_{p\alpha}|i\rangle\,\frac{\partial A_i}{\partial X_\alpha} \quad , \tag{40}$$

where $\bar{u}_p$ is the Grüneisen constant, defined by

$$\bar{u}_p = \frac{\bar{\rho}}{\bar{\epsilon}}\,\frac{\partial\bar{\epsilon}}{\partial\bar{\rho}} \tag{41}$$

and

$$\bar{v}_{p\alpha} = \frac{\partial\bar{\epsilon}}{\partial p_\alpha} \quad . \tag{42}$$

The matrix elements in Eq. (40) are defined in the conventional way. For example,

$$\langle i|\bar{v}_{p\alpha}|j\rangle = \int \psi_i(\vec{p})\bar{v}_{p\alpha}\psi_j(\vec{p})\,d\tau_p \quad . \tag{43}$$

The coefficients of the zero-eigenvalue eigenfunctions can be related to macroscopic quantities. Consider, for simplicity, a situation in which the density of the liquid is $\bar{\rho}$ everywhere. From Eq. (23) the α component of the momentum density is

$$j_\alpha = \rho\,v_{s\alpha} + \int n_p p_\alpha\,d\tau_p \quad .$$

Using Eqs. (38) and (39) and the orthogonality of the eigenfunctions, this may be put in the form

$$j_\alpha = \rho_s v_{s\alpha} + (\bar{\rho}_n/\bar{\beta})^{1/2}A_\alpha \quad , \tag{44}$$

where A_α is the expansion coefficient for the eigenfunction $\psi_\alpha(\vec{p})$ given by Eq. (38). By comparing this with the corresponding expression for the two-fluid model, Eq. (1), we see that the coefficients of the zero-eigenvalue eigenfunctions arising from momentum conservation are connected with the velocity of the normal fluid by

$$A_\alpha = (\bar\beta \bar\rho_n)^{1/2} v_{n\alpha} \ , \quad \alpha = 1, 2, 3 \ . \tag{45}$$

Similarly, one can show that A_0 is related to the local temperature T by

$$A_0 = (\bar C \bar\rho \bar\beta / \bar T)^{1/2} (T - \bar T) \ . \tag{46}$$

If we now take the equation for the acceleration of the superfluid, Eq. (25), and substitute into it n_p, Eqs. (31) and (39), we find

$$\frac{\partial v_{S\alpha}}{\partial t} + \frac{\partial G}{\partial X_\alpha} = -\left(\frac{\bar C \bar T}{\bar\rho \bar\beta}\right)^{1/2} \sum_i{}' \langle 0 | u_p | i \rangle \frac{\partial A_i}{\partial X_\alpha} \ . \tag{47}$$

The prime on the summation indicates that the sum is restricted to eigenfunctions with nonzero eigenvalues. If we now set $j = 0$ in Eq. (40), we obtain, after simplification,

$$\frac{\partial S}{\partial t} + \bar S \operatorname{div} \vec v_n = -\left(\frac{\bar C}{\bar\beta \bar\rho \bar T}\right)^{1/2} \sum_i{}' \langle 0 | v_{p\alpha} | i \rangle \frac{\partial A_i}{\partial X_\alpha} \ . \tag{48}$$

Similarly, by setting $j = \beta$ in Eq. (40), where $\beta = 1$, 2, or 3, we find

$$\bar\rho_n \frac{\partial}{\partial t} (v_{n\beta} - v_{S\beta}) = -\bar S \frac{\partial T}{\partial X_\beta} - \left(\frac{\bar\rho_n}{\bar\beta}\right)^{1/2} \times \sum_i{}' \langle \beta | v_{p\alpha} | i \rangle \frac{\partial A_i}{\partial X_\alpha} \ . \tag{49}$$

If we combine this result with Eq. (47) and use some thermodynamic relations, we obtain

$$\frac{\partial j_\alpha}{\partial t} + \frac{\partial P}{\partial X_\alpha} = -\left(\frac{\bar C \bar\rho \bar T}{\bar\beta}\right)^{1/2} \sum_i{}' \langle 0 | u_p | i \rangle \frac{\partial A_i}{\partial X_\alpha} - \left(\frac{\rho_n}{\bar\beta}\right)^{1/2} \sum_i{}' \langle \alpha | v_{p\beta} | i \rangle \frac{\partial A_i}{\partial X_\beta} \ . \tag{50}$$

III. VISCOSITY COEFFICIENTS

Consider the solution of the Boltzmann equation (40) when the density and the superfluid velocity are varying slowly in space and time. For nonzero λ_j we may write Eq. (40) as

$$A_j = -\frac{\dot A_j}{\lambda_j} + \frac{1}{\lambda_j}\left(\frac{\bar C \bar T \bar\beta}{\bar\rho}\right)^{1/2} \langle j | \bar u_p | 0 \rangle \frac{\partial \rho}{\partial t} - \frac{1}{\lambda_j} \sum_j{}' \langle j | \bar v_{p\alpha} | i \rangle \frac{\partial A_i}{\partial X_\alpha} - \frac{(\bar\beta \bar\rho_n)^{1/2}}{\lambda_j} \langle j | \bar v_{p\alpha} | \beta \rangle \frac{\partial v_{n\beta}}{\partial X_\alpha} - \frac{1}{\lambda_j}\left(\frac{\bar C \bar\rho \bar\beta}{\bar T}\right)^{1/2} \langle j | \bar v_{p\alpha} | 0 \rangle \frac{\partial T}{\partial X_\alpha} \ . \tag{51}$$

This equation may be solved by iteration. The result is a series expansion for A_j involving increasing powers of λ_j^{-1} multiplied by higher-order space or time derivatives of ρ, $\vec v_n$, or T. When these are slowly varying quantities, the series will converge rapidly. The lowest-order result is

$$A_j = \frac{1}{\lambda_j}\left(\frac{\bar C \bar T \bar\beta}{\bar\rho}\right)^{1/2} \langle j | u_p | 0 \rangle \frac{\partial \rho}{\partial t} - \frac{(\bar\beta \bar\rho_n)^{1/2}}{\lambda_j} \langle j | \bar v_{p\alpha} | \beta \rangle \frac{\partial v_{n\beta}}{\partial X_\alpha} - \frac{1}{\lambda_j}\left(\frac{\bar C \bar\rho \bar\beta}{\bar T}\right)^{1/2} \langle j | \bar v_{p\alpha} | 0 \rangle \frac{\partial T}{\partial X_\alpha} \ .$$

If we substitute this expression into Eqs. (47), (48), and (50), we find exactly the Landau–Khalatnikov results, Eq. (13)–(15), provided that we make the identifications

$$\zeta_1 = \zeta_4 = \left(\frac{\bar C \bar T \bar\rho_n}{\bar\rho}\right)^{1/2} \Sigma_4 + \bar C \bar T \Sigma_5 \ , \tag{52}$$

$$\zeta_2 = \tfrac{1}{3} \bar\rho_n (\Sigma_2 + 2\Sigma_3) + 2(\bar C \bar T \bar\rho \rho_n)^{1/2} \Sigma_4 + \bar C \bar\rho \bar T \Sigma_5 \ , \tag{53}$$

$$\zeta_3 = (\bar C \bar T / \bar\rho) \Sigma_5 \ , \tag{54}$$

$$\eta = \tfrac{1}{2} \bar\rho_n (\Sigma_2 - \Sigma_3) \ , \tag{55}$$

$$\kappa = \bar C \bar\rho \Sigma_1 \ , \tag{56}$$

where

$$\Sigma_1 = \sum_j{}' \frac{\langle 0 | \bar v_{px} | j \rangle^2}{\lambda_j} \ , \tag{57}$$

$$\Sigma_2 = \sum_j{}' \frac{\langle x | \bar v_{px} | j \rangle^2}{\lambda_j} \ , \tag{58}$$

$$\Sigma_3 = \sum_j{}' \frac{\langle x | \bar v_{px} | j \rangle \langle y | \bar v_{py} | j \rangle}{\lambda_j} \ , \tag{59}$$

$$\Sigma_4 = \sum_j{}' \frac{\langle 0 | u_p | j \rangle \langle x | \bar v_{px} | j \rangle}{\lambda_j} \ , \tag{60}$$

$$\Sigma_5 = \sum_j{}' \frac{\langle 0 | u_p | j \rangle^2}{\lambda_j} \ . \tag{61}$$

Here $\langle x|$ and $\langle y|$ denote the eigenfunctions arising from conservation of momentum in the x and y directions, and $\bar v_{px}$ and $\bar v_{py}$ are the components of the group velocity in these directions for an excitation with momentum $\vec p$.

The collision operator $\tilde C(\vec p, \vec p\,')$ only depends on the magnitudes of $\vec p$ and $\vec p\,'$, and on the angle between these vectors. It follows that the angular parts of the eigenfunctions are spherical harmonics. We adopt a spectroscopic notation in which the eigenfunction arising from energy conservation is called the $1S$ state and the momentum conservation eigenfunctions are $1P$ states. These eigenfunctions have no radial nodes. From the form of the matrix elements involved in the sums Σ_n, it can be seen that only S, P, and D states contribute. Let $S_n(p)$, $P_n(p)$, and $D_n(p)$ be the radial parts of the eigenfunctions for the nth S, P, or D states, respectively, and let the eigenvalues of these states be λ_{nS}, λ_{nP}, λ_{nD}. Then it is straightforward to show that

$$\zeta_1 = \tfrac{1}{2}\,\bar{C}\bar{T}(\sigma_4 + \sigma_5 - \sigma_3)\ , \tag{62}$$

$$\zeta_2 = \bar{C}\bar{\rho}\bar{T}\sigma_4\ , \tag{63}$$

$$\zeta_3 = (\bar{C}\,\bar{T}/\bar{\rho})\sigma_5\ , \tag{64}$$

$$\eta = \tfrac{3}{5}\,\bar{C}\bar{\rho}\bar{T}\sigma_2\ , \tag{65}$$

$$\kappa = (\bar{C}^2\bar{\rho}^2\bar{T}/\bar{\rho}_n)\sigma_1\ , \tag{66}$$

where

$$\sigma_1 = \sum_{nP}{}' \frac{[P_1|\tfrac{1}{3}(\bar{\epsilon}/p)(\partial\bar{\epsilon}/\partial p)|P_n]^2[P_1|\bar{\epsilon}^2/p^2|P_1]^{-2}}{\lambda_{nP}}\ , \tag{67}$$

$$\sigma_2 = \sum_{nD}{}' \frac{[S_1|\tfrac{1}{3}(p/\bar{\epsilon})(\partial\bar{\epsilon}/\partial p)|D_n]^2}{\lambda_{nD}}\ , \tag{68}$$

$$\sigma_3 = \sum_{nS}{}' \frac{[S_1|\tfrac{1}{3}(p/\bar{\epsilon})(\partial\bar{\epsilon}/\partial p)|S_n]^2}{\lambda_{nS}}\ , \tag{69}$$

$$\sigma_4 = \sum_{nS}{}' \frac{[S_1|\bar{u}_p + \tfrac{1}{3}(p/\bar{\epsilon})(\partial\bar{\epsilon}/\partial p)|S_n]^2}{\lambda_{nS}}\ , \tag{70}$$

$$\sigma_5 = \sum_{nS}{}' \frac{[S_1|\bar{u}_p|S_n]^2}{\lambda_{nS}}\ . \tag{71}$$

A sum over nP means that the sum is only over P states. The matrix elements are now radial integrals only. Thus, for example,

$$[P_1|f(p)|P_n] = h^{-3}\int_0^\infty P_1(p)f(p)P_n(p)p^2\,dp\ .$$

Notice that

$$[P_1|\bar{\epsilon}^2/p^2|P_1] = \bar{C}\bar{\rho}\,\bar{T}/3\bar{\rho}_n\ .$$

The sums σ_n as defined by Eqs. (67)–(71) all have the dimensions of time.

IV. RELATIONS BETWEEN VISCOSITIES

It follows from the definitions of the σ_n sums that they are positive definite. Therefore, ζ_2, ζ_3, η, and κ are all greater than or equal to zero. It is also easy to show that $\zeta_2\zeta_3 \geq \zeta_1^2$. Thus, we have rederived the Landau–Khalatnikov inequalities Eqs. (16)–(20).

It is clear that unless there is a very well-hidden relation between the matrix elements, there are no exact relations between the five dissipative coefficients. We now consider relations which may exist between the dissipative coefficients when the collision integral has special properties. In Sec. II it was shown that there exist four independent eigenfunctions with zero eigenvalues because energy and momentum are conserved in collisions between excitations. One therefore expects that there will be eigenfunctions with anomalously small eigenvalues if there are quantities that are "nearly conserved" in collisions between excitations. These eigenfunctions will then dominate the sums σ_n and possibly lead to approximate relations between the viscosities.

To decide what quantities are "nearly conserved" in a particular temperature range requires detailed quantitative calculations of the collision kernel $C(\vec{p}, \vec{p}')$. The only attempt that has been made so far in this direction is the calculation of Landau and Khalatnikov.[4–12] Unfortunately, their results must now be regarded as suspect on several grounds. Three important criticisms of their calculation are as follows:

(a) They assumed a δ-function interaction between rotons, thus obtaining s-wave scattering only. It is now known that the actual interaction has a considerable d-wave contribution and is attractive, leading to bound states.[16,17,19,20]

(b) The phonon and roton parts of the distribution were assumed to come to equilbrium by "one-step" four-particle processes, either

$$P + R \rightleftharpoons P' + R'$$

or

$$P + R \rightleftharpoons R' + R''\ ,$$

where P and R denote phonon and roton, respectively. Donnelly and Roberts[15] have proposed that it is more likely that equilibrium is achieved by a sequence of collisions enabling a roton to "diffuse" over the maximum in the dispersion curve which separates it from the phonons.

(c) Landau and Khalatnikov thought that the five-phonon process was an important mechanism for changing the number of phonons. It is now known that the three-phonon process is allowed[25] (at least for small momenta) and this constitutes a far faster mechanism for changing the number of phonons than the five-phonon process.

We therefore restrict attention to giving some examples of relations that should exist between the viscosities if certain quantities happen to be nearly conserved.

A. Small-Angle Collisions

If all the collisions occur between excitations with exactly parallel momenta, there is an eigenfunction

$$\psi_{1lm} = (\text{const})\bar{\epsilon}\,\bar{n}_p^{1/2}(\bar{n}_p + 1)^{1/2}Y_{lm}(\theta\phi)\ ,$$

where $Y_{lm}(\theta\phi)$ is the spherical harmonic. The eigenvalue λ_{1l} corresponding to this eigenfunction will be zero for all l. If the collisions are of finite but small angle, the states $|100\rangle$ and $|11m\rangle$ will still have zero eigenvalues because of conservation of energy and momentum. States with $l \geq 2$ will have nonzero but small eigenvalues. Thus, we expect λ_{1D} to be anomalously small. Then, σ_2 should be much greater than any of the other sums, and η will be large in the sense that

$$\eta \gg \zeta_1\bar{\rho}\ ,\ \zeta_2,\ \zeta_3\bar{\rho}^2,\ \kappa\rho_n/\overline{C\rho}\ . \tag{72}$$

B. Energy Conserved Separately for Phonons and Rotons

Then we find

$$\int_{\mathrm{ph}} \left(\frac{\partial n_p}{\partial t}\right)_{\mathrm{coll}} \overline{\epsilon}\, d\tau_p = 0 \ , \tag{73}$$

$$\int_{\mathrm{r}} \left(\frac{\partial n_p}{\partial t}\right)_{\mathrm{coll}} \overline{\epsilon}\, d\tau_p = 0 \ , \tag{74}$$

where the integrals are over the phonon and roton parts of momentum space, respectively. Proceeding as in Sec. II we find

$$\psi(\vec{\mathrm{p}}) = a_{\mathrm{ph}} \overline{\epsilon}\, \overline{n}_p^{1/2} (\overline{n}_p + 1)^{1/2} \quad \text{(phonon region)} \tag{75}$$

$$= a_{\mathrm{r}} \overline{\epsilon}\, \overline{n}_p^{1/2} (\overline{n}_p + 1)^{1/2} \quad \text{(roton region)} \ , \tag{76}$$

where a_{ph} and a_{r} are constants independent of $\vec{\mathrm{p}}$. This wave function must be correctly normalized and must be orthogonal to the 1S state. It follows that

$$a_{\mathrm{ph}} = (\overline{\beta}\overline{C}_{\mathrm{r}} / \overline{C}\overline{\rho}\overline{T}\overline{C}_{\mathrm{ph}})^{1/2} \ ,$$

$$a_{\mathrm{r}} = -(\overline{\beta}\overline{C}_{\mathrm{ph}} / \overline{C}\overline{\rho}\overline{T}\overline{C}_{\mathrm{r}})^{1/2} \ ,$$

where $\overline{C}_{\mathrm{ph}}$ and $\overline{C}_{\mathrm{r}}$ are the phonon and roton contributions to the specific heat. We call this eigenfunction the 2S state. If the energies of the phonon and roton systems are nearly conserved separately, the eigenvalue λ_{2S} will be small. Then, σ_3, σ_4, and σ_5 should be much larger than σ_1 and σ_2. If only the 2S term is retained in the sums σ_3, σ_4, σ_5, one finds

$$\sigma_3 = I_1^2/\lambda_{2S} \ , \tag{77}$$

$$\sigma_4 = I_2^2/\lambda_{2S} \ , \tag{78}$$

$$\sigma_5 = (I_2 - I_1)^2/\lambda_{2S} \ , \tag{79}$$

where

$$I_1 = \left[S_1 \left| \tfrac{1}{3} \frac{p}{\overline{\epsilon}} \frac{\partial \overline{\epsilon}}{\partial p} \right| S_2 \right] ,$$

$$I_2 = \left[S_1 \left| \overline{u}_p + \tfrac{1}{3} \frac{p}{\overline{\epsilon}} \frac{\partial \overline{\epsilon}}{\partial p} \right| S_2 \right] \ .$$

These results lead immediately to the relations

$$\zeta_1^2 = \zeta_2 \zeta_3 \ , \tag{80}$$

$$\zeta_2 \gg \eta \ , \tag{81}$$

$$\zeta_2 \overline{C}\overline{\rho}/\overline{\rho}_n \gg \kappa \ . \tag{82}$$

More complete results may be obtained by estimating I_1 and I_2. The thermal expansion coefficient $\overline{\alpha}$ at density $\overline{\rho}$ and temperature $\overline{T}$ may be calculated from the free energy, Eq. (9), giving

$$\overline{\alpha} = \frac{\overline{\beta}}{\overline{T}\overline{B}_i} \int \overline{\epsilon}^2 \overline{n}_p (\overline{n}_p + 1) \left(\overline{u}_p + \tfrac{1}{3} \frac{p}{\overline{\epsilon}} \frac{\partial \overline{\epsilon}}{\partial p} \right) d\tau_p \ ,$$

where $\overline{B}_i$ is the isothermal bulk modulus. We may divide this into roton and phonon contributions, $\overline{\alpha}_{\mathrm{r}}$ and $\overline{\alpha}_{\mathrm{ph}}$, respectively. Similarly, the entropy $\overline{S}$ per unit mass, Eq. (10), can be divided into $\overline{S}_{\mathrm{r}}$ and $\overline{S}_{\mathrm{ph}}$. It is then straightforward to show that

$$I_1 = \frac{(\overline{C}_{\mathrm{r}}/\overline{C}_{\mathrm{ph}})^{1/2} \overline{S}_{\mathrm{ph}} - (\overline{C}_{\mathrm{ph}}/\overline{C}_{\mathrm{r}})^{1/2} \overline{S}_{\mathrm{r}}}{\overline{C}} \ , \tag{83}$$

$$I_2 = [(\overline{C}_{\mathrm{r}}/\overline{C}_{\mathrm{ph}})^{1/2} \overline{\alpha}_{\mathrm{ph}} - (\overline{C}_{\mathrm{ph}}/\overline{C}_{\mathrm{r}})^{1/2} \overline{\alpha}_{\mathrm{r}}](\overline{B}_i / \overline{C}\overline{\rho}) \ . \tag{84}$$

Below 0.4 °K $\overline{C}$ is dominated by phonons.[2] We may therefore make a rough estimate of $\overline{C}_{\mathrm{ph}}$ at all temperatures by making a T^3 extrapolation of $\overline{C}$ from 0.4 °K. The roton specific heat can be determined by subtracting $\overline{C}_{\mathrm{ph}}$ from the experimentally observed total specific heat. Proceeding in a similar way for the entropy and the thermal expansion, we can estimate values for I_1 and I_2. The ratio I_1/I_2 turns out to be surprisingly constant, varying only from 0.11 at 0.4 °K, to 0.06 at 0.7 °K and to 0.05 at 1.9 °K. Thus,

$$\sigma_4 \approx \sigma_5 \ , \tag{85}$$

$$\sigma_4, \sigma_5 \gg \sigma_3 \ . \tag{86}$$

Hence,

$$\overline{\rho}^2 \zeta_3 \approx \overline{\rho} \zeta_1 \approx \zeta_2 \ . \tag{87}$$

Although we have derived this result on the assumption that λ_{2S} is unusually small, it seems likely that it may be more general. This remark is based solely on the observation that for most values of p in the phonon or roton regions the absolute magnitude $|\overline{u}_p|$ of the Grüneisen parameter happens to be considerably greater than the quantity

$$f_p = \tfrac{1}{3} \frac{p}{\overline{\epsilon}} \frac{\partial \overline{\epsilon}}{\partial p} \ .$$

For example, in the phonon region[26] as $p \to 0$, $u_p \to 2.84$, whereas f_p is $\frac{1}{3}$. Moreover, f_p is zero at the roton minimum. Thus, one expects that Eqs. (85) and (86) will usually be valid and Eq. (87) will be a reasonable approximation.

C. Momentum Conserved Separately for Phonons and Rotons

Consider when

$$\int_{\mathrm{ph}} \left(\frac{\partial n_p}{\partial t}\right)_{\mathrm{coll}} p_\alpha\, d\tau_p = 0 \ , \tag{88}$$

$$\int_{\mathrm{r}} \left(\frac{\partial n_p}{\partial t}\right)_{\mathrm{coll}} p_\alpha\, d\tau_p = 0 \ . \tag{89}$$

There is then an eigenfunction $\psi_{2P}(\vec{\mathrm{p}})$ given by

$$\psi_{2P}(\vec{\mathrm{p}}) = (\overline{\beta}\overline{\rho}_{n\,\mathrm{r}}/\overline{\rho}_n\overline{\rho}_{n\,\mathrm{ph}})^{1/2} p \cos\theta\, \overline{n}_p^{1/2} (\overline{n}_p + 1)^{1/2} \quad \text{(phonon region)} \ , \tag{90}$$

$$= -(\overline{\beta}\overline{\rho}_{n\,\mathrm{ph}}/\overline{\rho}_n\overline{\rho}_{n\,\mathrm{r}})^{1/2} p \cos\theta\, \overline{n}_p^{1/2} (\overline{n}_p + 1)^{1/2} \quad \text{(roton region)} \ . \tag{91}$$

If the momentum of the phonon and roton gases are nearly conserved separately, λ_{2P} will be anomalously small. This state will dominate the σ_1 sum. We find

$$\sigma_1=\frac{\bar{S}_{\mathrm{ph}}(\bar{\rho}_{n\,\mathrm{r}}/\bar{\rho}_{n\,\mathrm{ph}})^{1/2}-\bar{S}_{\mathrm{r}}(\bar{\rho}_{n\,\mathrm{ph}}/\bar{\rho}_{n\,\mathrm{r}})^{1/2}}{\bar{C}^2\lambda_{2P}} \quad . \qquad (92)$$

Since σ_1 is much bigger than the other sums,

$$\kappa\bar{\rho}_n/\bar{C}\bar{\rho}\gg\eta\ ,\ \ \zeta_1\bar{\rho},\ \zeta_2,\ \zeta_3\bar{\rho}^2\ \ . \qquad (93)$$

It is rather artificial to consider Secs. IV B and IV C separately. If the phonon and roton gases interact so weakly that the energy of each is nearly conserved, it is likely that the momentum of each gas will also be nearly conserved. If both energy and momentum are nearly conserved, Eq. (87) relating the second viscosities still holds. However, since λ_{2S} and λ_{2P} may be of the same order of magnitude, inequalities (82) and (93) must be replaced by the weaker statement

$$\kappa\bar{\rho}_n/\bar{C}\bar{\rho},\ \zeta_1\bar{\rho},\ \zeta_2,\ \zeta_3\bar{\rho}^2\gg\eta\ \ . \qquad (94)$$

It is very instructive to compare our results with those obtained by Putterman.[23] He assumed that superfluid helium was characterized by an "internal macroscopic variable" (IMV), whose relaxation was the dominant mechanism in determining the second viscosities. From this he was able to derive relation (80), $\zeta_1^2=\zeta_2\zeta_3$. It is clear that the amplitude A_{2S} of the $2S$ state is an IMV in the sense used by Putterman and this is why his result agrees with ours. To obtain relation (80) it is only necessary to suppose that *one* of the S states has an anomalously small eigenvalue. Nothing needs be assumed about *why* this is true, and the eigenfunction may take any form, provided that the relevant matrix element does not vanish. Note that Putterman's theory takes the IMV to be a scalar, which is therefore to be associated with an S state. One could, however, assume that the IMV was a vector or a tensor. Following arguments similar to those given in Putterman's paper, one would arrive at some relations between the dissipative coefficients. It seems very likely that these would be the same relations as those we have found when the eigenvalue of one of the P states (vector IMV) or one of the D states (tensor IMV) is very small.

V. SUMMARY

The form of the dissipative terms that we have found is in complete agreement with the proposal of Landau and Khalatnikov. There is no evidence of extra dissipative terms of the form proposed by Schmidt. Although there appears to be no rigorous relation between the dissipative coefficients, we have shown that it may be possible to derive approximate relations between them if nearly conserved quantities exist. We have given examples of this procedure.

The formal results we have obtained for the viscosities are also useful for quantitative calculations. Since the angular dependence of the eigenfunctions is known, the integral equation (35) reduces to an equation for the radial part of the eigenfunction only. For an assumed form of the collision operator, this equation can be solved easily by matrix methods. At temperatures below 0.6 °K, where only phonons are excited, the collision operator is known.[25] We have calculated the eigenfunctions and eigenvalues in this case[27] and have used them to find the viscosity η from Eqs. (65) and (68). The result was in excellent agreement with experiment.[28] Similar calculations at higher temperatures where rotons are important would provide valuable tests of models for roton-roton scattering.

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Quadrupole Contributions to the Electron Broadening of Overlapping Neutral-Atom Lines in a Plasma

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The impact-parameter formalism for Stark broadening of partially degenerate neutral lines emitted by an atom immersed in a dense plasma is extended to include the quadrupole term of the electron-atom interaction, considered in the second order of the Dyson expansion. The corresponding nondiagonal broadening functions and their impact-parameter averages are derived and their properties are studied in detail. The nondiagonal quadrupole shift function is shown to be negligible with respect to the dipole term, for most situations of physical interest. These new functions are also useful in the collision theory of electrons with atoms and molecules.

I. INTRODUCTION

Within the framework of the generalized impact theory,[1] the impact-electron contribution to the complete profile of partially degenerate neutral lines arising from atoms immersed in a dense plasma is usually evaluated by retaining only the first (dipole) term in the multipole expansion of the potential energy for the system electron perturber-excited radiator. The expression is written

$$V(\vec{r},\vec{R})=-\frac{1}{|\vec{r}|}+\frac{1}{|\vec{r}-\vec{R}|} \tag{1.1}$$

in atomic units, where $\vec{r}=\vec{\rho}+\vec{v}t(\vec{\rho}\perp\vec{v})$ denotes the parametrized trajectory of the perturber referenced with respect to the nucleus of the emitter and $\vec{R}$ is the position vector of the optical electron. The use of a classical picture for the plasma surrounding the radiator allows us to evaluate (1.1) with the condition $r>R$, so that the Legendre expansion makes the 2^L-polar series

$$V(\vec{r},\vec{R})=\frac{\vec{r}\cdot\vec{R}}{|\vec{r}|^3}+\frac{1}{2}\left(3\,\frac{(\vec{r}\cdot\vec{R})^2}{|\vec{r}|^5}-\frac{\vec{R}\cdot\vec{R}}{|\vec{r}|^3}\right)+\cdots \tag{1.2}$$

appear in a straightforward way.

Only the first two terms with $L=1$ and $L=2$ have been worked out. The second (quadrupole) term has already been considered by Griem,[2] for the special situation of the fully degenerate hydrogen Ly_α line. Griem included dipole terms to every order of the Dyson time-dependent perturbative expansion

$$S=I-(i\hbar)^{-1}\int_{-\infty}^{+\infty}dt\,\tilde{V}(t)+(i\hbar)^{-2}\int_{-\infty}^{+\infty}dt_1\,\tilde{V}(t_1)\int_{-\infty}^{t_1}\tilde{V}(t_2)\,dt_2+\cdots \tag{1.3}$$

for the collision S matrix, given in terms of

$$\tilde{V}(t)=e^{iH_0t}\,V(\vec{r},\vec{R})\,e^{-iH_0t}\ , \tag{1.4}$$

where H_0 is the unperturbed atomic Hamiltonian. Cooper and Oertel[3] have also evaluated the quadrupole term for the particular situation of isolated lines, using the second-order approximation of (1.3), which gives the first nonvanishing term, once the plasma average over $\vec{\rho}$ and $\vec{v}$ has been taken into account. The more general problem of the quadrupole term for the intermediate situation of partially degenerate lines was treated briefly in a previous work,[4] giving a rough estimate for its real part, with the excitation transfer effect between the statically split sublevels taken into account by including nondiagonal matrix elements in the collision operator. The purpose of the present work is to give special attention to the full (not only diagonal) quadrupole contribution of the electron