Effect of Spatial Coherence on the Photoelectric Counting Statistics of Gaussian Light*

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Exact formulas are derived for the factorial cumulants of the photoelectric counting distribution of partially polarized Gaussian and Gaussian-plus-coherent light for arbitrary detector areas. The results make it possible to follow the transition from Bose–Einstein statistics in the small-area limit to Poisson statistics in the large-area limit. Numerical results are presented for a circular geometry. These methods, which enable experimentally measured photocount statistics to be extrapolated to zero detector area, are expected to be useful in measurements of small departures from Gaussian statistics. Other theoretical results make it possible to express the statistics of the sum of photocounts from different photocathodes, or the multiaperture, single-cathode (MASC) photocount statistics of light of arbitrary coherence properties, in terms of the multicathode (MC) counting statistics. Explicit expressions are derived for the MASC photocount cumulants for the special case of partially polarized Gaussian light.

I. INTRODUCTION

The photoelectric counting statistics of Gaussian[1–11] and Gaussian-plus-coherent[12–24] light at a single detector have been the subject of extensive experimental and theoretical investigation. However, the theoretical work to date has been incomplete in one respect: It is generally assumed that the area of the photoelectric detector employed is negligibly small compared to the coherence area of the Gaussian light. In actual, finite detectors, the tendency for the intensities at different points in the photocathode to be less fully correlated affects the observed statistical correlations among the photoelectrons. Closely related theoretical methods can be used to analyze both of the following experimental arrangements: (i) measurement of the statistics of the photoelectrons produced over the entire surface of a single photocathode; (ii) measurement of the statistics of the photoelectrons produced at several small, spatially separated areas on a single photocathode. Situation (i) will be called a single-cathode (SC) experiment, and (ii) will be called a multiaperture, single-cathode (MASC) experiment.

In this paper, the following results are obtained: (a) For partially polarized Gaussian light, and Gaussian-plus-coherent light, exact formulas are derived for the SC photocount cumulants observed with finite detectors, in terms of the mutual coherence function of the Gaussian light. Knowledge of the photocount cumulants is equivalent to knowledge of the full photocount probability distribution. The physical significance of the photocount cumulants has been discussed elsewhere.[25,26] (b) A numerical method is outlined and illustrated which can be used to calculate the SC photocount cumulants for partially polarized Gaussian light with arbitrary spatial-coherence properties. (c) A general relation, valid for light of *arbitrary* coherence properties, is obtained between the MASC photocount statistics and the statistics of photoelectric counts from N separate photocathodes. The MASC technique is, of course, equivalent to measuring the sum of the counts received at separate detectors in a multicathode (MC) experiment. (d) For partially polarized Gaussian light, an explicit expression for the MASC photocount cumulants is derived.

These results are useful in enabling normalization of the measured photocount statistics to zero detector area. This should be particularly important for experiments seeking to measure small deviations from Gaussian statistics, such as have recently been suggested for light scattered from a fluid near the critical point,[27] and observed in light scattered from a small number of particles,[28] because the effect of spatial coherence on the photocount cumulants increases with the order of the cumulant, and because the higher cumulants are generally more strongly affected than the lower by small departures from Gaussian statistics.[17,25]

II. SINGLE-CATHODE PHOTOELECTRON STATISTICS

The probability of counting n photoelectrons in the time interval $[t, t+\tau]$ if light of intensity $I(\vec{r}, t)$ is incident on a detector with surface area A is[3]

$$p(n, t, \tau) = \langle : (W^n/n!)e^{-W} : \rangle , \tag{1}$$

where W is the intensity, integrated over the counting time and the surface of the detector,

$$W = \int_t^{t+T} dt' \int_A d^2r' \, \eta(\vec{r}\,')I(\vec{r}\,', t') , \tag{2}$$

and $\eta(\vec{r})$ is the mean number of photoelectrons produced per unit area, for unit incident intensity. Quantum mechanically, the intensity

$$I(\vec{r}, t) = \vec{E}^{(-)}(\vec{r}, t) \cdot \vec{E}^{(+)}(\vec{r}, t) \quad (3)$$

is an operator, and the powers of $E^{(+)}$ and $E^{(-)}$ which occur in equations such as (1) must be normally ordered, as indicated by the symbol: : . However, the calculations in this paper depend on a statistical theorem (the Gaussian moment theorem) which is valid whether I is regarded as an operator or as the intensity of a classical field. In what follows, it will always be correct to interpret $\vec{E}$ classically.

The effects of spatial coherence on the photocounting statistics have a simple physical interpretation in terms of the coherence time T_c and the coherence area A_c of the Gaussian field. If $T \ll T_c$ and $A \ll A_c$, then $W = \eta ATI$; in this approximation the photoelectron factorial cumulants are identical to the intensity cumulants. However, if W depends on the values of $I(\vec{r}, t)$ over a range of positions and times such that $A \gtrsim A_c$ or $T \gtrsim T_c$, then the intensity and photoelectron cumulants are different. In this case, it is convenient, both theoretically and experimentally, to describe the statistics in terms of the cumulants of the intensity, $K_N(W)$, and the factorial cumulants of n,[27]

$$k_N(n) = K_N(W) . \quad (4)$$

If the incident light were fully coherent, the photoelectron distribution would be Poisson, and the factorial cumulants of order $N > 1$ would vanish. Nonzero values for $k_N(n)$ $(N > 1)$ are thus a measure of the departure of the field from full coherence. In general, the Nth-order cumulant represents the "true" Nth-order correlation among a set of variables, with all accidental correlations and correlations of lower order subtracted away.[25] This property is reflected in the behavior (derived below) of $k_N(n)$ as the detector area A is increased to a point where $A_c \ll A$. In this case the factorial cumulants are all proportional to A/A_c, i.e., to the number of cells of phase space observed by the detector. This is consistent with a picture in which the Nth factorial cumulant is the sum of Nth-order correlations from individual cells of phase space with no "accidental" correlations between different cells.

The desired cumulant $K_N(W)$ is obtained by spatial and temporal integration of the N-fold intensity cumulant $K_{11\cdots1}(I(\vec{r}_1, t_1), \ldots, I(\vec{r}_N, t_N))$. In Ref. 23, Eq. (11), the latter cumulant has been calculated for superposed Gaussian and coherent light,

$$\vec{E} = \vec{E}^G + \vec{E}^c .$$

For simplicity we assume from now on that the Gaussian light is cross-spectrally pure, and is partially polarized with degree of partial polarization P. In this case it is possible to choose a coordinate system for $\vec{E}$ in which the mutual coherence matrix is

$$\langle E_1^{G(-)}(\vec{r}_1, t_1) E_1^{G(+)}(\vec{r}_2, t_2) \rangle = \tfrac{1}{2}(1+P) \langle I^G \rangle \gamma(\vec{r}_1, \vec{r}_2; 0) \gamma(t_2 - t_1) , \quad (5)$$

$$\langle E_2^{G(-)}(\vec{r}_1, t_1) E_2^{G(+)}(\vec{r}_2, t_2) \rangle = \tfrac{1}{2}(1-P) \langle I^G \rangle \gamma(\vec{r}_1, \vec{r}_2; 0) \gamma(t_2 - t_1) , \quad (6)$$

$$\langle E_1^{G(-)}(\vec{r}_1, t_1) E_2^{G(+)}(\vec{r}_2, t_2) \rangle = 0 , \quad (7)$$

where E_1 and E_2 are the components of $\vec{E}$ in the chosen coordinates; $\gamma(\vec{r}_1, \vec{r}_2; 0)$ is the usual two-point mutual-coherence function, evaluated for zero time delay; and $\gamma(t)$ is the Fourier transform of the spectral distribution of the Gaussian light. We also assume that the coherent field is monochromatic:

$$\vec{E}^{c(+)}(\vec{r}, t) = \hat{e} (\langle I^c \rangle)^{1/2} e^{i(\vec{k} \cdot \vec{r} - \omega_c t)} , \quad (8)$$

where $\hat{e}$ is a unit vector with components e_1 and e_2 in the coordinates of Eqs. (5)–(7). Then

$$k_N(n) = K_N(W) = \mathcal{G}_N + \mathcal{H}_N , \quad (9)$$

where the contribution from the Gaussian light alone is

$$\mathcal{G}_N = \{[\tfrac{1}{2}(1+P)]^N + [\tfrac{1}{2}(1-P)]^N\} \times \langle n^G \rangle^N (N-1)! \, \mathcal{S}_{N,\mathcal{G}} \mathcal{T}_{N,\mathcal{G}} , \quad (10)$$

$$\mathcal{S}_{N,\mathcal{G}} = (\bar{\eta} A)^{-N} \int_A d^2 r_1 \cdots d^2 r_N \eta(\vec{r}_1) \cdots \eta(\vec{r}_N) \times \gamma(\vec{r}_1, \vec{r}_2; 0) \gamma(\vec{r}_2, \vec{r}_3; 0) \cdots \gamma(\vec{r}_N, \vec{r}_1; 0) , \quad (11)$$

$$\mathcal{T}_{N,\mathcal{G}} = T^{-N} \int_0^T dt_1 \cdots dt_N \gamma(t_1 - t_2) \times \gamma(t_2 - t_3) \cdots \gamma(t_N - t_1) , \quad (12)$$

and the contribution of spatial and temporal heterodyning of the Gaussian and coherent light is

$$\mathcal{H}_N = \langle n^c \rangle \langle n^G \rangle^{N-1} N! \, \{[\tfrac{1}{2}(1+P)]^N (e_1)^2 + [\tfrac{1}{2}(1-P)]^N (e_2)^2\} \times \mathcal{S}_{N,\mathcal{H}} \mathcal{T}_{N,\mathcal{H}} , \quad (13)$$

$$\mathcal{S}_{N,\mathcal{H}} = (\bar{\eta} A)^{-N} \int_A d^2 r_1 \cdots d^2 r_N \eta(\vec{r}_1) \cdots \eta(\vec{r}_N) \times \gamma(\vec{r}_1, \vec{r}_2; 0) \gamma(\vec{r}_2, \vec{r}_3; 0) \cdots \gamma(\vec{r}_N, \vec{r}_1; 0) \times e^{i\vec{k} \cdot (\vec{r}_N - \vec{r}_1)} , \quad (14)$$

$$\mathcal{T}_{N,\mathcal{H}} = T^{-N} \int_0^T dt_1 \cdots dt_N \gamma(t_1 - t_2) \gamma(t_2 - t_3) \cdots \times \gamma(t_{N-1} - t_N) e^{-i\omega_c (t_N - t_1)} . \quad (15)$$

In (10)–(15), $\langle n^G \rangle$ and $\langle n^c \rangle$ are the mean numbers of counts due to the Gaussian and coherent light, respectively, and $\bar{\eta}$ is the mean quantum efficiency, such that $\langle n \rangle = \bar{\eta} A T \langle I \rangle$. It has been assumed that $\langle I^G \rangle$ is constant in space and time. All of the spherical assumptions made in deriving (10)–(15)

may be removed if more general formulas are needed, by use of the equations of Ref. 23.

III. DISCUSSION

The factors $\mathcal{S}$ and $\mathcal{T}$ in Eqs. (10)–(15) represent the modification of the photoelectron statistics by the spatial and temporal coherence properties of the incident light. The effects of temporal coherence are well known, and we shall concentrate on discussing spatial effects. When $A \ll A_c$, $\gamma_{ij} \approx 1$ and $\mathcal{S}_N \approx 1$, so that $k_N(n) \propto A^N$. For larger areas, the fact that the integrands in $\mathcal{S}_N$ have a "ring" structure (i.e., the integrand has the form $\gamma_{12}\gamma_{23}\cdots\gamma_{N1}$) implies that whenever any of the points $\vec{r}_1, \ldots, \vec{r}_N$ lie in different coherence areas, the integrand tends to zero. Thus for $A \gg A_c$, $k_N(n)$ increases as A/A_c, since different coherence areas contribute independently. (This is true only of the factorial cumulants, and not of the factorial moments.)

This result implies a modified central limit theorem for photocount statistics. When $A \gg A_c$, the Nth normalized factorial cumulant

$$k_N(n)/[k_1(n)]^N$$

tends to zero as $(A_c/A)^{N-1}$, for $N > 1$. Thus as A/A_c becomes infinite, the statistics of n approach the Poisson distribution, for which $k_1(n) = \langle n \rangle$, $k_N(n) = 0$ $(N > 1)$. The physical reason for this is the averaging out of uncorrelated intensity fluctuations in different coherence areas.

As is well known,[1,2] one coherence volume is defined as $A_c T_c = h^3/\Delta p_x \Delta p_y \Delta p_z$, where Δp_j is the momentum uncertainty along the j axis. Thus a single coherence area at the detector corresponds to a single h^3 cell in photon phase space. With this in mind, further insight into $k_N(n)$ can be gained by comparing (10)–(12) with the factorial cumulants $k_N^{(B)}(m)$ for bosons distributed in more than one cell of phase space. The Bose–Einstein distribution for $m = m_1 + \cdots + m_c$ photons distributed among c different cells is

$$p^{(B)}(m) = \prod_{j=1}^{c} [1 + \langle m_j \rangle]^{-1} [1 + 1/\langle m_j \rangle]^{-m_j}, \qquad (16)$$

where $\langle m_j \rangle$ is the mean number of photons in the jth cell. Evaluation of the generating function for (16) leads to

$$k_N^{(B)}(m) = (N-1)! \sum_{j=1}^{c} \langle m_j \rangle^N . \qquad (17)$$

A comparison with (10)–(12) indicates that the behavior of $k_N(n)$ for $A \gg A_c$ is the result of sampling the photon distribution over A/A_c cells in phase space. Such a comparison also facilitates a physical interpretation of the correction factors $\mathcal{S}_{N,\mathcal{G}}$ and $\mathcal{T}_{N,\mathcal{G}}$. If we assume that the occupation number of each cell of phase space is $\frac{1}{2}(1+P)\langle n^G \rangle$ for one polarization and $\frac{1}{2}(1-P)\langle n^G \rangle$ for the orthogonal polarization, then we can equate (17) to (10) to obtain an effective number of cells of phase space:

$$c^{N-1} = \mathcal{S}_{N,\mathcal{G}} \mathcal{T}_{N,\mathcal{G}} . \qquad (18)$$

As defined by this equation, c is also the number of coherence volumes sampled by the detector. In general, c will depend on N, and thus cannot be taken as a well-defined effective number of cells, characterizing the whole photocount distribution of the Gaussian component of the incident light.[29] This is clearly evident in Table I, where for a given geometry (corresponding to a given normalized aperture radius κ), c_N is seen to take on different values for different orders N.

Equation (17) also gives a physical interpretation to the fact that the photocount cumulants are (crudely speaking) additive in the number of coherence areas on the detector, and thus provides an interesting basis for the claim that the cumulants represent the "true" correlations.

The heterodyne terms $\mathcal{H}_N$ can be analyzed similarly.

IV. NUMERICAL RESULTS FOR SINGLE-CATHODE STATISTICS

The main computational problem in this method of correcting the photocount distribution for the effects of spatial coherence is the evaluation of the multiple integrals, Eqs. (14) and (15), of N-fold products of the mutual coherence function. In this section we summarize a calculation of the spatial correction $\mathcal{S}_{N,\mathcal{G}}$ for Gaussian light, for the special case of a uniformly efficient circular detector which is on the same axis as a uniform circular source (Fig. 1). Although this case is of considerable practical interest in itself, the methods used can be generalized easily to other (nonuniform or differently shaped) apertures and sources.

TABLE I. Numerical comparison between exact and approximate correction factors for factorial cumulants.

N	Exact	Approximate
	Aperture radius = 0.5 coherence radii	
2	0.8034	0.8034(fit)
3	0.7094	0.6455
4	0.6322	0.5186
5	0.5696	0.4166
6	0.5109	0.3347
	Aperture radius = 1.0 coherence radii	
2	0.4729	0.4729 (fit)
3	0.2875	0.2236
4	0.1835	0.1058
5	0.1139	0.0500
6	0.0730	0.0237

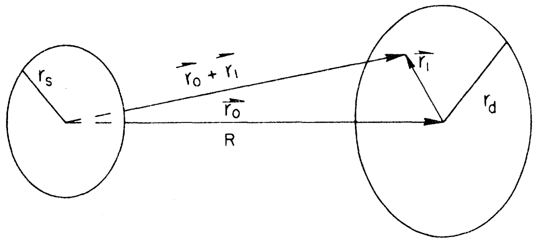

FIG. 1. Geometrical relations for the calculation of the finite-aperture corrections to the photocount distribution, for a uniform circular source (radius r_s) on the same axis as a uniform circular detector (radius r_d).

The mutual-coherence function in this case is[30]

$$\gamma(\vec{r}_1, \vec{r}_2; 0) = \frac{2J_1(\kappa' |\vec{r}_1 - \vec{r}_2|)}{\kappa' |\vec{r}_1 - \vec{r}_2|} e^{i\psi_{12}} , \tag{19}$$

where

$$k = 2\pi/\lambda ,$$
$$\psi_{12} = k(|\vec{r}_1 - \vec{r}_0| - |\vec{r}_2 - \vec{r}_0|) , \tag{20}$$
$$\kappa' = kr_s/R .$$

Combining (11) with (19) and scaling the variables of integration, one finds that

$$\mathcal{S}_{N,\mathcal{G}} = \left(\frac{2}{\pi}\right)^N \int_0^1 r_1 dr_1 \cdots \int_0^1 r_N dr_N \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \times \frac{J_1(\kappa z_1 r_{12})}{\kappa z_1 r_{12}} \cdots \frac{J_1(\kappa z_1 r_{N1})}{\kappa z_1 r_{N1}} \tag{21}$$

depends on the single parameter

$$\kappa = kr_s r_d / z_1 R . \tag{22}$$

The numerical factor $2z_1 = 3.83171\ldots$ is the first zero of the Bessel function $J_1(z)$ such that $z > 0$; and $r_{ij} = |\vec{r}_i - \vec{r}_j|$. The scaling has been chosen so that κ is the ratio of the radius of the aperture to the radius of an aperture occupying one coherence area. If we take the coherence area A_c to be the area of the circle $|\vec{r}_1 - \vec{r}_2| = \text{const}$, such that $\gamma(\vec{r}_1, \vec{r}_2) = 0$, then

$$A_c = \pi(z_1 R)^2/(kr_s)^2 \tag{23}$$

and

$$\kappa^2 = A/A_c . \tag{24}$$

For second order ($N=2$), one can represent the Bessel functions by power series which can be integrated term by term to give a power series for $\mathcal{S}_{2,\mathcal{G}}$.[31] However, this method does not generalize to larger N or to other geometries. Furthermore, since the dimension of the domain of the integral is $2N$, a direct numerical integration is cumbersome if not impractical even for third order, because of accumulated rounding errors. However, Monte Carlo integration is an attractive alternative since it can provide reasonably precise results, and is as well if not better suited for sixth order as for third.

In this example the points $\vec{r}_j = (x_j, y_j)$ were selected by a random-number generator which produced values of x_j and y_j, each uniformly distributed between -1 and 1. If the resulting point $\vec{r}_j$ did not fall within the unit disk, the point was rejected and another point drawn. The uncertainty in the Monte Carlo estimate of the integral was estimated using standard techniques; the details can be found in Ref. 26.

The results of the calculation are presented in Fig. 2. Second-order values computed from the Jakeman–Pike formula[31] are included for comparison, as a check on the accuracy of the Monte Carlo calculation. The results of the two methods are in agreement, within the calculated statistical limits of the Monte Carlo method. Of the eight values which were compared, six of the Monte Carlo values are within one standard deviation of the exact result.

V. MASC PHOTOCOUNTING STATISTICS

In an interesting series of papers[29,32–34] Bures, Delisle, and Zardecki have reported measurements of the multiaperture, single-cathode (MASC) photoelectric counting statistics of Gaussian light, and have obtained good agreement with theoretical calculations. This work is of interest because it represents the first measurement of the higher photocount factorial cumulants at more than two space-time points, and provides the basis for a new method of measuring the coherence area of

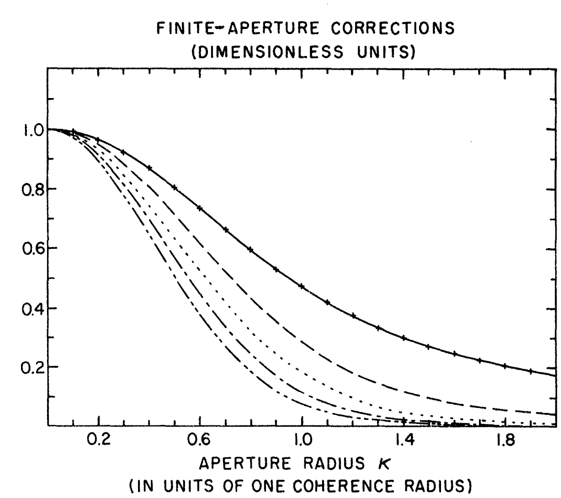

FIG. 2. Finite-aperture correction $\mathcal{S}_{N,\mathcal{G}}$, for N running from 2 to 6, as functions of the aperture radius. Solid line: $N=2$. The curve is calculated from the Jakeman–Pike formula, Ref. 29; the points are the results of the Monte Carlo calculation. Dashed line: $N=3$. Dotted line: $N=4$. Single-dot–dashed line: $N=6$.

Gaussian light.[33] The new technique, which employs a single photocathode illuminated by the light which is transmitted through a set of N apertures in a screen covering the photocathode, is substantially easier experimentally, but gives somewhat less information, than earlier techniques employing N separate photocathodes.[5,6,10,11,17]

We shall now derive an expression [Eq. (37)] for the MASC photocount factorial cumulants in terms of the MC factorial cumulants, which are well known. Suppose that N nonoverlapping areas on a photocathode are illuminated with Gaussian light. The total number n of photoelectric counts registered in a time interval T is the sum of the counts produced in the N different illuminated areas, and has the probability distribution

$$p(n)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}p_N(n_1,\ldots,n_N)\,\delta_{n,n_1+\cdots+n_N}\,, \tag{25}$$

where p_N is the N-fold joint photocount probability distribution. Because of the spatial coherence properties of the light, p_N will not in general be a product of onefold distributions. The generating function for $p(n)$ is

$$\begin{aligned}\mathcal{G}(s)&=\sum_{n=0}^{\infty}(1-s)^n p(n)\\&=\langle(1-S)^n\rangle\\&=\mathcal{G}_N(s_1,\ldots,s_N)\big|_{s_1=s_2=\cdots=s_N=s}\,,\end{aligned}\tag{26}$$

where $\mathcal{G}_N$ is the N-fold generating function,

$$\mathcal{G}_N(s_1,\ldots,s_N)=\langle(1-s_1)^{n_1}\cdots(1-s_N)^{n_N}\rangle\,. \tag{27}$$

The factorial-cumulant generating functions are

$$\mathcal{F}(s)=\ln\mathcal{G}(s) \tag{28}$$

for the MASC case, and

$$\mathcal{F}_N(s_1,\ldots,s_N)=\ln\mathcal{G}_N(s_1,\ldots,s_N) \tag{29}$$

for the MC case.

Using Eq. (26), we can express the MASC factorial moments

$$F_M=\frac{1}{M!}\left(\frac{d}{ds}\right)^M\mathcal{G}(s)\big|_{s=0} \tag{30}$$

and factorial cumulants

$$k_M=\frac{1}{M!}\left(\frac{d}{ds}\right)^M\mathcal{F}(s)\big|_{s=0} \tag{31}$$

in terms of the corresponding MC quantities

$$F^{(n)}_{n_1,\ldots,n_N}=\frac{1}{n_1!n_2!\cdots n_N!}\left(\frac{\partial}{\partial s_1}\right)^{n_1}\cdots\left(\frac{\partial}{\partial s_N}\right)^{n_N}\times\mathcal{F}_N(s_1,\ldots,s_N)\big|_{s_1=s_2=\cdots=s_N=0} \tag{32}$$

and

$$k^{(N)}_{n_1,\ldots,n_N}=\frac{1}{n_1!n_2!\cdots n_N!}\left(\frac{\partial}{\partial s_1}\right)^{n_1}\cdots\left(\frac{\partial}{\partial s_N}\right)^{n_N}\times\mathcal{F}_N(s_1,\ldots,s_N)\big|_{s_1=s_2=\cdots=s_N=0}\,. \tag{33}$$

We make use of Leibniz's rule,

$$\left(\frac{d}{ds}\right)^M f(s)=\sum_{\{a_j\}}\frac{1}{a_1!a_2!\cdots a_M!}\left(\frac{d}{ds}\right)^{a_1}f_1(s)\cdots\times\left(\frac{d}{ds}\right)^{a_N}f_N(s)\,, \tag{34}$$

where

$$f(s)=f_1(s)\cdots f_N(s)\,,$$

and the summation runs over all sets of non-negative integers $a_1,\ldots,a_N$ such that

$$\sum_{j=1}^{N}a_j=M\,. \tag{35}$$

When (34) is used to evaluate (30) and (31), making use of (26) and (27), the resulting Mth-order MASC factorial moments and factorial cumulants are

$$F_M=\sum_{\{a_j\}}F^{(N)}_{a_1,\ldots,a_N}\,, \tag{36}$$

$$k_M=\sum_{\{a_j\}}k^{(N)}_{a_1,\ldots,a_N}\,, \tag{37}$$

where the summations are restricted by (35).

For example, for $N=2$, the third-order MASC factorial cumulant is

$$k_3=k^{(2)}_{3,0}+k^{(2)}_{2,1}+k^{(2)}_{1,2}+k^{(2)}_{0,3}\,. \tag{38}$$

The cumulants $k^{(2)}_{3,0}$ and $k^{(2)}_{0,3}$ can be measured by occulting one or the other of the two holes in the aperture. Thus, it is also possible to determine the symmetric quantity $k^{(2)}_{2,1}+k^{(2)}_{1,2}$, but not $k^{(2)}_{2,1}$ or $k^{(2)}_{1,2}$ separately. Similar conclusions apply to higher-order cumulants. For many experiments,[33] however, the full information given by the separate N-fold cumulants such as $k^{(2)}_{1,2}$ may be unnecessary, and the MASC result k_3 may be sufficient.

From the derivation it is evident that the equations in this section are valid for light of arbitrary coherence properties; no special assumptions have been made regarding the ratio of the counting time to the coherence time, or the ratio of the areas of the N apertures to a coherence area.

VI. PARTIALLY POLARIZED GAUSSIAN LIGHT

When the known results[11,23,35] for the N-fold photocount cumulants of cross-spectrally pure partially polarized Gaussian light are substituted in (37) one finds that

$$k_M=\alpha^M\int_0^T dt_1'\cdots\int_0^T dt_N'\int_{A_1}d^2r_1'\cdots\int_{A_N}d^2r_N'\times\sum_{\{a_1,\ldots,a_M\}}\sum_C\sum_{\mu=1}^{2}\Gamma^{(\mu,\mu)}{}_{l_1,Cl_1}\cdots\Gamma^{(\mu,\mu)}{}_{l_M,Cl_M}\,, \tag{39}$$

where $C: l_j \to Cl_j$ is any cyclic permutation of the set of integers $l_1, \ldots, l_M$, where

$$\begin{aligned} l_1 = l_2 = \cdots = l_{a_1} = 1 \\ \vdots \\ l_{a_1+\cdots+a_{N-1}+1} = \cdots = l_M = N \end{aligned} \tag{40}$$

and

$$\Gamma^{(\mu,\nu)}{}_{j,k} = \langle V_\nu(\vec{r}_k', t_k')^* V_\mu(\vec{r}_j', t_j')\rangle \;. \tag{41}$$

In (39), the counting-time interval is $0 \le t_j' \le T$, and the detector-aperture areas are $A_1, \ldots, A_N$. In (39) and (41) $V_\mu(\vec{r}, t)$ is the μ component of the electric field amplitude with respect to a polarization basis. For cross-spectrally pure light, one can choose the polarization basis so that[8]

$$\Gamma^{(\mu,\nu)}{}_{j,k} = \delta_{\mu\nu}\langle I^{(\mu)}\rangle \gamma_{jk} \;, \tag{42}$$

where the normalized mutual-coherence function γ_{jk} depends only on $\vec{r}_j$, $\vec{r}_k$, and $t_k - t_j$, not on μ or ν; and

$$\langle I^{(1)}\rangle = \tfrac{1}{2}(1+P)\langle I\rangle \;, \quad \langle I^{(2)}\rangle = \tfrac{1}{2}(1-P)\langle I\rangle \tag{43}$$

are the intensities of the two statistically independent polarizations of Eq. (42), expressed in terms of the degree of polarization P and the total intensity $\langle I\rangle$. Relations (42) and (43) have been used in deriving (39).

Interchanging summations in (39) and using (42) and (43), we find

$$k_M = \alpha^M\{[\tfrac{1}{2}(1+P)]^M + [\tfrac{1}{2}(1-P)]^M\}\langle I\rangle^M \times \int_0^T dt_1' \cdots \int_0^T dt_N' \int_{A_1} d^2r_1' \cdots \int_{A_N} d^2r_N' \times \sum_{l_1=1}^N \cdots \sum_{l_M=1}^N \gamma_{l_1,l_2}\gamma_{l_2,l_3}\cdots\gamma_{l_M,l_1} \;, \tag{44}$$

which reduces to

$$k_M = (M-1)!\,\alpha^M\{[\tfrac{1}{2}(1+P)]^M + [\tfrac{1}{2}(1-P)]^M\}\langle I\rangle^M \times \int_0^T dt_1' \cdots \int_0^T dt_N' \int_{A_1} d^2r_1' \cdots \int_{A_N} d^2r_N' \times \sum_{l_1=1}^N \cdots \sum_{l_M=1}^N \gamma_{l_1,l_2}\gamma_{l_2,l_3}\cdots\gamma_{l_M,l_1} \tag{45}$$

since different cyclic permutations of the summation indices $l_1, \ldots, l_M$ can all be reduced to the same permutation by renaming the indices. When the light is fully polarized ($P=1$) and all the aperture areas $A_1, \ldots, A_N$ are negligible compared to a coherence area, Eq. (45) reduces to the results obtained by Bures *et al.*[33] using a special technique for Gaussian light.

The property of cross-spectral purity implies that the spatial and temporal coherence properties can be separated:

$$\gamma_{jk}(\vec{r}_j, \vec{r}_k; t_k - t_j) = \gamma_{jk}(\vec{r}_j, \vec{r}_k; 0)\gamma_{jj}(\vec{r}_j, \vec{r}_j; t_k - t_j) \;, \tag{46}$$

where $\gamma_{jj}(\vec{r}_j, \vec{r}_j; t_k - t_j)$ is the same function of t for all points $\vec{r}_j$ in the aperture. When (46) is substituted into (45), one obtains the full dependence of k_M on the degree of polarization P and on the temporal and spatial coherence properties of the incident light. The resulting integrals can be evaluated in a straightforward way, as outlined in Sec. IV. These corrections may be significant in MASC experiments.[33]

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Multiphoton Ionization Induced by Circularly Polarized Radiation

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The ratio μ_N of the cross section for circularly polarized light to that for linearly polarized is numerically investigated for N-photon ionization of atomic hydrogen. For $N<4$, it appears that the gain factor is greater than unity. For $N\geq 4$, the ratio μ_N is generally smaller than unity and decreases when the multiphoton order N increases.

I. INTRODUCTION

A number of discussions[1-7] have recently appeared concerning the question of whether circularly polarized light ionizes atoms more strongly than linearly polarized light. Experiments[8,9] on two- and three-photon ionization of atomic Cs have shown that the ratio of circular- to linear-polarization cross sections $\mu_N=\sigma_c^N/\sigma_l^N$ took values equal to $\mu_2=1.28$ and $\mu_3=2.15$, respectively.

Though this result seems to be in contradiction with the general conclusions of theoretical computations performed a few years ago,[10] Lambropoulos[3] has given a clear interpretation of these data. Klarsfeld and Maquet[5] have found an upper bound for the ratio μ_N, suggesting that circularly polarized radiation may often be more efficient in multiphoton ionization than in linearly polarized radiation. By using the momentum-translation method Reiss[6] has improved this bound and shown, in contrast with the previous conclusion, the strong dominance of linear- over circular- polarization cross section for high-order multiphoton processes.

Within the framework of time-dependent perturbation theory, precise values for linear- and circular-polarization cross sections, as a function of radiation wavelength, have been computed for $N=2$[4] and $N=3$.[2] But, a lack of information continues to exist concerning the situation under dispute in the case of high-order multiphoton processes. The purpose of this paper is to give the additional results, for $N=2-8$, which should provide a clear interpretation of the effect of light polarization on the cross sections, and which should contribute to a better understanding of the ionization process in gases. To achieve this we present a detailed analysis of the dependence of the gain factor μ_N on the photon wavelength and multiphoton order N. Section II contains the analytical formula for the multiphoton ionization cross section of hydrogen atom with circularly polarized light. A comparison between the values found for the linear- and circular-polarization case is given in Sec. III, where we discuss the reasons causing the decrease of the values which are calculated for the ratio μ_N as N increases.

II. IONIZATION CROSS SECTION FOR CIRCULARLY POLARIZED LIGHT

In previous papers[11] the computation was reported for the multiphoton ionization cross section σ_l^N of atomic hydrogen by linearly polarized light.