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# Retarded van der Waals Forces at All Distances Derived from Classical Electrodynamics with Classical Electromagnetic Zero-Point Radiation

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The van der Waals forces between a polarizable particle and a conducting wall and between two polarizable particles are calculated within the theory of classical electrodynamics with classical electromagnetic zero-point radiation. This theory assumes the differential equations of traditional classical electrodynamics but changes the homogeneous boundary condition on Maxwell's equations to correspond to the presence of random classical electromagnetic radiation with a Lorentz-invariant spectrum. The van der Waals force calculations are performed exactly within the nonrelativistic equations of motion for the particles represented as point-dipole oscillators. The classical results are found to agree identically to all orders in the fine-structure constant $\alpha$ with the nonrelativistic quantum electrodynamic calculations of Renne. To fourth order, there is agreement with the perturbation-theory work of Casimir and Polder.

## I. INTRODUCTION

In work published recently,[1,2] it has been shown that the short- and long-range asymptotic limits for interatomic van der Waals forces can be calculated from a simple classical model. Picturing atoms or molecules in the Drude–Lorentz approximation as classical dipole oscillators, one can understand the van der Waals forces between a neutral polarizable particle and a conducting wall or between two neutral polarizable particles as due to classical electromagnetic interactions when the particles are immersed in random classical radiation with a Lorentz invariant spectrum. In two previous articles, the unretarded London force[2] when the particle separation $R\to 0$, and the asymptotic retarded force[1] when $R\to\infty$ were evaluated and found to agree exactly with quantum calculations; the present work extends the agreement between the theories for these forces to the entire range of separations $R$. The full fourth-order Casimir–Polder formula[3] from quantum electrodynamics has thus been obtained from a purely classical theory of electromagnetism.

There has been continuing interest in van der Waals forces within both physics and chemistry, and recently even attention to applications of van der Waals forces in biological systems. However, the present article does not produce any new formula for application to a specific situation. Rather the calculation here represents a further step in a general program in theoretical physics. The program is intended to discover just how much of the physics which is presently regarded as dependent upon the notion of discrete quanta can actually be understood within a specific theory of purely classical electromagnetism. The theory, which we have termed classical electrodynamics with classical electromagnetic zero-point radiation, adopts the differential equations of traditional classical electrodynamics but changes the homogeneous boundary condition on Maxwell's equations to correspond to the presence of random classical electromagnetic radiation with a Lorentz invariant spectrum. Thus far, a number of phenomena within statistical thermodynamics have been analyzed in terms of this theory—including the blackbody-radiation spectrum,[4] the fluctuations usually ascribed to photon statistics,[5] the third law of thermodynamics,[6] and oscillator and rotator specific heats.[7] Marshall has applied the theory to the van der Waals forces between macroscopic objects.[8]

The present article extends the theory to van der Waals forces between microscopic particles described as point dipole oscillators.

The quantum-mechanical work on van der Waals forces was initiated by London[9] in 1930 when he obtained the force between two quantum dipole oscillators interacting through the Coulomb field. In 1948 Casimir and Polder[3] performed the full fourth-order quantum-electrodynamic calculation of the force between two neutral polarizable particles. A number of recalculations have followed the Casimir-Polder result. Some involving dispersion theory[10] or canonical transformations[11] within perturbation theory have obtained the full fourth-order expression; some employing quantum zero-point energy[12] have obtained only the asymptotic limit forms. Recently, Renne[13] has provided a quantum-electrodynamic calculation to all orders.

It is apparent that the two-particle van der Waals force involves a relatively simple system which is also fundamental. It is a natural place to investigate the predictions of an unexplored theory such as classical electrodynamics with classical electromagnetic zero-point radiation. In earlier papers[1,2] the asymptotic limits were considered; in the present work, a calculation valid for all particle separations is provided. The calculation here is exact for the nonrelativistic classical theory. It turns out to agree precisely with Renne's results from nonrelativistic quantum electrodynamics.

The material to follow is broken into three basic sections: a calculation of the force between a polarizable particle and a conducting wall, a calculation of the force between two polarizable particles, and finally a sketch of Renne's procedure for connecting the exact expressions with the formula of Casimir and Polder.

## II. POLARIZABLE PARTICLE NEAR A CONDUCTING WALL

### A. Basic Model

The calculation for the van der Waals force between a neutral polarizable particle and a perfectly conducting wall was first carried out by Casimir and Polder,[3] who found that the attractive potential varied asymptotically as $R^{-3}$ when the separation $R$ between the particle and the wall became small, and as $R^{-4}$ when this separation became large. In previous papers, we have shown that the purely classical theory involving zero-point radiation exactly reproduces the Casimir–Polder asymptotic limits. In the present work we will show the agreement for all separations $R$.

Our model pictures the atom as a dipole oscillator, and this oscillator will intact with its dipole image in the conducting wall. The random zero-point radiation forces the dipole oscillator into random motion. In free space the Lorentz-invariant spectrum[14] of random zero-point radiation is given as a sum over plane waves

$$\vec{E}_{ZP}(\vec{r},t)=\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\hat{\epsilon}(\vec{k},\lambda)\mathfrak{h}(\vec{k},\lambda)\times\exp[i(\vec{k}\cdot\vec{r}-\omega t+\vartheta(\vec{k},\lambda))]\,, \tag{1}$$

$$\hat{\epsilon}(\vec{k},\lambda)\cdot\hat{\epsilon}(\vec{k},\lambda')=\delta_{\lambda\lambda'}\,,\qquad \vec{k}\cdot\hat{\epsilon}(\vec{k},\lambda)=0,\quad \omega=ck\,, \tag{2}$$

and with the scale set by Planck's constant

$$\pi^2\mathfrak{h}^2=\tfrac{1}{2}\hbar\omega\,. \tag{3}$$

The random character of the radiation is provided by the use of the random phase $\vartheta(\vec{k},\lambda)$, as in the work on random classical radiation by Planck[15] and Einstein and Hopf.[16]

The equation for the oscillator on the $z$ axis at a distance $R$ from a conducting wall in the $xy$ plane given by

$$m\frac{d^2\vec{\xi}}{dt^2}=-m\omega_0^2\vec{\xi}+\frac{2}{3}\frac{e^2}{c^3}\frac{d^3\vec{\xi}}{dt^3}+e\vec{E}_{\mathrm{IMG}}+e\vec{E}_{\mathrm{RZP}}\,, \tag{4}$$

where $\vec{\xi}$ is the displacement of the oscillator of mass $m$ and charge $e$, while $-m\omega_0^2\vec{\xi}$ represents the harmonic restoring force, $\frac{2}{3}(e^2/c^3)(d^3\vec{\xi}/dt^3)$ is the radiation damping self-force, and $\vec{E}_{\mathrm{IMG}}$ is the field due to the image dipole. The zero-point radiation is reflected from the conducting wall and, hence, must satisfy this boundary condition in the form

$$\begin{aligned}\vec{E}_{\mathrm{RZP}}(\vec{r},t)&=\mathrm{Re}\sum_{\lambda=1}^{2}\int_{k_z<0}d^3k\,\mathfrak{h}\,\{\hat{\epsilon}\exp[i(\vec{k}\cdot\vec{r}-\omega t+\vartheta)]+(-\hat{i}\epsilon_x-\hat{j}\epsilon_y+\hat{k}\epsilon_z)\exp[i(k_xx+k_yy-k_zz-\omega t+\vartheta)]\}\\&=\mathrm{Re}\sum_{\lambda=1}^{2}\int_{k_z<0}d^3k\,\mathfrak{h}2[i(\hat{i}\epsilon_x+\hat{j}\epsilon_y)\sin k_zz+\hat{k}\epsilon_z\cos k_zz]\exp[i(k_xx+k_yy-\omega t+\vartheta)]\,.\end{aligned} \tag{5}$$

We will use the dipole approximation which evaluates the fields $\vec{E}_{\mathrm{IMG}}$ and $\vec{E}_{\mathrm{RZP}}$ at the center of mass of the oscillator.

It seems convenient to change from the oscillator displacement $\vec{\xi}$ over to the dipole moment of the oscillator

$$\vec{p}=e\vec{\xi}\,. \tag{6}$$

We may then rewrite the oscillator equation of motion as

$$\ddot{\vec{p}} = -\omega_0^2\vec{p} + \Gamma\dddot{\vec{p}} + \tfrac{3}{2}\Gamma c^3\vec{E}_{\rm IMG} + \tfrac{3}{2}\Gamma c^3\vec{E}_{\rm RZP}\,, \tag{7}$$

where

$$\Gamma = \frac{2}{3}\frac{e^2}{mc^3}\ . \tag{8}$$

The oscillator equation of motion (7) refers only to the natural frequency $\omega_0$ and damping constant $\Gamma$, and does not involve the individual mechanical properties of the oscillator such as charge or mass.

### B. Fourier Decomposition of the Equation of Motion

Introducing a Fourier decomposition for $\vec{p}(t)$ which formally follows that for $\vec{E}_{\rm RZP}$

$$\vec{p}(t) = {\rm Re}\sum_{\lambda=1}^{2}\int_{k_z<0} d^3k\,\vec{P}(\vec{k},\lambda)\,e^{-i\omega t}\,, \tag{9}$$

$$\vec{E}_{\rm RZP}(\vec{r},t) = {\rm Re}\sum_{\lambda=1}^{2}\int_{k_z<0} d^3k\,\vec{E}_R(\vec{r},\vec{k},\lambda)\,e^{-i\omega t}\,, \tag{10}$$

the equation of motion becomes

$$(C+\Sigma_x)P_x = \beta E_{Rx}\,, \tag{11}$$

with analogous equations for the $y$ and $z$ components. Here we have introduced the symbols

$$C = -\omega^2 + \omega_0^2 - i\Gamma\omega^3\,, \tag{12}$$

$$\beta = \tfrac{3}{2}\Gamma c^3 = e^2/m\,, \tag{13}$$

and

$$\Sigma_x = \Sigma_y = +\tfrac{3}{2}\Gamma\omega^3\left(\frac{1}{2kR} + \frac{i}{(2kR)^2} - \frac{1}{(2kR)^3}\right)e^{i2kR}\,, \tag{14}$$

$$\Sigma_z = -\tfrac{3}{2}\Gamma\omega^3\,2\left(\frac{-i}{(2kR)^2} + \frac{1}{(2kR)^3}\right)e^{i2kR}\ . \tag{15}$$

The terms in $\Sigma$ provide the fields of the image dipole.[17] Note that the signs involved in $\Sigma_x$ and $\Sigma_y$ recognize that the image dipole for orientations along the $x$ and $y$ axes is reversed in orientation from the initial dipole. Clearly the solution to Eq. (11) is

$$P_x = \beta E_{Rx}/(C+\Sigma_x)\ . \tag{16}$$

### C. Force on a Dipole

The force on a dipole is given from classical theory as

$$\vec{F} = (\vec{p}\cdot\nabla)\vec{E} + \frac{1}{c}\frac{d\vec{p}}{dt}\times\vec{B}\,, \tag{17}$$

where $\vec{E}$ and $\vec{B}$ are the total electric and magnetic fields. The time-averaged force can be transformed as

$$\langle\vec{F}\rangle = \left\langle(\vec{p}\cdot\nabla)\vec{E} + \frac{1}{c}\frac{d\vec{p}}{dt}\times\vec{B}\right\rangle = \left\langle(\vec{p}\cdot\nabla)\vec{E} - \frac{\vec{p}}{c}\times\frac{\partial\vec{B}}{\partial t}\right\rangle$$

$$= \langle(\vec{p}\cdot\nabla)\vec{E} + \vec{p}\times(\nabla\times\vec{E})\rangle\ . \tag{18}$$

The first step here involves an integration by parts, noting that the end-point terms do not contribute to the time average; the second step uses Maxwell's equations. In the present case, symmetry dictates that the average force on the oscillator will be normal to the wall,

$$\langle\vec{F}\rangle = \hat{z}\langle\vec{F}_z\rangle = \hat{z}\left\langle p_x\frac{\partial E_x}{\partial z} + p_y\frac{\partial E_y}{\partial z} + p_z\frac{\partial E_z}{\partial z}\right\rangle\ . \tag{19}$$

The self-fields of an oscillator do not contribute an average force so that the electric field $\vec{E}$ at the oscillator is that due to the image dipole and to zero-point radiation. Thus terms such as $\langle p_x\partial E_x/\partial z\rangle$ become

$$\left\langle p_x\frac{\partial E_x}{\partial z}\right\rangle = \left\langle p_x\frac{\partial}{\partial z}E_{\rm RZP\it x}\right\rangle + \left\langle p_x\frac{\partial}{\partial z}E_{\rm IMG\it x}\right\rangle\ . \tag{20}$$

### D. Force Due to Incident and Reflected Fields

The first term in (20) is

$$\left\langle p_x\frac{\partial}{\partial z}E_{\rm RZP\it x}\right\rangle = \tfrac{1}{2}{\rm Re}\left\langle\sum_{\lambda=1}^{2}\int_{k_z<0}d^3k\times\frac{\beta E_{Rx}(0,0,R;\vec{k},\lambda)}{C+\Sigma_x}e^{-i\omega t}\times\sum_{\lambda'=1}^{2}\int_{k_z'<0}d^3k'\times\frac{\partial}{\partial z}E_{Rz}^*(0,0,z;\vec{k}',\lambda')\Big|_{z=R}e^{i\omega' t}\right\rangle, \tag{21}$$

where from (5) and (10)

$$E_{Rx}(0,0,R;\vec{k},\lambda) = \mathfrak{h}(\vec{k},\lambda)2i\epsilon_x(\vec{k},\lambda)\sin k_zR\,e^{i\vartheta(\vec{k},\lambda)} \tag{22}$$

and

$$\frac{\partial}{\partial z}E_{Rx}(0,0,z;\vec{k}',\lambda')\big|_{z=R} = \mathfrak{h}(\vec{k}',\lambda')2i\epsilon_x(\vec{k}',\lambda')k_z'\cos k_z'R\,e^{i\vartheta(\vec{k},\lambda)}\ . \tag{23}$$

The expression $\frac{1}{2}$Re is introduced because of the use of complex notation in the averaging. Treating the averaging over the random phases $\vartheta(\vec{k},\lambda)$ and $\vartheta(\vec{k}',\lambda')$,

$$\langle e^{i\vartheta(\vec{k},\lambda)}e^{-i\vartheta(\vec{k}',\lambda')}\rangle = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\,, \tag{24}$$

$$\langle e^{i\vartheta(\vec{k},\lambda)}e^{i\vartheta(\vec{k}',\lambda')}\rangle = 0\ . \tag{25}$$

Thus

$$\left\langle p_x\frac{\partial}{\partial z}E_{\rm RZP\it x}\right\rangle = \tfrac{1}{2}{\rm Re}\sum_{\lambda=1}^{2}\int_{k_z<0}d^3k\frac{\beta\mathfrak{h}^2 4}{C+\Sigma_x}\times\epsilon_x^2k_z\sin k_zR\cos k_zR\ . \tag{26}$$

Since the integrand is even in $k_z$, we may introduce a further factor of $\frac{1}{2}$ and carry the integral over all values of $\vec{k}$. The expression for $\langle p_y\, \partial E_{\mathrm{RZP}z}/\partial z\rangle$ is the same as (26) with the subscript $x$ replaced everywhere by $y$, while the expression for $\langle p_z\, \partial E_{\mathrm{RZP}z}/\partial z\rangle$ is the same as (26) with the subscript $x$ replaced everywhere by $z$ and an additional over-all minus sign from differentiating $\cos k_z R$ in the equation analogous to (23).

The sum over the two polarizations gives

$$\sum_{\lambda=1}^{2} \epsilon_i^{\,2} = 1 - \frac{k_i^{\,2}}{k^2} \ . \tag{27}$$

The angular integrations needed for Eq. (26) can be carried out before differentiating with respect to $R$ by noting

$$\epsilon_x^{\,2} k_z \sin k_z R \cos k_z R = \frac{1}{2}\, \frac{\partial}{\partial R}\, (\epsilon_x^{\,2} \sin^2 k_z R) \ . \tag{28}$$

Then the angular integrations are

$$\sum_{\lambda=1}^{2} \int d\Omega\, \epsilon_x^{\,2} \sin k_z R = \int_{\theta=0}^{\pi} d\theta \sin\theta \times \int_{\phi=0}^{2\pi} d\phi\, (1 - \cos^2\phi \sin^2\theta) \sin^2(kR\cos\theta)$$
$$= \frac{2\pi}{\beta k^3}\, (\Gamma - \mathrm{Im}\Sigma_x) = -\frac{2\pi}{\beta k^3}\, \mathrm{Im}(C + \Sigma_x) \ , \tag{29}$$

$$\sum_{\lambda=1}^{2} \int d\Omega\, \epsilon_y^{\,2} \sin^2 k_z R = \frac{2\pi}{\beta k^3}\, (\Gamma - \mathrm{Im}\Sigma_y) = -\frac{2\pi}{\beta k^3}\, \mathrm{Im}(C + \Sigma_y) \ , \tag{30}$$

$$\sum_{\lambda=1}^{2} \int d\Omega\, \epsilon_z^{\,2} \cos^2 k_z R = \frac{2\pi}{\beta k^3}\, (\Gamma - \mathrm{Im}\Sigma_z) = -\frac{2\pi}{\beta k^3}\, \mathrm{Im}(C + \Sigma_z) \ . \tag{31}$$

The expression (26) becomes

$$\left\langle p_x \frac{\partial}{\partial z} E_{\mathrm{RZP}x} \right\rangle = \frac{1}{2}\, \frac{1}{2} \int_{k=0}^{\infty} dk\, k^2 \beta \mathfrak{h}^2 \times 4\left( -\frac{2\pi}{\beta k^3}\, \frac{1}{2}\, \frac{\partial}{\partial R}\, \mathrm{Im}\Sigma_x \right) \mathrm{Re}\, \frac{1}{C + \Sigma_x} \ , \tag{32}$$

and the contributions involving $y$ and $z$ are found by merely replacing the subscript $x$.

### E. Force Due to Field of Image Dipole

The second term in (20) involves the field due to the image dipole $\langle p_x \partial E_{\mathrm{IMG}x}/\partial z\rangle$. Here we must be careful to understand the original classical force expression in (20). All derivatives are with respect to the field coordinate, and not with respect to the source of the field. Thus recalling that for $p_x$ the image dipole is $-p_x$,

$$E_{\mathrm{IMG}x}(0, 0, z, t) = \mathrm{Re} \sum_{\lambda=1}^{2} \int_{k_z<0} d^3k\, \frac{1}{\beta}\, \frac{3}{2}\, \Gamma \omega^3 \times \left( \frac{1}{k(z+R)} + \frac{i}{k^2(z+R)^2} - \frac{1}{k^3(z+R)^3} \right) \times e^{ik(z+R)} (-P_x) e^{-i\omega t} \ , \tag{33}$$

so that

$$\frac{\partial}{\partial z} E_{\mathrm{IMG}x}(0, 0, z, t)\big|_{z=R} = \mathrm{Re} \sum_{\lambda=1}^{2} \int_{k_z<0} d^3k \times P_x e^{-i\omega t} \left( -\frac{1}{\beta}\, \frac{1}{2}\, \frac{\partial \Sigma_x}{\partial R} \right) \ , \tag{34}$$

with analogous expressions in $y$ and $z$. Observing this one delicate point,

$$\left\langle p_x \frac{\partial}{\partial z} E_{\mathrm{IMG}x} \right\rangle = \tfrac{1}{2} \mathrm{Re} \left\langle \sum_{\lambda=1}^{2} \int_{k_z<0} d^3k\, \frac{\beta E_{Rx}}{C + \Sigma_x}\, e^{-i\omega t} \times \sum_{\lambda'=1}^{2} \int_{k_z'<0} d^3k'\, \frac{\beta E_{Rx}^*}{C^* + \Sigma_x^*} \left( -\frac{1}{2\beta}\, \frac{\partial \Sigma_x^*}{\partial R} \right) \right\rangle \ . \tag{35}$$

Averaging over the random phases as in (24) and (25), this becomes

$$\left\langle p_x \frac{\partial}{\partial z} E_{\mathrm{IMG}x} \right\rangle = \frac{1}{2} \sum_{\lambda=1}^{2} \int_{k_z<0} d^3k\, \frac{\beta^2 \mathfrak{h}^2 4}{|C + \Sigma_x|^2} \times (\epsilon_x^{\,2} \sin^2 k_z R) \left( -\frac{1}{2\beta}\, \frac{\partial}{\partial R}\, \mathrm{Re}\Sigma_x^* \right). \tag{36}$$

The term in $p_y$ is found merely by changing the $x$ subscripts to $y$, and that for $p_z$ by changing subscripts and also replacing $\sin^2 k_z R$ by $\cos^2 k_z R$. Again the expression is even in $k_z$ and so the integral may be converted to all $\vec{k}$ values while introducing a compensating factor of $\frac{1}{2}$.

The angular integration needed is exactly that listed in Eq. (29), so that

$$\left\langle p_x \frac{\partial}{\partial z} E_{\mathrm{IMG}x} \right\rangle = \frac{1}{2}\, \frac{1}{2} \int_{k=0}^{\infty} dk\, k^2\, \frac{\beta \mathfrak{h}^2 4}{|C + \Sigma_x|^2} \times \left( -\frac{2\pi}{\beta k^3}\, \mathrm{Im}(C + \Sigma_x) \right) \left( -\frac{1}{2}\, \frac{\partial}{\partial R}\, \mathrm{Re}\Sigma_x^* \right). \tag{37}$$

### F. Total Force

It is clear that Eqs. (32) and (37) may be combined in a symmetrical form if we rewrite part of (32) noting

$$\mathrm{Re}\, \frac{1}{C + \Sigma_x} = \frac{1}{|C + \Sigma_x|^2}\, \mathrm{Re}(C^* + \Sigma_x^*) \ . \tag{38}$$

Then

$$\left\langle p_x \frac{\partial}{\partial z} E_x \right\rangle = -\pi \int_{k=0}^{\infty} dk\, \frac{\mathfrak{h}^2}{k} \left[ \frac{1}{|C + \Sigma_x|^2} \left( \mathrm{Re}(C^* + \Sigma_x^*) \right. \right.$$

$$\times \frac{\partial}{\partial R}\,\mathrm{Im}\Sigma_x - \mathrm{Im}(C+\Sigma_x)\,\frac{\partial}{\partial R}\,\mathrm{Re}\Sigma_x^*\Big)\Big] . \qquad (39)$$

Inasmuch as $C$ is independent of the distance $R$ so that

$$\frac{\partial C}{\partial R} = \frac{\partial C^*}{\partial R} = 0 , \qquad (40)$$

the factor in square brackets in Eq. (39) becomes

$$\begin{aligned}
&\frac{1}{|C+\Sigma_x|^2}\left(\mathrm{Re}(C^*+\Sigma_x^*)\,\frac{\partial}{\partial R}\,\mathrm{Im}\Sigma_x - \mathrm{Im}(C+\Sigma_x)\,\frac{\partial}{\partial R}\,\mathrm{Re}\Sigma_x^*\right)\\
&\quad = \frac{1}{|C+\Sigma_x|^2}\,\frac{1}{2i}\left((C^*+\Sigma_x^*)\,\frac{\partial}{\partial R}\,(C+\Sigma_x) - (C+\Sigma_x)\,\frac{\partial}{\partial R}\,(C^*+\Sigma_x^*)\right) = \mathrm{Im}\left(\frac{1}{(C+\Sigma_x)}\,\frac{\partial}{\partial R}\,(C+\Sigma_x)\right)\\
&\quad = \frac{\partial}{\partial R}\,\mathrm{Im}\ln(C+\Sigma_x) = \frac{\partial}{\partial R}\,\mathrm{Im}\ln\left(1+\frac{\Sigma_x}{C}\right) .
\end{aligned} \qquad (41)$$

The terms in $y$ and $z$ are obtained by replacing the subscript $x$ by the appropriate $y$ or $z$. Adding the three terms on the right-hand side of Eq. (20), using the forms (39), (41), while noting $\mathfrak{h}^2$ from (3), we find the particle is attracted to the wall by a force

$$F_{\mathrm{PW}z} = -\frac{\hbar c}{2\pi}\int_{k=0}^{\infty} dk\,\frac{\partial}{\partial R}\left\{\mathrm{Im}\ln\left[\left(1+\frac{\Sigma_x}{C}\right)\left(1+\frac{\Sigma_y}{C}\right)\left(1+\frac{\Sigma_z}{C}\right)\right]\right\}. \qquad (42)$$

### G. Result for Potential Energy Function

The associated potential-energy function $U(R)$ for which

$$F_z = -\frac{\partial}{\partial R}\,U(R) \qquad (43)$$

is

$$U_{\mathrm{PW}}(R) = \frac{\hbar c}{2\pi}\int_{k=0}^{\infty} dk\,\mathrm{Im}\ln\left[\left(1+\frac{\Sigma_x}{C}\right)\left(1+\frac{\Sigma_y}{C}\right)\left(1+\frac{\Sigma_z}{C}\right)\right] . \qquad (44)$$

The expression (44) which we have obtained for the van der Waals potential between a polarizable particle and conducting wall is exact beyond the nonrelativistic approximation used in the original dipole equation of motion (4). However, we would like to demonstrate analytically that this exact expression agrees to order $\alpha^2$ with the result obtained by Casimir and Polder using quantum-electrodynamic perturbation theory. Renne has shown how this may be done conveniently using contour integrations, and we will review his treatment in Sec. IV.

## III. TWO NEUTRAL POLARIZABLE PARTICLES

### A. Basic Model

The van der Waals force between two neutral polarizable particles arises from the same basic mechanism as considered in the previous case of a polarizable particle and a conducting wall. The classical zero-point radiation in the universe forces the oscillators into random motion and these then interact through classical electromagnetic interactions.

The equations of motion for the dipole oscillators representing the atoms are

$$m\,\frac{d^2\vec{\xi}_A}{dt^2} = -m\omega_0^2\vec{\xi}_A + \frac{2}{3}\,\frac{e^2}{c^3}\,\frac{d^3\vec{\xi}_A}{dt^3} + e\vec{\mathrm{E}}_{DB}(\vec{\mathrm{r}}_A, t) + e\vec{\mathrm{E}}_{\mathrm{ZP}}(\vec{\mathrm{r}}_A, t) , \qquad (45)$$

$$m\,\frac{d^2\vec{\xi}_B}{dt^2} = -m\omega_0^2\vec{\xi}_B + \frac{2}{3}\,\frac{e^2}{c^3}\,\frac{d^3\vec{\xi}_B}{dt^3} + e\vec{\mathrm{E}}_{DA}(\vec{\mathrm{r}}_B, t) + e\vec{\mathrm{E}}_{\mathrm{ZP}}(\vec{\mathrm{r}}_B, t) , \qquad (46)$$

where, analogous to Eq. (4), $\vec{\xi}_A$ and $\vec{\xi}_B$ are the oscillator displacements. Now $\vec{\mathrm{E}}_{DB}(\vec{\mathrm{r}}_A, t)$ and $\vec{\mathrm{E}}_{DA}(\vec{\mathrm{r}}_B, t)$ are the dipole fields at the position of each particle due to the other particle. We will assume that dipole $A$ is at the origin of coordinates while $B$ is displaced at distance $R$ along the positive $z$ axis. The zero-point radiation $\vec{\mathrm{E}}_{\mathrm{ZP}}$ is that of Eq. (1) appropriate for free space.

### B. Fourier Decomposition of the Equations of Motion

Again introducing the electric dipole moments

$$\vec{\mathrm{p}}_A = e\vec{\xi}_A , \quad \vec{\mathrm{p}}_B = e\vec{\xi}_B , \qquad (47)$$

the Fourier decompositions analogous to (9), (10)

$$\vec{\mathrm{p}}_A(t) = \mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\vec{\mathrm{P}}_A(\vec{\mathrm{k}}, \lambda)\,e^{-i\omega t} , \qquad (48)$$

$$\vec{\mathrm{p}}_B(t) = \mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\vec{\mathrm{P}}_B(\vec{\mathrm{k}}, \lambda)e^{-i\omega t} , \qquad (49)$$

$$\vec{\mathrm{E}}_{\mathrm{ZP}}(\vec{\mathrm{r}}, t) = \mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\vec{\mathrm{E}}_Z(\vec{\mathrm{r}}, \vec{\mathrm{k}}, \lambda)e^{-i\omega t} , \qquad (50)$$

and denoting

$$\vec{\mathrm{E}}_Z(\vec{\mathrm{r}}_A, \vec{\mathrm{k}}, \lambda) = \vec{\mathfrak{E}}_A , \quad \vec{\mathrm{E}}_Z(\vec{\mathrm{r}}_B, \vec{\mathrm{k}}, \lambda) = \vec{\mathfrak{E}}_B , \qquad (51)$$

the equations of motion become

$$CP_{Ax} + \eta_x P_{Bx} = \beta \mathfrak{E}_{Ax} , \tag{52}$$

$$\eta_x P_{Ax} + CP_{Bx} = \beta \mathfrak{E}_{Bx} , \tag{53}$$

where $C$ and $\beta$ are just as defined in (12) and (13), but $\eta_x$ arises from the dipole fields $\vec{E}_D$ as

$$\eta_x = \eta_y = -\tfrac{3}{2}\Gamma\omega^3\left(\frac{1}{kR} + \frac{i}{(kR)^2} - \frac{1}{(kR)^3}\right)e^{ikR} , \tag{54}$$

$$\eta_z = -\tfrac{3}{2}\Gamma\omega^3 2\left(\frac{-i}{(kR)^2} + \frac{1}{(kR)^3}\right)e^{ikR} . \tag{55}$$

Equations (52) and (53) are specified for the $x$ component of the dipoles, with similar terms understood for the $y$ and $z$ components.

The solutions to the equations of motion are

$$P_{Ax} = \beta\frac{(C\mathfrak{E}_{Ax} - \eta_x \mathfrak{E}_{Bx})}{C^2 - \eta_x^2} = \frac{\beta}{2}\left(\frac{\mathfrak{E}_{Ax} + \mathfrak{E}_{Bx}}{C + \eta_x} + \frac{\mathfrak{E}_{Ax} - \mathfrak{E}_{Bx}}{C - \eta_x}\right) , \tag{56}$$

$$P_{Bx} = \beta\frac{(C\mathfrak{E}_{Bx} - \eta_x \mathfrak{E}_{Ax})}{C^2 - \eta_x^2} = \frac{\beta}{2}\left(\frac{\mathfrak{E}_{Ax} + \mathfrak{E}_{Bx}}{C + \eta_x} - \frac{\mathfrak{E}_{Ax} - \mathfrak{E}_{Bx}}{C - \eta_x}\right) , \tag{57}$$

where the last expressions are given in the normal coordinate form.

### C. Force on One Dipole

The arguments for the force given earlier in Sec. II C also hold true here, so that the average force upon dipole $B$ is along the $z$ axis:

$$F_z = \left\langle p_x \frac{\partial E_x}{\partial z} + p_y \frac{\partial E_y}{\partial z} + p_z \frac{\partial E_z}{\partial z}\right\rangle , \tag{58}$$

where the electric field at $B$ has contributions from the dipole at $A$ and the zero-point radiation,

$$\left\langle p_{Bx}\frac{\partial}{\partial z}E_x(\vec{r}_B, t)\right\rangle = \left\langle p_{Bx}\frac{\partial}{\partial z}E_{ZPx}(\vec{r}_B, t)\right\rangle + \left\langle p_{Bx}\frac{\partial}{\partial z}E_{Dx}(\vec{r}_B, t)\right\rangle . \tag{59}$$

### D. Force Due to the Homogeneous Radiation Field

The first term on the right-hand side of (59) gives the force due to the homogeneous zero-point radiation field,

$$\left\langle p_{Bx}\frac{\partial}{\partial z}E_{ZPx}(\vec{r}_B, t)\right\rangle = \tfrac{1}{2}\mathrm{Re}\left\langle \sum_{\lambda=1}^{2}\int d^3k\,\frac{\beta}{2}\left(\frac{\mathfrak{E}_{Ax} + \mathfrak{E}_{Bx}}{C + \eta_x} - \frac{\mathfrak{E}_{Ax} - \mathfrak{E}_{Bx}}{C - \eta_x}\right)e^{-i\omega t} \times \sum_{\lambda'=1}^{2}\int d^3k'\,\frac{\partial}{\partial z}E^*_{Zx}(0, 0, z, \vec{k}', \lambda')|_{z=z_B=R}\, e^{i\omega' t}\right\rangle , \tag{60}$$

where from (1) and (50)

$$E_{Zx}(0, 0, z, \vec{k}', \lambda') = \epsilon_x(\vec{k}', \lambda')\mathfrak{h}(\vec{k}', \lambda') \times \exp[ik_z'z + i\vartheta(\vec{k}', \lambda')] \tag{61}$$

and

$$\frac{\partial}{\partial z}E_{Zx}(0, 0, z, \vec{k}', \lambda')|_{z=z_B=R} = \epsilon_x(\vec{k}', \lambda')\mathfrak{h}(\vec{k}', \lambda')ik_z' \times \exp[ik_z'z + i\vartheta(\vec{k}', \lambda')] . \tag{62}$$

Here we have written $P_{Bx}$ using the normal coordinate form in (57). Averaging over the random phases as in (24) and (25),

$$\left\langle p_{Bx}\frac{\partial}{\partial z}E_{ZPx}(\vec{r}_B, t)\right\rangle = \tfrac{1}{2}\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\frac{\beta}{2}\,\mathfrak{h}^2\epsilon_x^2(-ik_z) \times\left(\frac{(e^{-ik_zR}+1)}{C+\eta_x} - \frac{(e^{-ik_zR}-1)}{C-\eta_x}\right) . \tag{63}$$

The integral in (63) is over all values of $\vec{k}$. Hence the terms which are odd in $k_z$ vanish in the angular integrations, leaving the terms

$$\left\langle p_{Bx}\frac{\partial}{\partial z}E_{ZPx}(\vec{r}_B, t)\right\rangle = \tfrac{1}{2}\mathrm{Re}\sum_{\lambda=1}^{2}\int d^3k\,\frac{\beta}{2}\,\mathfrak{h}^2\epsilon_x^2(-k_z) \times \sin k_zR\left(\frac{1}{C+\eta_x} - \frac{1}{C-\eta_x}\right) . \tag{64}$$

The analogous expressions involving $p_{By}$ and $p_{Bz}$ are obtained by replacing the subscript $x$ in (64) by $y$ and $z$.

As in Sec. II D, we will remove the derivative from the angular integration, noting

$$\epsilon_x^2(-k_z)\sin k_zR = +\frac{\partial}{\partial R}(\epsilon_x^2\cos k_zR) . \tag{65}$$

The required angular integrations then give

$$\sum_{\lambda=1}^{2}\int d\Omega\,\epsilon_x^2\cos k_zR = \int_{\theta=0}^{\pi}d\theta\sin\theta \times\int_{\phi=0}^{2\pi}d\phi\,(1-\cos^2\phi\sin^2\theta)\cos(kR\cos\theta) = -\frac{4\pi}{\beta k^3}\mathrm{Im}\eta_x , \tag{66}$$

$$\sum_{\lambda=1}^{2}\int d\Omega\,\epsilon_y^2\cos k_zR = -\frac{4\pi}{\beta k^3}\mathrm{Im}\eta_y , \tag{67}$$

$$\sum_{\lambda=1}^{2}\int d\Omega\,\epsilon_z^2\cos k_zR = -\frac{4\pi}{\beta k^3}\mathrm{Im}\eta_z . \tag{68}$$

Introducing the result (66) into (64) and writing the real part as

$$\mathrm{Re}\left(\frac{1}{C+\eta_x} - \frac{1}{C-\eta_x}\right) = \left(\frac{\mathrm{Re}(C^*+\eta_x^*)}{|C+\eta_x|^2} - \frac{\mathrm{Re}(C^*-\eta_x^*)}{|C-\eta_x|^2}\right) , \tag{69}$$

it follows that

$$\left\langle p_{Bx}\frac{\partial}{\partial z}E_{ZPx}(\vec{r}_B, t)\right\rangle$$

$$=\frac{1}{2}\int_{k=0}^{\infty} dk\, k^2 \frac{\beta}{2}\, \mathfrak{h}^2 \left(-\frac{4\pi}{\beta k^3}\; \frac{\partial}{\partial R}\, \mathrm{Im}\eta_x\right)$$

$$\times\left(\frac{\mathrm{Re}(C^* + \eta_x^*)}{|C+\eta_x|^2} - \frac{\mathrm{Re}(C^* - \eta_x^*)}{|C-\eta_x|^2}\right) , \quad (70)$$

with analogous expressions for the $y$ and $z$ components.

### E. Force Due to the Field of the Other Dipole

The second term in (59) involves the field at $\vec{p}_B$ due to the first dipole $\vec{p}_A$,

$$\langle p_{Bx} \partial E_{DAx}(\vec{r}_B, t)/\partial z \rangle .$$

Carrying out the derivative with respect to the coordinate of the field point, we have

$$E_{DAx}(0, 0, z, t) = \mathrm{Re} \sum_{\lambda=1}^{2} \int d^3k \frac{1}{\beta}\; \frac{3}{2}\, \Gamma\omega^3$$

$$\times\left(\frac{1}{kz} + \frac{i}{(kz)^2} - \frac{1}{(kz)^3}\right) e^{ikz} P_{Ax}\, e^{-i\omega t} , \quad (71)$$

$$\frac{\partial}{\partial z} E_{DAx}(0, 0, z, t)\big|_{z=z_B=R}$$

$$= \mathrm{Re} \sum_{\lambda=1}^{2} \int d^3k\, P_{Ax}\, e^{-i\omega t}\; \frac{-1}{\beta}\; \frac{\partial \eta_x}{\partial R} . \quad (72)$$

The force term then becomes

$$\left\langle p_{Bx} \frac{\partial}{\partial z} E_{DAx}(\vec{r}_B, t)\right\rangle$$

$$= \tfrac{1}{2}\mathrm{Re}\left\langle \sum_{\lambda=1}^{2} \int d^3k\, \frac{\beta}{2} \left(\frac{\mathfrak{E}_{Ax} + \mathfrak{E}_{Bx}}{C+\eta_x} - \frac{\mathfrak{E}_{Ax} - \mathfrak{E}_{Bx}}{C-\eta_x}\right) e^{-i\omega t} \sum_{\lambda'=1}^{2} \int d^3k'\, \frac{\beta}{2} \left(\frac{\mathfrak{E}_{Ax}^* + \mathfrak{E}_{Bx}^*}{C^*+\eta_x^*} + \frac{\mathfrak{E}_{Ax}^* - \mathfrak{E}_{Bx}^*}{C^*-\eta_x^*}\right) e^{i\omega' t}\; \frac{-1}{\beta}\; \frac{\partial \eta_x^*}{\partial R}\right\rangle$$

$$= \tfrac{1}{2}\mathrm{Re} \sum_{\lambda=1}^{2} \int d^3k\, \frac{\beta^2}{4}\, \epsilon_x^{\,2} \mathfrak{h}^2 2 \left(\frac{1+\cos k_z R}{|C+\eta_x|^2} + \frac{i \sin k_z R}{(C+\eta_x)(C^*-\eta_x^*)} + \frac{i \sin k_z R}{(C-\eta_x)(C^*+\eta_x^*)} - \frac{1-\cos k_z R}{|C-\eta_x|^2}\right)\; \frac{-1}{\beta}\; \frac{\partial \eta_x^*}{\partial R} , \quad (73)$$

where in proceeding to the last line we have averaged over the random phases as in (24) and (25).

The two terms involving $\sin k_z R$ are odd in $k_z$ and vanish in the angular integrations. The remaining angular integrations may be read off from equation (67), together with the observation

$$\sum_{\lambda=1}^{2} \int d\Omega\, \epsilon_x^{\,2} = \int_{\theta=0}^{\pi} d\theta\, \sin\theta$$

$$\times \int_{\phi=0}^{2\pi} d\phi\, (1 - \cos^2\phi \sin^2\theta) = \tfrac{8}{3}\pi , \quad (74)$$

$$\sum_{\lambda=1}^{2} \int d\Omega\, \epsilon_y^{\,2} = \sum_{\lambda=1}^{2} \int d\Omega\, \epsilon_z^{\,2} = \tfrac{8}{3}\pi . \quad (75)$$

It follows

$$\sum_{\lambda=1}^{2} \int d\Omega\, \epsilon_x^{\,2}\, (1 \pm \cos k_z R) = -\frac{4\pi}{\beta k^3}\, \mathrm{Im}(C \pm \eta_x) , \quad (76)$$

with the expressions for $\epsilon_y$, $\epsilon_z$ found by replacing the $x$ subscript.

The expression (73) for the contribution to the force is then simplified to

$$\left\langle p_{Bx} \frac{\partial}{\partial z} E_{DAx}(\vec{r}_B, t)\right\rangle$$

$$= \frac{1}{2}\int_{k=0}^{\infty} dk\, k^2 \frac{\beta^2}{4}\, \mathfrak{h}^2 2 \left(-\frac{4\pi}{\beta k^3}\, \frac{\mathrm{Im}(C+\eta_x)}{|C+\eta_x|^2}\right.$$

$$\left. + \frac{4\pi}{\beta k^3}\, \frac{\mathrm{Im}(C-\eta_x)}{|C-\eta_x|^2}\right)\; \frac{-1}{\beta}\; \frac{\partial}{\partial R}\, \mathrm{Re}\eta_x^* . \quad (77)$$

### F. Total Force and Potential-Energy Function

The total force involves the contributions of (77) and (70) as

$$\left\langle p_{Bx} \frac{\partial}{\partial z} E_x(\vec{r}_B, t)\right\rangle = -\pi \int_{k=0}^{\infty} dk\, \frac{\mathfrak{h}^2}{k} \left[\frac{1}{|C+\eta_x|^2} \left(\mathrm{Re}(C^* + \eta_x^*)\, \frac{\partial}{\partial R}\, \mathrm{Im}\eta_x - \mathrm{Im}(C+\eta_x)\, \frac{\partial}{\partial R}\, \mathrm{Re}\eta_x^*\right)\right.$$

$$\left. - \frac{1}{|C-\eta_x|^2} \left(\mathrm{Re}(C^* - \eta_x^*)\, \frac{\partial}{\partial R}\, \mathrm{Im}\eta_x - \mathrm{Im}(C-\eta_x)\, \frac{\partial}{\partial R}\, \mathrm{Re}\eta_x^*\right)\right] . \quad (78)$$

The expression in the bracket is analogous to that appearing in Eq. (41). Following the same procedure as given there, we have

$$\left[\frac{1}{|C+\eta_x|^2} \left(\mathrm{Re}(C^* + \eta_x^*)\, \frac{\partial}{\partial R}\, \mathrm{Im}\eta_x - \mathrm{Im}(C+\eta_x)\, \frac{\partial}{\partial R}\, \mathrm{Re}\eta_x^*\right) - \frac{1}{|C-\eta_x|^2} \left(\mathrm{Re}(C^* - \eta_x^*)\, \frac{\partial}{\partial R}\, \mathrm{Im}\eta_x - \mathrm{Im}(C-\eta_x)\, \frac{\partial}{\partial R}\, \mathrm{Re}\eta_x^*\right)\right]$$

$$= \mathrm{Im}\left(\frac{1}{C+\eta_x}\ \frac{\partial}{\partial R}(C+\eta_x) + \frac{1}{C-\eta_x}\ \frac{\partial}{\partial R}(C-\eta_x)\right) = \frac{\partial}{\partial R}\,\mathrm{Im}\ln[(C+\eta_x)(C-\eta_x)] = \frac{\partial}{\partial R}\,\mathrm{Im}\ln\left(1-\frac{\eta_x^2}{C^2}\right) \quad . \tag{79}$$

Thus from Eqs. (59), (78), and (79), the force of attraction between the two particles is

$$F_{2\mathrm{PZ}} = -\frac{\hbar c}{2\pi}\int_{k=0}^{\infty} dk\,\frac{\partial}{\partial R}\,\mathrm{Im}\ln\left[\left(1-\frac{\eta_x^2}{C^2}\right)\times\left(1-\frac{\eta_y^2}{C^2}\right)\left(1-\frac{\eta_z^2}{C^2}\right)\right], \tag{80}$$

and the associated potential energy function $U(R)$ as in (43) is

$$U_{2\mathrm{P}}(R) = \frac{\hbar c}{2\pi}\int_{k=0}^{\infty} dk\,\mathrm{Im}\ln\left[\left(1-\frac{\eta_x^2}{C^2}\right)\times\left(1-\frac{\eta_y^2}{C^2}\right)\left(1-\frac{\eta_z^2}{C^2}\right)\right] \quad . \tag{81}$$

## IV. COMPARISON WITH RESULTS OF QUANTUM ELECTRODYNAMICS

### A. Agreement with Renne's Results

The expressions (44) and (81) for the potentials between a polarizable particle and a conducting wall, and between two polarizable particles correspond exactly to the results found by Renne[18] from quantum-electrodynamic calculations using a nonrelativistic-quantum-oscillator model for the polarizable particles. Thus there seems to be agreement between the two nonrelativistic theories which extends to all orders in perturbation theory for van der Waals forces.

### B. Comparison with Casimir–Polder Form

In order to convert the exact results over to the form obtained by Casimir and Polder, in fourth order, we may proceed as follows. Since both (44) and (81) involve taking imaginary parts, they can be split into two integrals, for example,

$$U_{\mathrm{PW}}(R) = \frac{\hbar c}{4\pi i}\int_{k=0}^{\infty} dk\ln\left[\left(1+\frac{\Sigma_x}{C}\right)\left(1+\frac{\Sigma_y}{C}\right)\left(1+\frac{\Sigma_z}{C}\right)\right] - \frac{\hbar c}{4\pi i}\int_{k=0}^{\infty} dk\ln\left[\left(1+\frac{\Sigma_x^*}{C^*}\right)\left(1+\frac{\Sigma_y^*}{C^*}\right)\left(1+\frac{\Sigma_z^*}{C^*}\right)\right]. \tag{82}$$

The integrands are analytic functions in the right half-plane for $k$ now regarded as a complex variable. Moreover, since $\Sigma_x$, $\Sigma_y$, and $\Sigma_z$ all involve $e^{i2kR}$, these expressions are exponentially decreasing in the upper half-plane, while $\Sigma_x^*$, $\Sigma_y^*$, $\Sigma_z^*$ are exponentially decreasing in the lower half-plane. Hence, for the first integral, it is possible to consider a closed contour out the real axis, along a quarter circle in the upper half-plane, and down a line in the right half-plane parallel to the positive imaginary axis. No singularities are enclosed and hence the contour integral vanishes. Also the contribution along the quarter circle vanishes as the contour is pushed to infinity. Exactly analogous comments can be made for the second integral where the contour is in the lower half-plane. Thus the original integral is converted to an integral along a line to the right of the imaginary $k$ axis. The integrands still have singularities on the imaginary axis corresponding to the two points where $C$ and $C^*$ vanish,

$$C = -c^2k^2 + \omega_0^2 - i\Gamma c^3k^3 = 0\ , \tag{83}$$

$$C^* = -c^2k^2 + \omega_0^2 + i\Gamma c^3k^3 = 0\ . \tag{84}$$

Renne shows that the contributions from these singularities are negligible compared to the approximately equal integral obtained by changing the radiation damping term, replacing $C$ by

$$C' = -c^2k^2 + \omega_0^2 + i\Gamma c\omega_0^2 k\ . \tag{85}$$

The integrals along the positive and negative imaginary axes give equal contributions. Hence writing $k = iu$, we obtain

$$U_{\mathrm{PW}}(R) \cong \frac{\hbar c}{2\pi}\int_{u=0}^{\infty} du\ln\left[\left(1+\frac{\Sigma_x(iu)}{C'(iu)}\right)\times\left(1+\frac{\Sigma_y(iu)}{C'(iu)}\right)\left(1+\frac{\Sigma_z(iu)}{C'(iu)}\right)\right], \tag{86}$$

$$U_{2\mathrm{P}}(R) \cong \frac{\hbar c}{2\pi}\int_{u=0}^{\infty} du\ln\left[\left(1-\frac{\eta_x^2(iu)}{C'^2(iu)}\right)\times\left(1-\frac{\eta_y^2(iu)}{C'^2(iu)}\right)\left(1-\frac{\eta_z^2(iu)}{C'^2(iu)}\right)\right]. \tag{87}$$

Next, expanding the logarithm as

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots \tag{88}$$

and keeping only the first term, we obtain

$$U_{\mathrm{PW}}(R) \cong \frac{\hbar c}{\pi}\int_{u=0}^{\infty} du\left(\frac{\Sigma_x(iu)}{C'(iu)} + \frac{\Sigma_y(iu)}{C'(iu)} + \frac{\Sigma_z(iu)}{C'(iu)}\right) \tag{89}$$

and

$$U_{2\mathrm{P}}(R) \cong -\frac{\hbar c}{\pi}\int_{u=0}^{\infty} du\left(\frac{\eta_x^2(iu)}{C'^2(iu)} + \frac{\eta_y^2(iu)}{C'^2(iu)} + \frac{\eta_z^2(iu)}{C'^2(iu)}\right). \tag{90}$$

Finally in the limit that the radiation damping is neglected $\Gamma \to 0$, we obtain the Casimir–Polder results

$$U_{\mathrm{PW}}(R) \cong -\alpha\,\frac{\hbar c}{\pi}\int_{u=0}^{\infty} du\,\frac{u^2\omega_0^2}{c^2u^2+\omega_0^2}\times\frac{e^{-2uR}}{2R}\left(1+\frac{1}{uR}+\frac{1}{2(uR)^2}\right), \tag{91}$$

$$U_{2P}(R) \cong -\alpha^2 \frac{\hbar c}{\pi} \int_{u=0}^{\infty} du \frac{u^4 \omega_0^4}{(c^2u^2+\omega_0^2)^2} \times \frac{e^{-2uR}}{R^2}\left(1+\frac{2}{uR}+\frac{5}{(uR)^2}+\frac{6}{(uR)^3}+\frac{3}{(uR)^4}\right), \quad (92)$$

where

$$\alpha = e^2/m\omega_0^2 \quad (93)$$

is the static polarizability of each particle.

## V. CLOSING SUMMARY

In 1948 Casimir and Polder[3] calculated the van der Waals forces between a polarizable particle and a conducting wall, and between two polarizable particles using up to fourth-order perturbation theory in quantum electrodynamics. These authors were struck by the apparent simplicity of the asymptotic forms for the forces, and Casimir[12] went on to show that the asymptotic retarded forces, $R \to \infty$, could be obtained within a semiclassical picture which treated seriously the zero-point energy $\frac{1}{2}\hbar\omega$ per normal mode in the quantum electromagnetic field.

As a gradual outgrowth of Casimir's ideas on quantum zero-point energy, the present author has been led to consider[1,2,4-7] a purely classical electrodynamic theory which changes the homogeneous boundary condition on Maxwell's equations so as to include random classical electromagnetic radiation with a Lorentz invariant spectrum. It is natural to set the scale of this classical zero-point radiation to correspond to the $\frac{1}{2}\hbar\omega$ familiar in quantum theory. In the present article we have found that this purely classical electromagnetic theory gives a complete classical understanding of the Casimir–Polder result. The classical random radiation sets the polarizable particles into motion and these then interact via classical electromagnetism. The classical expressions are in precise agreement with the quantum electrodynamic calculations to all orders by Renne,[13] and they recover the original quantum electrodynamic perturbation-theory results of Casimir and Polder.

## ACKNOWLEDGMENT

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