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Multiple-Scattering Expansions for Nonrelativistic Three-Body Collision Problems. VIII. Eikonal Approximation for the Faddeev–Watson Multiple-Scattering Expansion*

Joseph C. Y. Chen, Lars Hambro,† and A.-L. Sinfailam‡

Department of Physics and Institute for Pure and Applied Physical Sciences, University of California, San Diego, La Jolla, California 92037

Kwong T. Chung

Department of Physics, North Carolina State University, Raleigh, North Carolina 27607

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The Faddeev–Watson multiple-scattering (FWMS) expansion is investigated in the eikonal approximation. For the case with the classical trajectories taken to lie along straight lines, the second-order FWMS expansion reduces to the well-known Glauber-eikonal approximation. Application of the second-order FWMS expansion in the eikonal approximation is carried out for elastic and inelastic (e,H) scattering with the classical trajectories taken as intersecting two straight-line segments. The result suggests that there is a substantial cancellation between the first-order and second-order terms in the FWMS expansion for pair Coulomb interactions.

I. INTRODUCTION

The Faddeev–Watson multiple-scattering (FWMS) expansion for nonrelativistic three-body systems[1] has been applied to high-energy atomic-collision problems.[2–4] Calculations have been carried out in the first-order FWMS approximation for both the scattering and rearrangement collisions.[5,6] The results converge to the first-order Born results at energies higher than normally expected. For rearrangement collisions, the total cross section in the first-order FWMS approximation is found to lie between the first-order Born [Jackson–Schiff[7] (JS)] cross section and the Brinkman–Kramers[8] (BK) cross section obtained with the repulsive pair interaction neglected. The cross section approaches the first-order Born cross section in the high-energy limit, where the nonrelativistic approximation is no longer expected to be valid. For scatterings, the first-order FWMS approximation yields results which converge to the first-order Born approximation at energies one magnitude higher than normally expected; the magnitude of the cross section is much larger than the first-order Born approximation. This then gives rise to the question whether the second-order FWMS terms are of importance. The purpose of the present paper is to investigate the second-order FWMS approximation. The use of the eikonal approximation permits the evaluation of the FWMS expansions to second order for scattering processes.

II. FADDEEV–WATSON MULTIPLE-SCATTERING EXPANSION

The multiple-scattering expansion for a three-body scattering process

$$1+(2,3)\rightarrow 1+(2,3) \tag{2.1}$$

has been considered both for the on-shell and off-shell cases.[1,9] The transition operator T_s, in the transition amplitude between asymptotic states $\psi_i^{(1)}$ and $\psi_f^{(1)}$

$$\mathcal{T}_s=\langle\psi_f^{(1)}|T_s|\psi_i^{(1)}\rangle, \tag{2.2}$$

takes the form in the FWMS exapnsion

$$T_3=T_2+T_3+T_2G_0T_3+T_3G_0T_2+\cdots, \tag{2.3}$$

with

$$T_i=V_i+V_iG_0T_i,\quad G_0=(E-H_0+i\delta)^{-1}, \tag{2.4}$$

where V_i is the two-body potential V_{jk}, G_0 is the Green's function for the three-body system in the absence of interaction, and T_i is the two-body T matrix in the presence of a spectator particle i.

The asymptotic states in coordinate representation are of the form

$$\langle\vec{r}_1\vec{R}_1|\psi_i^{(1)}\rangle=(2\pi)^{-3/2}\,\chi_i^{(1)}(\vec{r}_1)\,e^{i\vec{k}_i\cdot\vec{R}_1}, \tag{2.5}$$

$$\langle\psi_f^{(1)}|\vec{r}_1,\vec{R}_1\rangle=(2\pi)^{-3/2}\chi_f^{(1)}(\vec{r}_1)^*\,e^{-i\vec{k}_f\cdot\vec{R}_1}, \tag{2.6}$$

where $\vec{r}_1$ is the internal coordinates of the two-body subsystem (2, 3) pointing from particle 2 to particle 3, with bound-state wave function $\chi^{(1)}(\vec{r}_1)$, and where $\vec{R}_1$ and $\vec{k}$ are the coordinates and asymptotic momenta of particle 1 with respect to the center of mass of the subsystem (2, 3). Utilizing these asymptotic states, the transition amplitude given by Eq. (2.2) can be rewritten as

$$\mathcal{T}_s=\langle\,\chi_f^{(1)}(\vec{r}_1)\,|\,\Upsilon_s(\vec{r}_1,\vec{r}_1')\,|\,\chi_i^{(1)}(\vec{r}_1')\rangle, \tag{2.7}$$

with

$$\Upsilon_s(\vec{r}_1,\vec{r}_1')=(2\pi)^{-3}\int d\vec{R}_1\,d\vec{R}_1'\,e^{i\vec{k}_i\cdot\vec{R}_1'-i\vec{k}_f\cdot\vec{R}_1}\times\langle\vec{r}_1\vec{R}_1|T_s|\vec{r}_1'\vec{R}_1'\rangle, \tag{2.8}$$

where $\Upsilon_s(\vec{r}_1,\vec{r}_1')$ is the Fourier transform of the transition operator T_s in the coordinate representation.

Substitution of T_s given by Eq. (2.3) into Eq. (2.8) yields the multiple expansion of $\Upsilon_s(\vec{r}_1, \vec{r}_1')$:

$$\Upsilon_s(\vec{r}_1, \vec{r}_1') = \sum_{\lambda=1} \Upsilon_s^{(\lambda)}(\vec{r}_1, \vec{r}_1'), \tag{2.9}$$

with

$$\Upsilon_s^{(1)}(\vec{r}_1, \vec{r}_1') = (2\pi)^{-3} \int d\vec{R}_1 d\vec{R}_1' e^{i\vec{\kappa}_i \cdot \vec{R}_1' - i\vec{\kappa}_f \cdot \vec{R}_1} \times \langle \vec{r}_1 \vec{R}_1 | T_2 + T_3 | \vec{r}_1' \vec{R}_1' \rangle, \tag{2.10}$$

$$\Upsilon_s^{(2)}(\vec{r}_1, \vec{r}_1') = (2\pi)^{-3} \int d\vec{R}_1 d\vec{R}_1' e^{i\vec{\kappa}_i \cdot \vec{R}_1' - i\vec{\kappa}_f \cdot \vec{R}_1} \times \langle \vec{r}_1 \vec{R}_1 | T_2 G_0 T_3 + T_3 G_0 T_2 | \vec{r}_1' \vec{R}_1' \rangle, \tag{2.11}$$

etc. The differential cross section takes the form

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \mu_1^2 \frac{\kappa_f}{\kappa_i} \left| \sum_{\lambda=1} \langle \chi_f^{(1)}(\vec{r}_1) | \Upsilon_s^{(\lambda)}(\vec{r}_1, \vec{r}_1') | \chi_i^{(1)}(\vec{r}_1') \rangle \right|^2, \tag{2.12}$$

with

$$E = \frac{\kappa_i^2}{2\mu_1} + \epsilon_i^{(1)} = \frac{\kappa_f^2}{2\mu_1} + \epsilon_f^{(1)}, \tag{2.13}$$

$$\mu_1 = \frac{m_1(m_2 + m_3)}{m_1 + m_2 + m_3}, \tag{2.14}$$

where m_1, m_2, and m_3 are the particle masses and $\epsilon_i^{(1)}$ and $\epsilon_f^{(1)}$ are the energies of the two-body system (2, 3).

III. EIKONAL APPROXIMATION FOR FWMN EXPANSION

We consider the reduction of the multiple-scattering expansion of $\Upsilon_s(\vec{r}_1, \vec{r}_1')$ in the eikonal approximation.[10–12] The eikonal approximation will be valid if

$$\hbar / \kappa a_0 \ll 1, \tag{3.1}$$

where κ is the magnitude of the relative momentum of the colliding particles. For the energy region $E - \epsilon^{(1)} \gg 1$, the eikonal criterion is satisfied.

The channel coordinate for the elastic scattering process is $\vec{R}_1$. We consider that the pair potentials are of the form

$$V_2(r_2) = V_2(|\vec{R}_1 + \gamma_2 \vec{r}_1|), \tag{3.2}$$

$$V_3(r_3) = V_3(|\vec{R}_1 + \gamma_3 \vec{r}_1|), \tag{3.3}$$

with

$$\gamma_2 = -\frac{m_2}{m_2 + m_3}, \quad \gamma_3 = \frac{m_3}{m_2 + m_3}. \tag{3.4}$$

In the eikonal approximation we have[12]

$$\langle \vec{r}_1 \vec{R}_1 | T_j | \vec{r}_1' \vec{R}_1' \rangle e^{i\vec{\kappa}_i \cdot \vec{R}_1'} \cong V_j(r_j) e^{iS_j^+(\vec{R}_1)} \times \delta(\vec{R}_1 - \vec{R}_1') \delta(\vec{r}_1 - \vec{r}_1'), \tag{3.5}$$

with the eikonal defined by the path integrals

$$S_j^+(\vec{R}_1) = \int^{\vec{R}_1} [\kappa_i^2 - 2\mu_1 V_j(|\vec{R}_1 + \gamma_j \vec{r}_1|)]^{1/2} ds, \tag{3.6}$$

where the path integrals are taken along the classical trajectory, with ds denoting the element of path. In Eq. (3.5), we have taken the eikonal amplitude to be unity and neglected the energy of the bound two-body system. These are reasonable approximations for the energy region of our interest.

It is worthwhile to recall that we are here dealing with the scattering of a particle by a two-body target through pair potentials V_2 and V_3. The Newtonian equations of motion for the determination of the classical trajectories must include both potentials. In the multiple-scattering approximation, these two potentials are treated one at a time. It is therefore convenient to introduce eikonals for each potential such as S_j^+ given by Eq. (3.6). Care must be taken, however, in the evaluation of the path integral along the classical trajectories provided by the two potentials. Of course, for the simple straight-line approximation for the classical trajectories, the problem is trivial.

When the eikonal approximation given by Eq. (3.5) is adopted in the multiple expansion for $\Upsilon_s(\vec{r}_1, \vec{r}_1')$, we obtain for the first-order term

$$\Upsilon_s^{(1)}(\vec{r}_1, \vec{r}_1') = \frac{\delta(\vec{r}_1 - \vec{r}_1')}{(2\pi)^3} \int d\vec{R}_1 \times (V_2 e^{i(S_2^+ - \vec{\kappa}_f \cdot \vec{R}_1)} + V_3 e^{i(S_3^+ - \vec{\kappa}_f \cdot \vec{R}_1)}). \tag{3.7}$$

It is convenient to evaluate the integral in the cylindrical-coordinate system (ρ, z, φ). To relative order $|\theta_c|$ (θ_c is the classical scattering angle) we have

$$d\mathrm{R}_1 = \rho\, d\rho\, dz\, d\varphi \cong b\, db\, dz\, d\varphi. \tag{3.8}$$

The eikonal can be written in the form[12]

$$S_j^+ - \vec{\kappa}_f \cdot \vec{R}_1 = q_\parallel z + q_\perp b \cos\varphi + \Phi_j(z, b), \tag{3.9}$$

with

$$q_\parallel = \kappa_i - \kappa_f \cos\theta, \quad q_\perp = -\kappa_f \sin\theta, \tag{3.10}$$

where $q_\parallel$ and $q_\perp$ are the longitudinal and transverse momentum transfer, θ is the scattering angle, and $\Phi_j(z, b)$ are the local eikonal phases, which depend also on the internal coordinates of the target.

To order $\kappa a_0 |\theta_c|^3$, $\Phi_j(z, b)$ takes the form[12]

$$\Phi_j(z, b) = -\kappa_i \int_{-\infty}^{z} \{\tfrac{1}{4}[\beta^2 - \beta^2(z')] + (\mu_1/\kappa_i^2) V_j\} dz', \tag{3.11}$$

with $\beta = \lim \beta(z)$ as $z \to \infty$ and,

$$\beta(z) = -d \int_0^z \frac{d}{dR_1} \left[1 - \left(1 - \frac{2\mu_1}{\kappa_i^2} (V_2 + V_3) \right)^{1/2} \right] \frac{dz'}{R_1} + O(\beta^2). \tag{3.12}$$

Here d is the distance of closest approach. If we assume the classical trajectories lie along straight lines, the eikonal phase takes the simple form

$$\Phi_j^{(s)}(z, b) = -(\mu_1/\kappa_i)\int_{-\infty}^{z} dz'\, V_j\,. \qquad (3.13)$$

A simple generalization of the straight-line trajectory consists of a trajectory with two straight-line segments intersecting at an angle 2β. In this angle approximation for the trajectory, the eikonal phase takes the form

$$\Phi_j^{(a)}(z, b) = \Phi_j^{(s)}(z, b) + 2(\mu_1/\kappa_i)\int_0^{d\sin\beta} V_j\, dz'\,. \qquad (3.14)$$

This simple generalization of the straight-line approximation has been found to be more accurate.[12–14]

To evaluate the integral in Eq. (3.7), we take the z axis to be parallel to the incident direction and write

$$\vec{R}_1 = \vec{b} + \hat{z}z, \quad \vec{r}_1 = \vec{s} + \hat{z}\xi, \qquad (3.15)$$

where $\vec{b} = (b, \varphi)$ and $\vec{s} = (s, \varphi_s)$ are two-dimensional vectors in the xy plane. From Eq. (3.7), we obtain

$$\Upsilon_s^{(1)}(\vec{r}_1, \vec{r}_1') = \frac{\delta(\vec{r}_1 - \vec{r}_1')}{(2\pi)^3}\int d^2b\, e^{iq_\perp b\cos\varphi}\, Q_s^{(1)}(b, \theta, s, \xi), \qquad (3.16)$$

with

$$Q_s^{(1)}(b, \theta, s, \xi) = \int_{-\infty}^{\infty} dz (V_2 e^{i\Phi_2(z,b)} + V_3 e^{i\Phi_3(z,b)}) e^{iq_\parallel z}. \qquad (3.17)$$

Similarly, the eikonal approximation can also be applied to the second-order multiple-scattering term $\Upsilon_s^{(2)}(\vec{r}_1, \vec{r}_1)$. We have

$$\langle \vec{r}_1\vec{R}_1 | T_j G_0 T_{j'} | \vec{r}_1'\vec{R}_1'\rangle e^{i\vec{\kappa}_i\cdot\vec{R}_1'}$$
$$\cong (2\pi)^{-3}\int d\vec{\kappa}\, d\vec{r}_1''\, d\vec{r}_1'''\, d\vec{R}_1''\, d\vec{R}_1'''\, \delta(\vec{r}_1'' - \vec{r}_1''')$$
$$\times \langle \vec{r}_1\vec{R}_1 | T_j | \vec{r}_j''\vec{R}_j''\rangle \frac{e^{i\vec{\kappa}\cdot\vec{R}_1'' - i\vec{\kappa}\cdot\vec{R}_1'''}}{E + i\eta - \kappa^2/2\mu_1}$$
$$\times \langle \vec{r}_1'''\vec{R}_1''' | T_j' | \vec{r}_1'\vec{R}_1'\rangle e^{i\vec{\kappa}_i\cdot\vec{R}_1'}, \qquad (3.18)$$

where we have again assumed the energy of the bound-state particles (i.e., particles 2 and 3) to be negligible in comparison with the energy of the projectile particles. Applying the eikonal approximation given by Eq. (3.5) to Eq. (3.18), we obtain from Eq. (2.11)

$$\Upsilon_s^{(2)} = \frac{\delta(\vec{r}_1 - \vec{r}_1')}{(2\pi)^3}\int d\vec{R}_1\, d\vec{R}_1' \left\{\left[V_j\left(\int d\vec{\kappa}\, \frac{e^{iS_j^+(\vec{R}_1) - i\vec{\kappa}\cdot\vec{R}_1'}}{E + i\eta - \kappa^2/2\mu_1}\right)\right.\right.$$
$$\left.\left.\times V_{j'}\, e^{iS_{j'}^+(\vec{R}_1') - i\vec{\kappa}_f\cdot\vec{R}_1}\right] + (j \leftrightarrow j')\right\}. \qquad (3.19)$$

At high energies, the Green's function propagates essentially in the forward direction. We may expand $\vec{\kappa}$ about $\vec{\kappa}_i$ and, to order $|\theta_c|$, neglect terms of second order in $\vec{\kappa} - \vec{\kappa}_i$. We then obtain[15]

$$\int d\vec{\kappa}\, \frac{e^{iS_j^+(\vec{R}_1) - i\vec{\kappa}\cdot\vec{R}_1'}}{E + i\eta - \kappa^2/2\mu_1} = \mu_1 \int d\vec{\kappa}\, \frac{e^{iS_j^+(\vec{R}_1) - i\vec{\kappa}\cdot\vec{R}_1'}}{2\mu_1 E + i\eta - \vec{\kappa}\cdot\vec{\kappa}_i}$$
$$= -i(2\pi)^3 e^{i\Phi_j(z,b)}\int_0^{\infty} dt$$
$$\times e^{i2Et}\, \delta(\vec{R}_1 - \vec{R}_1' - \vec{\kappa}_i t/\mu_1). \qquad (3.20)$$

This then permits $\Upsilon_s^{(2)}$ to take the form

$$\Upsilon_s^{(2)}(\vec{r}_1, \vec{r}_1') = \frac{\delta(\vec{r}_1 - \vec{r}_1')}{(2\pi)^3}\int d^2b\, e^{iq_\perp b\cos\varphi}\, Q_s^{(2)}(b, \theta, s, \xi), \qquad (3.21)$$

with

$$Q_s^{(2)}(b, \theta, s, \xi) = \{[(e^{i\Phi_{j'}(b)} - 1)\int_{-\infty}^{\infty} dz\, V_j e^{i\Phi_j(z,b) + iq_\parallel z}] + (j \to j')\}, \qquad (3.22)$$

$$\Phi_j(b) = \lim_{z\to\infty}\Phi_j(z, b), \qquad (3.23)$$

where we have made use of the integral

$$2i\int_0^{\infty} dt\, V_{j'}(R_1 - \kappa_i t/\mu_1)\, e^{i\Phi_{j'}(R_1 - \kappa_i t/\mu_1)} = 1 - e^{i\Phi_{j'}(b)} \qquad (3.24)$$

coming from the action of the Green's function [Eq. (3.20)] in Eq. (3.19).

The second-order multiple scattering in the eikonal approximation can now be obtained by combining Eq. (3.16) with Eq. (3.21). We obtain

$$\Upsilon_s(\vec{r}_1, \vec{r}_1') \cong \frac{\delta(\vec{r}_1 - \vec{r}_1')}{(2\pi)^3}\int d^2b\, e^{iq_\perp b\cos\varphi}\, Q_s(b, \theta, s, \xi), \qquad (3.25)$$

$$Q_s(b, \theta, s, \xi) = \tfrac{1}{2}[(1 + e^{i\Phi_3(b)})\int_{-\infty}^{\infty} dz\, V_2 e^{i[q_\parallel z + \Phi_2(z,b)]}] + \tfrac{1}{2}(3 \leftrightarrow 2). \qquad (3.26)$$

The z integrals in Eq. (3.26) can be evaluated analytically, if the longitudinal momentum transfer can be neglected. The neglect of the longitudinal momentum transfer is a characteristic of the well-known Glauber-type eikonal approximation,[11] by taking the momentum transfer

$$\vec{q}_{fi} = \vec{\kappa}_i - \vec{\kappa}_f \qquad (3.27)$$

to be perpendicular to the incident direction $\hat{\kappa}_i$. This is equivalent to the approximations

$$q_\perp \approx q_{fi}, \quad q_\parallel \approx 0. \qquad (3.28)$$

In this case, the z integral can be evaluated to relative order $|\theta_c|$:

$$Q_s(b, s, \xi) = i(\kappa_i/\mu_1)(e^{i[\Phi_2(b,s,\xi) + \Phi_3(b,s,\xi)]} - 1). \qquad (3.29)$$

Substitution of Eqs. (3.28) and (3.29) into Eq. (3.26) yields

$$\Upsilon_s(r, r') = i\,\frac{\delta(r - r_1')}{(2\pi)^3}\,\frac{\kappa_i}{\mu_1}\int d^2b\, e^{iq_{fi}\cdot b}(e^{i(\Phi_2 + \Phi_3)} - 1). \qquad (3.30)$$

Equation (3.30) would reduce to the familiar expresssion of the Glauber type if Φ_2 and Φ_3 are evaluated in the straight-line approximation. The increased generality in the eikonal phase, Φ_2 and Φ_3 has not altered the simple form of Eq. (3.30).

IV. PROBLEMS WITH COULOMB INTERACTION

Owing to the long-range nature of the Coulomb interaction, the Coulomb T matrix has branch-point singularities on the energy shell.[16] At high energies, we have in momentum representation [see Eq. (VI4.2)]

$$T_i(\vec{p}_i, \vec{p}_i';\ E-q_i^2) \cong V_i(\vec{p}_i, \vec{p}_i')\, e^{-i\nu_i \ln(\epsilon_i/4)}, \quad (4.1)$$

with [see Eqs. (III2.14) and (III2.15)]

$$\nu_i = \frac{Z_i e^2(\frac{1}{2}\mu_{jk})^{1/2}}{(E-q_i^2)^{1/2}}, \quad \epsilon_i = \frac{(E-q_i^2-p_i^2)(E-q_i^2-p_i'^2)}{(E-q_i^2)|\vec{p}_i-\vec{p}_i'|^2}, \quad (4.2)$$

where Z_i is the product of the charges of particles j and k, and μ_{jk} is their reduced mass. The momentum variables in Eqs. (4.1) and (4.2) are defined in the previous papers of this series. From this equation, it is clear that in the high-energy limit, the Coulomb T matrix approaches the Coulomb potential with a phase factor which contains branch-point singularities. It is this singular phase factor in the two-body amplitude which gives rise to difficulties in the treatment of three-body amplitude involving pair Coulomb interactions.

Mathematically it is clear that the branch-point singularity in the two-body amplitude would give rise to integrals of discontinuity in the three-body amplitude. Such integrals can be handled and in certain cases be evaluated numerically in the first-order FWMS approximation. For elastic as well as excitation scatterings, the result depends on the difference of contributions coming from two pair Coulomb amplitudes. Substantial numerical accuracies are required for reasonably reliable results. The task of evaluating these integrals of discontinuity for the higher-order FWMS approximation numerically is not at all straightforward. It is therefore necessary to rely on approximations.

The difficulties with the branch-point singularity are completely avoided in the Born approximation by simply taking the phase factor to be unity. Thus, in the Born approximation one encounters in the pair interaction at the most a pole singularity at the forward-angle elastic scattering. For three-body Coulomb problems in which the two-body target is neutral, even this forward-angle singularity in the elastic scattering gets canceled. Consequently, the three-body Coulomb amplitudes are well behaved in the Born approximation. The situations, however, are not well defined once one goes beyond the Born approximation. This is not unexpected, since one must then take the singular phase into consideration.

The eikonal approximation considered in Sec. III also gives rise to singularities in the two-body amplitude for Coulomb interactions. The problem here, however, is not as subtle as in the other types of distorted-wave or impulse approximations. The interesting feature of the eikonal approximation lies in the fact that the singularity in the two-body Coulomb amplitude gets canceled when the first-order three-body amplitude is combined with the second-order three-body amplitude in the second-order FWMS approximation. It is this analytic cancellation of the branch-point singularities which permits the evaluation of the three-body amplitude for Coulomb problems (involving neutral two-body target) to the second-order FWMS approximation.

For Coulomb interactions, the pair potentials [see Eqs. (3.2) and (3.3)] take the form

$$V_j(\vec{r}_j) = \frac{Z_j}{[\,|\vec{b}+\gamma_j\vec{s}\,|^2+(z+\gamma_j\xi)^2]^{1/2}}, \quad (4.3)$$

where we have made use of Eq. (3.15) for $\vec{r}_1$. It can be easily shown that the eikonal phase for Coulomb interaction is singular. For simplicity, let us consider the eikonal phase in the straight-line approximation. From Eq. (3.13), we have

$$\Phi_j^{(3)}(z,b) = -\frac{\mu_1}{\kappa_i}\lim_{z_0\to\infty}\int_{-z_0}^{z}\frac{Z_j\,dz'}{[\,|\vec{b}+\gamma_j\vec{s}\,|^2+(z'+\gamma_j\xi)^2]^{1/2}} = -\frac{\mu_1 Z_j}{\kappa_i}\lim_{z_0\to\infty}\ln\left(\frac{[\,|\vec{b}+\gamma_j\vec{s}\,|^2+(z+\gamma_j\xi)^2]^{1/2}+(z+\gamma_j\xi)}{[\,|\vec{b}+\gamma_j\vec{s}\,|^2-(z_0-\gamma_j\xi)^2]^{1/2}-(z_0-\gamma_j\xi)}\right). \quad (4.4)$$

It is then clear that the local eikonal phase for the Coulomb interaction is not defined in the $z\to\infty$ and $z_0\to\infty$ limits. This difficulty with the long-range Coulomb interaction can be found in various impulse approximations.

As a consequence of these singular phases, neither the first-order nor the second-order multiple-scattering terms are defined in the eikonal approximation. These difficulties can be circumvented by taking these limits after the first-order and second-order terms are combined together in the form given, for example, by Eq. (3.30). This follows from the fact that the sum of the phases takes the form

$$\Phi_2^{(s)}(b)+\Phi_3^{(s)}(b) = \lim_{z\to\infty}[\Phi_2^{(s)}(z,b)+\Phi_3^{(s)}(z,b)]$$

$$= \frac{2\mu_1}{\kappa_i}\left[\ln\left(\frac{|\vec{b}+\gamma_2\vec{s}\,|^{|Z_2|}}{|\vec{b}+\gamma_3\vec{s}\,|^{|Z_3|}}\right)\right.$$

$$+\left(|Z_3|-|Z_2|\right)\lim_{z_0,z\to\infty}\left[\ln(4zz_0)\right]\Big] . \qquad (4.5)$$

For a neutral target, we have $|Z_3| = |Z_2|$ and the sum of the eikonal phases is well behaved.

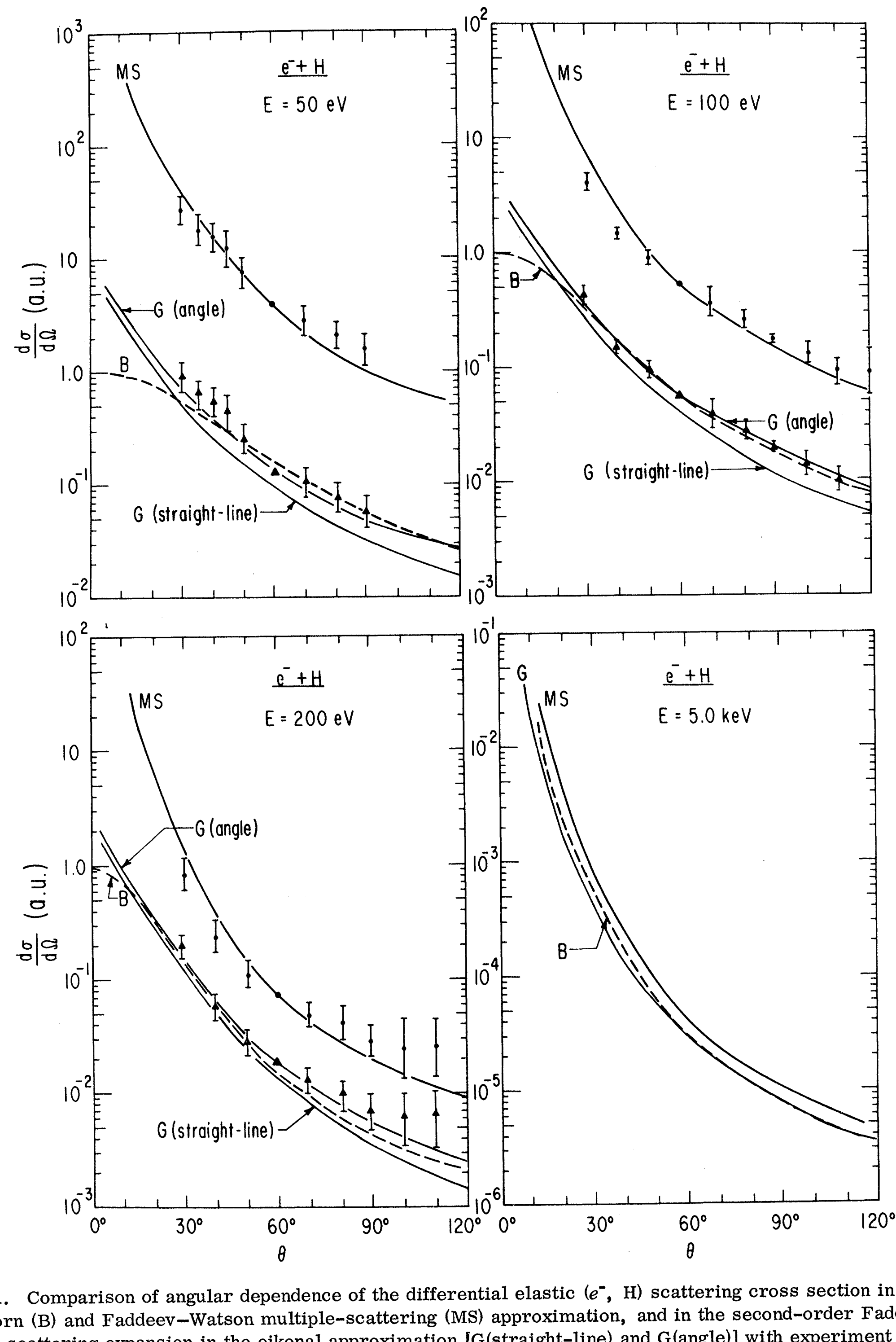

FIG. 1. Comparison of angular dependence of the differential elastic (e^-, H) scattering cross section in the first-order Born (B) and Faddeev–Watson multiple-scattering (MS) approximation, and in the second-order Faddeev–Watson multiple-scattering expansion in the eikonal approximation [G(straight-line) and G(angle)] with experiment (Ref. 20), at several fixed laboratory energies. The experimental data are normalized both to the MS and G (angle) theoretical results, at 60°, for each energy.

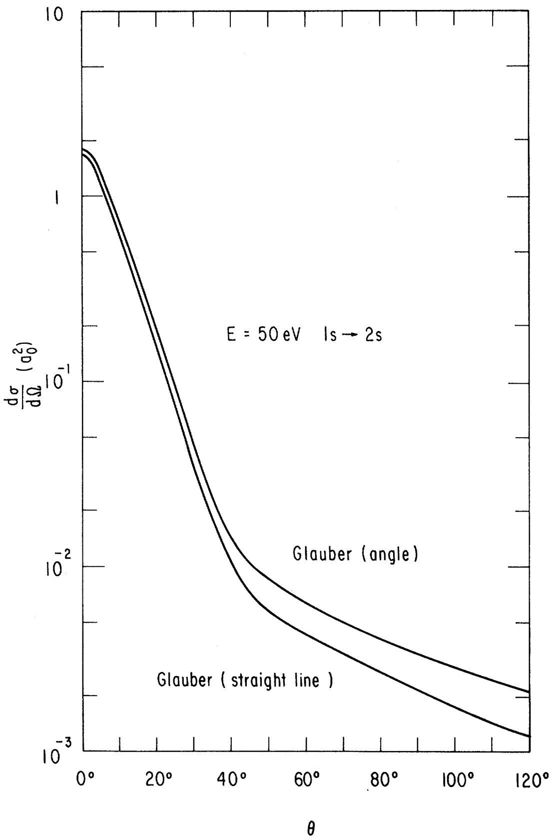

FIG. 2. Angular dependence of differential $1s$-$2s$-excitation cross section obtained in the second-order FWMS scattering expansion in the eikonal approximation, with straight-line and angle trajectories.

V. APPLICATION TO (e, H) SCATTERING

The three-body (e, H) problem involves Coulomb pair interactions. Applications of the first-order FWMS approximation to the (e, H) problem have been carried out in Paper VII. It is observed that the first-order FWMS result converges to the Born result at energies much higher than previously expected. The question was raised whether the inclusion of second-order FWMS would give rise to cancellations and bring the FWMS approximation in agreement with the first-order Born approximation at a somewhat lower energy. The eikonal approximation presented in Secs. III and IV permits an estimate of such cancellations.

We shall consider the application of the eikonal approximation given by Eq. (3.30) with the longitudinal momentum transfer neglected. This is a type of Glauber-eikonal approximation. Since the application of the Glauber-eikonal approximation to the (e, H) system has been carried out[17–19] in the straight-line trajectories, we should consider the "angle" approximation for the classical trajectories.

The additional phase in the "angle" approximation given in Eq. (3.14) can be evaluated for Coulomb interactions. We have

$$\Delta\Phi_{23}(b) \equiv 2\,\frac{\mu_1}{\kappa_i}\int_0^{d\sin\beta}(V_2+V_3)\,dz$$

$$\cong 2\,\frac{\mu_1}{\kappa_i}\,d\sin\beta\,(V_2+V_3)_{z=d\sin\beta}, \qquad (5.1)$$

with

$$\beta=(2\kappa_i)^{-1}\,\frac{d}{db}\,[\Phi_2^{(s)}(b)+\Phi_3^{(s)}(b)]. \qquad (5.2)$$

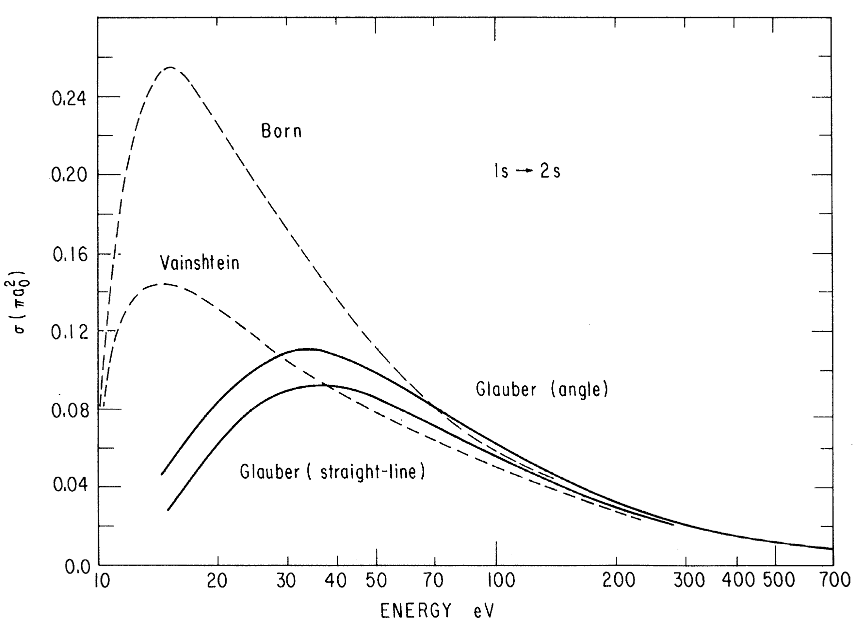

FIG. 3. Comparison of the energy dependence of the $1s$-$2s$-excitation cross section obtained in the first-order Born, Vainshtein, and the second-order FWMS scattering expansion in the eikonal approximation with straight-line and angle trajectories.

Thus, the eikonal phase in the "angle" approximation takes the form

$$\Phi_2^{(b)} + \Phi_3^{(b)}(b) = \Phi_2^{(s)}(b) + \Phi_3^{(s)}(b) + \Delta\Phi_{23}. \qquad (5.3)$$

Calculations have been carried out for the elastic and 2s-excitation differential and total cross sections in the "angle" approximation. The results are presented in graphic forms.

In Fig. 1 comparison of the present result with the first-order FWMS and Born results, as well as with the experimental data,[20] is made. The experimental data are normalized to both the first-order FWMS and the Glauber-eikonal (in the angle approximation) theoretical results at 60° for each incident energy. It is seen that the angle approximation does increase the differential cross section from the straight-line approximation. The magnitude of the Glauber-eikonal results is, in general, consistent with the Born approximation[21] and is much smaller than that of the first-order FWMS results. The difference between the Glauber-eikonal and the first-order FWMS result decreases with increasing energy. These results suggest that there is a substantial cancellation between the first-order and second-order terms in the FWMS expansion for pair Coulomb interactions.

It is, however, difficult to assess the accuracy of the eikonal approximation. Since the complete cancellation of the singularity in the eikonal phase took place only in the approximation in which the longitudinal momentum transfer is neglected, the Glauber-eikonal approximation, just like the Born approximation, is incapable of distinguishing the electron scattering from positron scattering. It is possible that the Glauber-eikonal approximation overestimates the cancellation between the first-order and second-order FWMS terms. The simple trajectory correction does increase the magnitude of the differential cross section. This is also true for the excitation differential cross section as shown in Fig. 2.

A comparison of the total 1s-2s-excitation cross section with other theoretical results[22] is shown in Fig. 3. It is seen that the inclusion of the second-order FWMS term has the importance of preventing the overestimation of the total cross section at low energies.

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†On leave from the University of Oslo, Norway.

‡Present address: IBM Research Laboratory, San Jose, Calif. 95114.

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