[21]J. G. Kirkwood, in *Phase Transformations in Solids*, edited by R. Smoluchowski, J. E. Mayer, and W. A. Weyl (Wiley, New York, 1951), p. 67.

[22]T. Schneider, Phys. Rev. A **3**, 2145 (1971); T. Schneider, R. Brout, H. Thomas, and J. Feder, Phys. Rev. Lett. **25**, 1423 (1970).

[23]W. L. McMillan, Phys. Rev. A **6**, 936 (1972); Phys. Rev. A **4**, 1238 (1971).

PHYSICAL REVIEW A VOLUME 7, NUMBER 5 MAY 1973

Why is the Laser Line So Narrow? A Theory of Single-Quasimode Laser Operation

Roy Lang* and Marlan O. Scully†‡

Department of Physics and Optical Sciences Center, University of Arizona, Tucson, Arizona 85721

Willis E. Lamb, Jr.§

Department of Physics, Yale University, New Haven, Connecticut 06520

(Received 31 March 1972; revised manuscript received 28 July 1972)

Previous theories of laser radiation have described the electromagnetic field in terms of a discrete set of quasimodes (Fox and Li "modes") of the laser cavity. Each quasimode was assumed to have a finite quality factor Q. In steady state (when gain due to lasing atoms cancels the losses) and in the absence of noise sources, the semiclassical theory predicts a δ-function line shape. In the present semiclassical analysis (fluctuations are ignored), this theory is generalized for a maser with a cavity having a semitransparent wall as one of the mirrors so that there are now many modes of the universe corresponding to each Fox-Li-type quasimode. It is demonstrated that the normal modes of the universe associated with a single Fox-Li "mode" may, under proper conditions, lock together and the δ-function laser line shape may be regained. Specifically, we investigate the conditions under which the coupled multimode equations associated with laser oscillation can be reduced to an equation of the form found in the usual single-quasimode theory.

I. INTRODUCTION

Theoretical investigations dealing with the behavior of the optical maser have been based on a model of a Fabry–Perot cavity[1–3] described by a discrete set[4] of quasimodes. Normal modes of such a cavity cannot be rigorously defined, because of diffraction losses and imperfect mirrors. This resonator problem was investigated numerically by Fox and Li.[5a] They showed that in the case of a Fabry–Perot-type cavity with perfectly reflecting mirrors, there exists a discrete set of quasimodes for which the diffraction leakage from the cavity is small. The usual laser theory begins with these Fox–Li modes and adds a phenomenological loss mechanism (e.g., ohmic currents smeared throughout the cavity) to represent the losses at the mirrors.

In the present paper[5b] we consider the theory of an optical maser based on a model of the cavity which is coupled to the outside world by means of a dielectric mirror. In this model there are a large (infinite) number of modes of the universe corresponding to each of the Fox–Li modes of the lasing cavity. In this way, the coupling to the outside world replaces the usual phenomenological loss which is introduced in order to represent the leakage of radiation from our lasing cavity. The motives for this work are as follows.

(i) The sense in which the present multimode approach reduces to the usual quasimode treatment, and therefore the sense in which the quasimode treatment has a rigorous foundation, is a problem of current interest. For example, in the quantum theory of the laser one begins by assuming that the modes of the Fox–Li cavity may be quantized, and at a later stage a loss mechanism is introduced into the calculation in order to simulate the leakage mentioned before. This approach is a good approach for most laser calculations but is not the most fundamental imaginable procedure, i.e.; the atoms in the mirrors have been ignored in the previously outlined calculation. They are taken into account by means of an explicit boundary value problem and in this sense we are not treating the radiation in free space but rather are treating the radiation as it is bounded by the atoms and electrons in the mirrors. A more rigorous calculation would involve the quantized field in vacuum and subsequent detailed consideration of these free field modes and their interaction with the atoms and electrons in the mirrors. This is a very complicated problem and so we have previously been satisfied with a less ambitious approach involving only the Fox–Li quasimodes and the subsequent quantization of these modes.

(ii) By understanding the sense in which the multimode approach reduces to the single-Fox–Li-mode treatment we hope to understand the mechanism leading to the extreme monochromaticity of the laser radiation. One often hears the argument that laser radiation is monochromatic as a consequence of gain narrowing. We regard this argument as incomplete[6a] at best, and a more fundamental approach involving many modes (of the universe) is desired. The present calculation indicates that the narrowness of the laser linewidth should be regarded as a consequence of a locking phenomenon between these modes.

(iii) It is of interest to investigate the sense in which the present calculation, which does not include the cavity dissipation as a phenomenological loss mechanism, still leads to and implies a fluctuation-dissipation theorem. That is to say, if we no longer consider the equation of motion for a single Fox–Li mode as being damped by a phenomenological damping mechanism (in addition to which there must be a Langevin noise source), then in what fashion do we recover the corresponding fluctuation-dissipation theorem? This point will be investigated in a future publication and the present work forms the foundation for that analysis.

(iv) The fully quantized theory of laser oscillation, as mentioned earlier, could be reasonably and readily carried out by quantizing a radiation field in the entire space. This approach has been carried out and will also be published elsewhere.

(v) Finally, there is at present interest in understanding the extent to which the laser threshold is related to the physics of phase-transition phenomena. It has been asserted by other workers that the Bose condensation of liquid helium below its λ point is analogous to the "condensation" of radiation into one mode in the case of laser behavior. It is clear that such a question can be analyzed only in the context of a many-mode laser analysis of the present variety.[6b]

In Sec. II and Appendix A the normal modes of the combined system of a maser cavity and the outside world to which it is coupled are obtained. In Sec. III and IV, equations for the field interacting with lasing atoms are derived. In Sec. V, linearized equations for the field are investigated. In Sec. VI, it is demonstrated that the nonlinear coupled multimode equations can be reduced to an equation of the form found in the usual quasimode theory.

II. NORMAL MODES OF COMBINED SYSTEM

Let us consider the normal modes for the combined system of a maser cavity coupled to the outside world. We represent the universe (in which the cavity is imbedded) by a much larger cavity having perfectly reflecting walls. A simple one-dimensional model which carries the essential features of such a combined system is illustrated in Fig. 1. The mirrors at $z=l$ and $-L$ are completely reflective, while the one at $z=0$ is semitransparent. Region 1 corresponds to a maser cavity and region 2 the rest of the world. The length L is allowed to go to infinity at the end of the calculation.

We represent a semitransparent mirror by a very thin plate with very large dielectric constant. As an idealization of such a mirror we choose the dielectric constant around $z=0$ to be

$$\epsilon(z)=\epsilon_0[1+\eta\delta(z)]\ , \qquad (2.1)$$

where η is a parameter with the dimensions of length which determines the transparency of this wall.

The normal mode functions of this system can be obtained by solving Maxwell's equations with the proper boundary conditions. The details of the calculation are given in Appendix A. For those normal modes having frequency $\Omega_k(=ck)$ close to a Fox–Li "resonant" frequency $\Omega(=ck_0)$, the eigenfunctions of the entire cavity are found to be

$$U_k(z)=M_k\sin k(z-l) \qquad (z>0) \qquad (2.2a)$$

$$=\xi_k\sin k(z+L) \qquad (z<0)\ , \qquad (2.2b)$$

where ξ_k is a phase factor which alternates between 1 and -1 as k increases from one value to the next.

The coefficients M_k in (2.2a) are defined as

$$M_k=\Gamma\Lambda[(\Omega_k-\Omega)^2+\Gamma^2]^{-1/2}\ , \qquad (2.3)$$

where Γ is the bandwidth associated with the mirror transparency and is given by

$$\Gamma=c/\eta^2k_0^2l=c/\Lambda^2l \qquad (2.4)$$

and

$$\Lambda=\eta\Omega/c=\eta k_0\ , \qquad (2.5)$$

while the frequency Ω of the nth Fox–Li quasimode

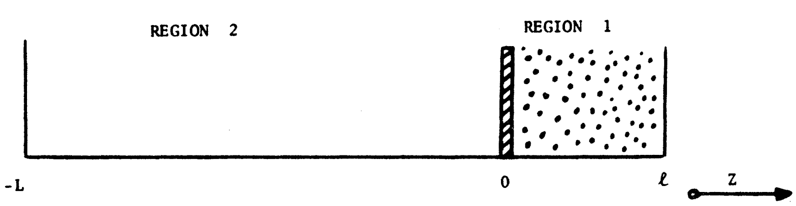

FIG. 1. Laser cavity with semitransparent wall imbedded in large ideal cavity.

is given by

$$\Omega = ck_0 = (n\pi + 1/\Lambda)c/l \ . \tag{2.6}$$

The density of states ρ in Ω_k space is found to be

$$\rho = L/c\pi \ . \tag{2.7}$$

An arbitrary undriven field in the entire cavity can be expressed as the real part of the complex field

$$E(z,t) = \sum_k E_k(0) U_k(z) e^{-i\Omega_k t} = \sum_k E_k(t) U_k(z) \ , \tag{2.8}$$

which is to be understood as a sum over modes of the large cavity, i.e., "the universe."

We now demonstrate that the semitransparency of the mirror leads to a damping of free oscillations in the laser cavity. Let us assume that, at $t=0$, the laser cavity (region 1) contains a field (in the complex notation) of the form

$$E(z,0) = |E_0| e^{-i\varphi} \sin k_0(z-l) \ , \tag{2.9}$$

while no field exists outside the cavity, i.e., in region 2. The coefficients $E_k(0)$ for this case are obtained in the usual fashion by multiplying (2.9) by $U_k(z)$ defined in (2.2a) and integrating[7] over z. We find

$$E_k(t) = (|E_0| M_k l/L) e^{-i(\Omega_k t + \varphi)} \ . \tag{2.10}$$

Therefore at later times, $t > 0$,

$$E(z,t) = (|E_0| l/L) \sum_k M_k U_k(z) e^{-i(\Omega_k t + \varphi)} . \tag{2.11}$$

The summation can be approximated by an integral if the frequency separation between the normal modes is small compared to Γ. Upon carrying out the integration over k in (2.11), the explicit form of $E(z,t)$ in the maser cavity turns out to be

$$E(z,t) = |E_0| \sin k_0(z-l) e^{i(\Omega t + \varphi) - \Gamma t} \ . \tag{2.12}$$

The approximation involved in the passage from (2.11) to (2.12) is discussed in Appendix B.

Equation (2.12) indicates that the field localized in the maser cavity decays exponentially owing to leakage through the mirror at a rate Γ. In addition to the mirror losses, there may exist some other damping mechanism (e.g., scattering from dirt or finite mirror conductivity); we must then add to Γ an extra contribution γ. The quality factor Q of the cavity is then defined by

$$\Omega/Q = 2(\Gamma + \gamma) \ . \tag{2.13}$$

III. ELECTROMAGNETIC FIELD EQUATIONS

Let us proceed to derive the equations of motion for the field interacting with an active lasing medium. We follow the usual prescription[1] except that we must expand the field in terms of the eigenfunctions of the combined system (Fabry–Perot cavity imbedded in the universe) as given by Eq. (2.2) rather than the Fox–Li quasimodes. The treatment here is semiclassical, as the radiation is described by Maxwell's equations (while the atomic states are to obey the quantum mechanical Schrödinger) equations of motions.

The atomic medium can be regarded as producing a macroscopic polarization $P(z,t)$. In the presence of such a polarization, Maxwell's equations may be combined to give an inhomogeneous wave equation

$$-\frac{\partial^2 E}{\partial z^2} + \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = -\mu_0 \frac{\partial^2 P}{\partial t^2} \ . \tag{3.1}$$

To simplify the problem, we have neglected the x and y dependence of the field, and the radiation has been assumed to be linearly polarized.

The field and the polarization can be expanded in terms of the eigenfunctions (2.2) as

$$E(z,t) = \sum_k A_k(t) U_k(z) \ , \tag{3.2}$$

where[7]

$$A_k(t) = (2/L) \int_{-L}^{l} dz\, E(z,t) U_k(z) \tag{3.3}$$

and

$$P(z,t) = \sum_k P_k(t) U_k(z) \ , \tag{3.4}$$

with

$$P_k(t) = (2/L) \int_{-L}^{l} dz\, P(z,t) U_k(z) \ . \tag{3.5}$$

Inserting (3.2) and (3.4) into (3.1), we easily obtain the following equations for each normal mode:

$$\frac{d^2 A_k}{dt^2} + \Omega_k^2 A_k = -\epsilon_0^{-1} \frac{d^2 P_k}{dt^2} \ . \tag{3.6}$$

Let us concentrate on the case where only one of the Fox–Li quasimodes is appreciably excited. We write P_k and A_k as

$$P_k(t) = \tfrac{1}{2}[\mathcal{P}_k(t) e^{-i\nu t} + \text{c.c.}] \ , \tag{3.7}$$

$$A_k(t) = \tfrac{1}{2}[E_k(t) e^{-i\nu t} + \text{c.c.}] \ , \tag{3.8}$$

where

$$E_k(t) = |E_k(t)| e^{i\varphi_k(t)} \ . \tag{3.9}$$

Using the above expressions and making the usual slowly varying phase and amplitude approximation, we obtain

$$\dot{E}_k(t) + i(\Omega_k - \nu) E_k(t) = (i\nu/2\epsilon_0) \mathcal{P}_k(t) \ . \tag{3.10}$$

IV. POLARIZATION OF MEDIUM

When the excited atoms are injected into the laser cavity and allowed to interact with the electromagnetic field, the atoms acquire a time-dependent dipole associated with the off-diagonal elements of the atomic density matrix. The collective effect of many atoms produces a macroscopic polarization which acts as the driving force for the radiation field.

The projection of the macroscopic polarization onto the kth mode has the following form[1]:

$$P_k(t) = \sum_\mu a_{k\mu} E_\mu(t) + \sum_{\mu\rho\sigma} b_{k\mu\rho\sigma} E_\mu E_\rho^* E_\sigma + \cdots \;. \tag{4.1}$$

The coefficients $a_{k\mu}$ and $b_{k\mu\rho\sigma}$ depend on properties of the lasing atoms and (in the case of no atomic motion) are given by

$$a_k = -\tfrac{1}{2}(i\wp^2/\hbar) N(k-\mu) D_{ab}(\omega - \nu) \;, \tag{4.2}$$

$$b_{k\mu\rho\sigma} = \tfrac{1}{32}(i\wp^4/\hbar^3)[N(\rho-\sigma+\mu-k) + N(\rho-\sigma-\mu+k) + N(\rho+\sigma-\mu-k)]\, \mathfrak{D}_{ab}(\omega-\nu)[\mathfrak{D}_a(0)+\mathfrak{D}_b(0)] \times[\mathfrak{D}_{ab}(\omega-\nu) + \text{c.c.}] \;, \tag{4.3}$$

where $\wp$ is the dipole matrix element and

$$\gamma_{ab} = \tfrac{1}{2}(\gamma_a + \gamma_b) \;, \tag{4.4}$$

$$\omega = (E_a - E_b)/\hbar \;, \tag{4.5}$$

with γ_a, E_a and γ_b, E_b the decay constants and the energies of the upper and the lower states of the two-level atom.

Also we have

$$\mathfrak{D}_{ab}(x) = (\gamma_{ab} + ix)^{-1}, \quad \mathfrak{D}_\alpha(x) = (\gamma_\alpha + ix)^{-1} \;, \tag{4.6}$$

$$N(k-\mu) = (2/L)\int_{-L}^{l} dz\, U_k(z)\, U_\mu(z)\, N(z) = (M_k M_\mu/L)\int_0^l dz \cos[(k-\mu)z - (k-\mu)l]\, N(z) \;, \tag{4.7}$$

where $N(z)$ is the population inversion density assumed to be time independent. The last expression has been obtained noting that the active medium exists only in the laser cavity and that terms which oscillate rapidly in space give negligible contributions to the integral.

Since $|k-\mu|l$ is small compared to unity, (4.7) can be integrated to give

$$N(k-\mu) = (l/L)\bar{N} M_k M_\mu \tag{4.8}$$

where $\bar{N}$ is the space average of $N(z)$. Similarly,

$$N(k-\mu+\rho-\sigma) = (2/8L) M_k M_\mu M_\rho M_\sigma \times \int_0^l \cos[(k-\mu+\rho-\sigma)z - (k-\mu+\rho-\sigma)l]\, N(z)\, dz = (l/4L) M_k M_\mu M_\rho M_\sigma \bar{N}. \tag{4.9}$$

In view of (4.8) and (4.10), the coefficients (4.2) and (4.3) simplify to the following when the Fox-Li frequency Ω is tuned to atomic resonance:

$$a_{k\mu} = -\tfrac{1}{2} i M_k M_\mu (\bar{N}\wp^2 l/\gamma_{ab}\hbar L) \equiv -ia M_k M_\mu \;, \tag{4.10a}$$

$$b_{k\mu\rho\sigma} = \tfrac{3}{32} i M_k M_\mu M_\rho M_\sigma (\bar{N}\wp^4 l/\gamma_a\gamma_b\gamma_{ab}\hbar^3 L) \equiv ib M_k M_\mu M_\rho M_\sigma \;. \tag{4.10b}$$

So far we have kept L (the length of region 2) finite. In this case, the radiation which escapes from the Fabry-Perot cavity through the semitransparent wall will be reflected at $z=-L$ and eventually come back into the laser cavity, if there were no loss in region 2. However in the limit $L\to\infty$ the escaped radiation will never return (within the lifetime of the experiment). In order to simulate the situation of an infinitely long cavity, when L is finite although large, we put some lossy material in the cavity: this we do by injecting nonresonant two-level atoms in their lower state uniformly over the entire cavity. The density of the damping atoms required is such that the cavity decay constant γ due to those atoms satisfies the relation

$$e^{-2\gamma L/c} \ll 1 \quad \text{or} \quad \gamma \gg c/2L \;. \tag{4.11}$$

The effect of damping atoms can be described in terms of a polarization, whose dependence to the field is again expressed by (4.1). Noting that these damping atoms are predominantly in their ground states, we may use (4.2) with the sign reversed,

$$a'_{k\mu} = \tfrac{1}{2} i (N'(k-\mu)\wp'^2)/\hbar\gamma'_{ab} \;, \tag{4.12}$$

where we made use of the fact $\gamma'_{ab} \gg |\omega' - \nu|$ for the damping atoms. Because of uniform distribution over the *entire cavity*, the orthonormality of $U_k(z)$ implies $N'(k-\mu) = \bar{N}'$, where $\bar{N}'$ is the number of the damping atoms per unit length. Hence (4.12) has the form

$$a'_{k\mu} = ia'\delta_{k\mu} \;. \tag{4.13}$$

The nonlinear polarization of the damping atoms is neglected. Using (4.1), (4.10), and (4.13), the field equation (3.10) can be written as

$$\dot{E}_k(t) + i(\Omega_k - \Omega)E_k(t) + \gamma E_k(t) \doteq \alpha M_k \sum M_\mu E_\mu(t) - \beta M_k \sum_{\mu\rho\sigma} M_\mu M_\rho M_\sigma E_\mu(t) E_\rho^*(t) E_\sigma(t) \;, \tag{4.14}$$

where

$$\alpha = \nu a/\epsilon_0 L, \quad \beta = \nu b/\epsilon_0 L, \quad \gamma = \nu a'/\epsilon_0 L \;. \tag{4.15}$$

It is to be noted that the range of k as well as μ, ρ, and σ has been restricted to the neighborhood of one Fox-Li mode corresponding to the single-quasimode laser operation. Equations (4.14) constitute our working equations for the rest of this paper.

V. LINEAR THEORY

In Sec. VI, it will be shown that the coupled multimode equations (4.14) can be transformed into a single equation which corresponds to the conventional quasimode theory of laser behavior. In order to motivate the analysis of Sec. VI and to gain some physical insight into the underlying physics, let us first discuss the solutions of the equations (4.14) neglecting the nonlinear terms. The solutions so obtained describe the transient state of the laser.

As is seen from Eq. (4.14), the linearized multimode equations take the following form:

$$\dot{E}_k(t)+i(\Omega_k-\Omega)E_k(t)+\gamma E_k(t)=\alpha M_k\sum_\mu M_\mu E_\mu(t)\ . \tag{5.1}$$

The most general solution Eq. (5.1) can be written as the superposition

$$E_k(t)=\sum_r C_{kr}e^{p_r t} \tag{5.2}$$

of particular solutions

$$E_k(t)=C_k e^{pt}. \tag{5.3}$$

Equation (5.1) now becomes

$$(p+i\delta\Omega_k+\gamma)C_k=\alpha M_k\sum_\mu M_\mu C_\mu\ , \tag{5.4}$$

where $\delta\Omega_k=\Omega_k-\Omega$. The usual procedure for solving Eq. (5.4) would involve writing down its secular determinant. In the present case it is simpler to obtain an equivalent solution by the following procedure. Solve (5.4) for C_k, multiply by M_k, and sum over k to obtain

$$\sum_k M_k C_k=\left(\sum_k \gamma M_k^2/(p+i\delta\Omega_k+\gamma)\right)\sum_\mu M_\mu C_\mu\ . \tag{5.5}$$

Dividing both sides of (5.5) by $\sum_k M_k C_k$, we obtain

$$\sum_k[\alpha M_k^2/(p+i\delta\Omega_k+\gamma)]=1\ . \tag{5.6}$$

It is convenient to write Eq. (5.6) in the form

$$S\equiv\sum_{s=-N}^{N}\alpha M_s^2/(p+is\Delta\Omega+\gamma)=1\ , \tag{5.7}$$

where we have renumbered the modes such that

$$\Omega_s-\Omega=s\Delta\Omega=s(c\pi/L)\ , \tag{5.8}$$

with $\Delta\Omega$ defined as

$$\Delta\Omega=c\pi/L \tag{5.9}$$

being approximately the very small mode spacing of the entire cavity. The number of modes coupled is taken to be $2N+1$, where N is an integer sufficiently large so that frequencies (5.8) cover the cavity bandwidth.

Since M_s^2 rapidly decreases as $(\Omega_s-\Omega)$ increases, we calculate S by extending the limit N to infinity and using Poisson's sum formula.[8] The result is

$$S=\frac{i\alpha M^2\Gamma}{\Gamma^2-(p+\gamma)^2}\left[\cot\left(\frac{i\pi(p+\gamma)}{\Delta\Omega}\right)-\frac{(p+\gamma)}{\Gamma}\cot\left(\frac{i\pi\Gamma}{\Delta\Omega}\right)\right]=1, \tag{5.10}$$

where

$$M^2=\sum_{s=-\infty}^{\infty}M_s^2\ . \tag{5.11}$$

As shown in Appendix C, the roots of this transcendental equation have the following properties. If $\alpha M^2>\Gamma$, there is a root equal to $\alpha M^2-\Gamma-\gamma$, which we refer as p_0. The other roots have negative real parts which approach $-\gamma$ as the frequency separation $\Delta\Omega$ decreases. In the case $\alpha M^2<\Gamma$, all the roots have negative real parts and approach $-\gamma$ as $\Delta\Omega$ approaches zero. The complex amplitudes $E_k(t)$ are expressed in terms of these normal mode solutions as

$$E_k(t)=\alpha M_k\sum_r C_r e^{p_r t}/(p_r+i\delta\Omega_k+\gamma)\ . \tag{5.12}$$

where the C_r are arbitrary complex constants to be determined by initial conditions.

If the real root p_0 is positive, the term in (5.12) proportional to $e^{p_0 t}$ grows in time, while the rest of the terms are damped out. Neglecting those decaying terms, $E_k(t)$ can be written as

$$\begin{aligned}E_k(t)&=\alpha M_k C_0 e^{p_0 t}/(p_0+i\delta\Omega_k+\gamma)\\&=e^{(\alpha M^2-\Gamma-\gamma)t}\alpha M_k C_0/(\alpha M^2-\Gamma+i\delta\Omega_k)\ .\end{aligned} \tag{5.13}$$

Equation (5.13) implies that all the $A_k(t)$ of (3.8) oscillate with the same frequency $\nu(=\omega=\Omega)$, and the phases of these modes are fixed in such a way that the phase relative to the center mode $(\delta\Omega_k=0)$ is

$$\varphi_k-\varphi_0=-\tan^{-1}[\delta\Omega_k/(\alpha M^2-\Gamma)]\ ; \tag{5.14}$$

hence, a "phase locking" has occurred. The relation of the present treatment to the usual theory[1] of mode locking is further discussed in Sec. VII.

Looking at the denominator of the equation (5.13), we find that only the modes which satisfy the inequality

$$\alpha M^2-\Gamma\gtrsim\delta\Omega_k\gtrsim-(\alpha M^2-\Gamma) \tag{5.15}$$

are appreciably excited. Furthermore as we are interested in the region $0<z<l$, we may write

$$\begin{aligned}E(z,t)&=\sum_k U_k(Z)A_k(t)\\&\approx\sum_k M_k E_k(t)e^{-i\nu t}\sin k_0(z-l)\\&=C_0 e^{(\alpha M^2-\Gamma-\gamma)t-i\nu t}\sin k_0(z-l)\ ,\end{aligned} \tag{5.16}$$

where k_0 is given by (2.6). The last equality in (5.16) suggests that the quantity

$$A(t)=\sum M_k E_k(t) \tag{5.17}$$

can be identified as the complex amplitude of a Fox–Li quasimode growing at a rate $\alpha M^2-\Gamma-\gamma$.

VI. NONLINEAR THEORY

Once the intensity of the radiation becomes large enough, the nonlinear terms in Eq. (4.14) are important. As shown in Sec. V, the modes are "locked" by this time, and the quasimode approximation might be expected to continue to be good even in the nonlinear problem. However, it is necessary to show that our coupled differential equations for the multimode field can be recombined to give a single equation when nonlinearities are important.

In order to show this, let us begin by using (5.17) to rewrite equation (4.14) as

$$\dot{E}_k + (i\delta\Omega_k + \gamma)E_k = M_k G(A) , \tag{6.1}$$

where

$$G(A) = \alpha A - \beta |A|^2 A . \tag{6.2a}$$

We could include higher powers of the electromagnetic field in the definition above, since the following derivation does not depend on the structure of $G(A)$. In fact, after summing up all powers of A, (6.2a) becomes[9]

$$G(A) = \alpha A/[1 + (\beta/\alpha)|A|^2] . \tag{6.2b}$$

A formal solution of (6.1) can be written as

$$E_k(t) = E_k(0)e^{-(i\delta\Omega_k+\gamma)t} + \int_0^t dt' \, M_k G(A(t'))\, e^{-(i\delta\Omega_k+\gamma)(t-t')} \tag{6.3}$$

Using this expression for $E_k(t)$, let us calculate the following quantity:

$$\sum_k (i\delta\Omega_k)M_k E_k(t) = \sum_k (i\delta\Omega_k)M_k E_k(0)e^{-(i\delta\Omega_k+\gamma)t} + \sum_k \int_0^t dt' \, (i\delta\Omega_k)M_k^2 G(A(t'))e^{-(i\delta\Omega_k+\gamma)(t-t')} \equiv I(t) + H(t) , \tag{6.4}$$

where

$$I(t) = \sum_k (i\delta\Omega_k)M_k E_k(0)e^{-(i\delta\Omega_k+\gamma)t} \tag{6.5}$$

depends only on the initial values of E_k's, and

$$H(t) = \sum_k \int_0^t dt' \, (i\delta\Omega_k)M_k^2 G(A(t'))\, e^{-(i\delta\Omega_k+\gamma)(t-t')} . \tag{6.6}$$

Exchanging the order of the summation and integration, and replacing the summation by an integration, we obtain

$$H(t) = \Gamma \int_0^t dt' \, M^2 G(A(t'))\, e^{-(\gamma+\Gamma)(t-t')} . \tag{6.7}$$

Next we multiply both sides of (6.1) by M_k and sum over k,

$$\dot{A}(t) + \gamma A(t) + \sum_k (i\delta\Omega_k) M_k E_k(t) = M^2 G(A(t)) . \tag{6.8}$$

Replacing the third term in the left-hand side of the equation above by (6.4), we obtain

$$\dot{A}(t) + \gamma A(t) + \Gamma \int_0^t dt' \, M^2 G(A(t'))e^{-(\gamma+\Gamma)(t-t')} = M^2 G(A(t)) - I(t) . \tag{6.9}$$

Let us write A as

$$A(t) = \int_0^t dt' \, M^2 G(A(t'))\, e^{-(\gamma+\Gamma)(t-t')} + \mathfrak{a}(t) . \tag{6.10}$$

Putting (6.10) into (6.9), we find

$$\dot{\mathfrak{a}}(t) + \gamma\, \mathfrak{a}(t) = -I(t) , \tag{6.11}$$

which is easily integrated to give

$$\mathfrak{a}(t) = \mathfrak{a}(0)e^{-\gamma t} - \int_0^t dt' \, I(t')\, e^{-\gamma(t-t')} . \tag{6.12}$$

Using (6.10), we find

$$A(0) = \mathfrak{a}(0) . \tag{6.13}$$

Combining expressions (6.9)–(6.13), we obtain

$$\dot{A}(t) + (\gamma + \Gamma)A(t) = M^2 G(A(t)) + \Gamma A(0)e^{-\gamma t} - \Gamma \int_0^t dt' \, I(t')e^{-\gamma(t-t')} - I(t) . \tag{6.14}$$

Using the explicit form of $I(t)$ given by (6.5), the integral on the right-hand side becomes

$$\Gamma \sum_k M_k [E_k(0)e^{-i(\delta\Omega_k+\gamma)t} - E_k(0)e^{-\gamma t}] . \tag{6.15}$$

Putting (6.15) into (6.14), we finally obtain

$$\dot{A}(t) + (\gamma + \Gamma)A(t) = M^2 G(A(t)) + \sum_k (\Gamma - i\delta\Omega_k)M_k E_k(0)e^{-i(\delta\Omega_k+\gamma)t} . \tag{6.16}$$

The second term on the right-hand side of this equation decays exponentially independent of $A(t)$. It will be shown in a future publication that this last term acts as a noise source associated with the mirror transparency. The mean motion of $A(t)$ may be examined by neglecting this term. Using the explicit form of $G(A)$, (6.16) is then reduced to

$$\dot{A}(t) + (\gamma + \Gamma)A(t) = \alpha M^2 A(t) - \beta M^2 |A(t)|^2 A(t) . \tag{6.17}$$

Separating real part from imaginary part, we get the usual equation for phase and amplitude of single quasimode operation. From Eq. (6.17), we have the steady-state intensity

$$|A|^2 = [\alpha M^2 - (\Gamma + \gamma)]/\beta M^2 , \tag{6.18}$$

which is the familiar result of the quasimode theory.

VII. DISCUSSION

A. Condition for the Quasimode Approximation

In Sec. VI we have seen that the coupled multimode equations (4.15) can be rigorously reduced to a single quasimode equation (6.16). Let us now summarize the conditions which allowed us to reduce the multimode equations to an equation for a single quasimode. First, the conditions for the atomic decay constants

$$\gamma_a, \gamma_b \gg \nu/Q \tag{7.1}$$

have been implicitly utilized in writing expressions (4.2) and (4.3). Secondly, approximations are made in (4.9) and (4.10), which are of the form

$$\int_0^l dz \, U_k(z)\, U_{k'}(z) \simeq M_k M_{k'} \, x \text{ (const)} . \tag{7.2}$$

This approximation is valid since, for any k and k' of interest

$$|k - k'|\, l \ll 1 \tag{7.3}$$

so that in the integrand we could approximate $U_k(z)$ as

$$U_k(z)=M_k\sin[k(z-l)]\simeq M_k\sin[k_0(z-l)] \quad . \tag{7.4}$$

In addition, the fact that

$$\sum_k M_k^2 e^{-i\delta\Omega_k t}=\left(\sum_k M_k^2\right)e^{-\Gamma t} \tag{7.5}$$

has been utilized in the passage from (6.6) to (6.7). It is to be noted that the derivation of the quasimode equation (6.16) depends only on the relations (7.1), (7.4), and (7.5). Similar relations are expected to hold in a wide range of practical lasers. However, the calculation does not depend on the detailed structure of the mode function or of the laser cavity.

B. Mode-Locking Aspects

We have seen in Sec. V that the dominance of a single exponential term in the general solution (5.2) leads all the modes to oscillate at a single frequency with a definite phase relationship between them. The close analogy between the "mode locking" here and that in the conventional sense can be demonstrated in the following rather unphysical but illuminating example of two-mode interaction.

Consider a case where only two modes are coupled by Eq. (5.1). To simplify the problem, we further assume

$$\Omega_1-\Omega=\Omega-\Omega_2\equiv\Delta\Omega/2 \tag{7.6}$$

and

$$M_1=M_2\equiv\mathfrak{M} \quad . \tag{7.7}$$

The Eq. (5.1) can be written in this case as

$$\dot{E}_1+(\tfrac{1}{2}i\Delta\Omega+\gamma)E_1=\alpha\mathfrak{M}^2(E_1+E_2) \quad , \tag{7.8a}$$

$$\dot{E}_2+(-\tfrac{1}{2}i\Delta\Omega+\gamma)E_2=\alpha\mathfrak{M}^2(E_1+E_2) \quad . \tag{7.8b}$$

These equations can be solved directly by applying the method used in Sec. V. The secular equation (5.7) for these equations is

$$2\alpha\mathfrak{M}^2(p+\gamma)/[(p+\gamma)^2+(\tfrac{1}{2}\Delta\Omega)^2]=1 \quad , \tag{7.9}$$

with eigenvalues $p_\pm$ given by

$$p_\pm=\alpha\mathfrak{M}^2-\gamma\pm(\alpha^2\mathfrak{M}^4-\tfrac{1}{4}\Delta\Omega^2)^{1/2} \quad . \tag{7.10}$$

We find that the condition for appearance of the dominant exponential term is

$$\alpha^2\mathfrak{M}^4-\tfrac{1}{4}\Delta\Omega^2>0 \quad . \tag{7.11}$$

Let us now examine the time change of the phases of the individual modes. For this purpose we introduce the real amplitudes $\mathcal{E}_j$ and φ_j by writing

$$E_j=\mathcal{E}_j e^{-i\varphi_j}, \quad j=1,2 \quad . \tag{7.12}$$

Inserting the expressions (7.12) into (7.8a) and (7.8b) and multiplying them by $e^{i\varphi_1}$ and $e^{i\varphi_2}$, respectively, we obtain as the imaginary parts of these equations

$$-\dot{\varphi}_1\mathcal{E}_1+(\tfrac{1}{2}\Delta\Omega)\mathcal{E}_1=\alpha\mathfrak{M}^2\mathcal{E}_2\sin\Delta\varphi \quad , \tag{7.13a}$$

$$-\dot{\varphi}_2\mathcal{E}_2-(\tfrac{1}{2}\Delta\Omega)\mathcal{E}_2=-\alpha\mathfrak{M}^2\mathcal{E}_1\sin\Delta\varphi \quad , \tag{7.13b}$$

where

$$\Delta\varphi=\varphi_1-\varphi_2 \quad . \tag{7.14}$$

As we have seen in Sec. V, E_i may be approximated by a single exponential term which grows rapidly in time. Because of the symmetry of the problem as expressed by Eqs. (7.6) and (7.7), we find through (5.12)

$$\mathcal{E}_1=\mathcal{E}_2 \quad . \tag{7.15}$$

Dividing (7.13a) and (7.13b) by $\mathcal{E}_j$ and subtracting one equation from the other, we obtain

$$\frac{d(\Delta\varphi)}{dt}=\Delta\Omega-2\alpha\mathfrak{M}^2\sin\Delta\varphi \tag{7.16}$$

This is a basic equation encountered in the theory of mode locking. The solution of this equation is well known, [10] and the condition for locking is

$$2\alpha\mathfrak{M}^2>|\Delta\Omega| \quad , \tag{7.17}$$

which is exactly the same as (7.11).

APPENDIX A: NORMAL MODE ANALYSIS OF EMBEDDED CAVITY

The electromagnetic field in the entire cavity (regions 1 and 2 in Fig. 1) is governed by the Maxwell wave equation

$$\frac{\partial E^2}{\partial z^2}-\mu_0\epsilon_0[1+\eta\delta(z)]\frac{\partial^2 E}{\partial t^2}=0 \quad . \tag{A1}$$

We may separate the variables by writing E as

$$E=U_k(z)e^{-i\Omega_k t} \tag{A2}$$

to find for $U_k(z)$

$$\frac{d^2U_k(z)}{dz^2}+\mu_0\epsilon_0[1+\eta\delta(z)]\Omega_k^2U_k(z)=0 \quad . \tag{A3}$$

The mode functions are found by solving (A3) with boundary conditions $U_k(z)=0$ at $z=l,\ -L$. The $\delta(z)$ function in (A3) can be eliminated by integrating (A3) over a small interval around $z=0$ to give an additional boundary condition

$$U_k'(0^+)-U_k'(0^-)=-\mu_0\epsilon_0\eta\Omega^2U_k(0) \quad . \tag{A4}$$

In deriving this equation we have used the continuity condition at $z=0$; i.e.,

$$U_k(0^+)=U_k(0^-) \quad . \tag{A5}$$

We have replaced Ω_k on the right-hand side of (A4) by the "resonance" frequency Ω to be defined later. This approximation is made possible because the thickness parameter η is small ($\sim 10^{-2}$ cm to account for $\nu/Q\sim 1$ MHz) and, as we shall see later,

$U_k(0)$ is appreciable only in the vicinity of $k = k_0 = \Omega/c$.

The solutions of (A4) have the form

$$U_k(z) = A_k \sin[k(z-l)] \quad \text{in region 1} \;, \tag{A6a}$$

$$U_k(z) = b_k \sin[k(z+L)] \quad \text{in region 2} \;, \tag{A6b}$$

where

$$k = \Omega_k/c \;. \tag{A7}$$

The conditions (A4) and (A5) with the form of $U_k(z)$ given by (A6a) and (A6b) result in a secular equation for the wavenumbers of the form

$$\tan kL = \tan kl/(\Lambda \tan kl - 1) \tag{A8}$$

where

$$\Lambda = \mu_0 \epsilon_0 \eta \Omega^2/k \simeq \eta k \tag{A9}$$

is a dimensionless parameter of the order of 10^2 (for $\eta \sim 10^{-2}$ cm, $\Omega \sim 10^{14}$ Hz) associated with the reflectivity of the wall at $z=0$. The transcendental equation (A8) determines the discrete wave numbers k of the normal modes. In the limit $L\to\infty$, however, k becomes continuous.

For each allowed value of k, the ratio of the amplitudes A_k and b_k is found from (A5) as

$$A_k/b_k = -\sin kL/\sin kl \;. \tag{A10}$$

Combining (A8) and (A10), we find

$$A_k^2/b_k^2 = [\tan^2 kl + 1]/[\tan^2 kl + (\Lambda \tan kl - 1)^2] \;. \tag{A11}$$

Examining the equation above, we find that, as a function of a (almost) continuous variable $\tan kl$, the ratio A_k^2/b_k^2 has a very sharp peak at $k = k_0$, where

$$\tan k_0 l = 1/\Lambda \;, \tag{A12}$$

with the peak value

$$(A_k^2/b_k^2)_{k=k_0} = \Lambda^2 + 1 \;. \tag{A13}$$

The values of $\tan kl$ at half-height points are

$$\tan kl = (1/\Lambda) \pm (1/\Lambda^2) \;. \tag{A14}$$

Equations (A12) and (A14) indicate that, for those wave numbers with the ratio A_k^2/b_k^2 appreciably large compared to unity, we may write kl as

$$kl = n\pi + \theta_k (|\theta_k| \ll 1) \;, \tag{A15}$$

where n is an integer fixed for those k in the neighborhood of k_0. Equation (A15) may be expressed as

$$\tan kl \simeq \theta_k \;. \tag{A16}$$

Inserting (A16) into (A11), and noting the narrowness of the peak at $k = k_0$, we find approximately for those modes around the peak

$$A_k^2/b_k^2 = 1/[(\Lambda\theta_k - 1)^2 + 1/\Lambda^2] \;. \tag{A17}$$

Noting that

$$\Omega_k = ck = (n\pi + \theta_k)c/l \tag{A18}$$

and introducing Fox–Li "resonance" frequency Ω as

$$\Omega = ck_0 = [n\pi + (1/\Lambda)]c/l \;, \tag{A19}$$

we rewrite (A17) as

$$A_k^2/b_k^2 = \Gamma^2\Lambda^2/[(\Omega_k - \Omega)^2 + \Gamma^2] \;, \tag{A20}$$

where

$$\Gamma = c/\Lambda^2 l \tag{A21}$$

plays the role of the bandwidth of the laser cavity.

The normalization condition gives another relationship between A_k and b_k. The orthogonality condition for the eigenfunctions is found directly from (A3) to be

$$(\Omega_k^2 - \Omega_{k'}^2)\int_{-L}^{l} dz\, U_k(z) U_{k'}(z) \epsilon(z) = 0 \;. \tag{A22}$$

Hence, it is convenient to normalize the eigenfunctions by

$$\int_{-L}^{l} dz\, U_k(z) U_k(z) \epsilon(z)/\epsilon_0 = \tfrac{1}{2} L \;. \tag{A23}$$

Substituting (A6a) and (A6b) into (A23), we obtain

$$\tfrac{1}{2} A_k^2 l + \tfrac{1}{2} b_k^2 L + A_k^2 \eta \sin kl = \tfrac{1}{2} L \;, \tag{A24}$$

where we have made use of the facts $kl \gg 1$ and $kL \gg 1$. Assuming $L \gg l \gg \eta$ and noting that the peak value of A_k^2/b_k^2 is finite and L independent, we find from (A24)

$$b_k^2 \simeq 1 \;. \tag{A25}$$

Combining (A20) and (A25), we obtain

$$A_k = \Gamma\Lambda/[(\Omega_k - \Omega)^2 + \Gamma^2]^{1/2} = M_k \;. \tag{A26}$$

In calculating the density of states, it is convenient to write kL as

$$kL = (N + \tfrac{1}{2})\pi - \chi_k \tag{A27}$$

where N is a large integer that labels the modes of the entire cavity. As seen from (A8), $|\chi_k|$ is small compared to unity for those k close to k_0. Using (A16) and (A27), (A8) is written as

$$\cot\chi_k = \theta_k/(\Lambda\theta - 1) \simeq 1/\Lambda(\Lambda\theta_k - 1) \;. \tag{A28}$$

Noting (2.3), we obtain from (A27)

$$\frac{\partial N}{\partial \Omega_k} = \frac{(\partial N/\partial k)}{c} = \frac{L}{\pi c} + \frac{(\partial\chi_k/\partial\Omega_k)}{\pi} \;. \tag{A29}$$

The last term in (A29) can be calculated through (A15), (A27), and (A28) and is found to be

$$\frac{\partial\chi_k/\partial\Omega_k}{\pi} = \frac{l/\pi c}{\theta_k^2 + (\Lambda\theta_k - 1)^2} \ll \frac{L}{\pi c} \;. \tag{A30}$$

Hence, we find the density of states to be

$$\rho = \frac{\partial N}{\partial \Omega_k} \simeq \frac{L}{\pi c} \;. \tag{A31}$$

The average mode spacing is

$$\Delta\Omega = \pi c/L \quad . \tag{A32}$$

Finally, by substituting (A15) and (A27) into (A10), and noting (A25), we find

$$b_k \simeq (-1)^{n-N-1} \quad . \tag{A33}$$

APPENDIX B: DERIVATION LEADING TO EQ. (2.12)

Using the explicit form of $U_k(z)$ and M_k given by (2.2a) and (2.3), Eq. (2.11) can be written in region 1 as

$$E(z,t) = (|E_0| l/L) \sum_k \{\Gamma^2\Lambda^2/[(\Omega-\Omega_k)^2+\Gamma^2]\} \times \sin[k(Z-l)]\, e^{-(\Omega_k t+\varphi)} \quad . \tag{B1}$$

Because of the Lorentzian factor in the curly bracket, only those terms in the summation for which

$$|\Omega - \Omega_k| \lesssim \Gamma \tag{B2}$$

have appreciable contribution. We find for these modes

$$|k - k_0|\, l = |\Omega_k - \Omega|\, l/c \ll 1 \ , \tag{B3}$$

since $\pi c/l$, the frequency separation between two neighboring Fox–Li modes, is usually large compared to the cavity bandwidth. Hence we may replace $\sin k(z-l)$ in (B1) with $\sin k_0(z-l)$. Taking the limit of $L\to\infty$, the summation can be transformed to an integral:

$$E(z,t) = \frac{l|E_0|}{L}\int d\Omega' \,\frac{\partial N}{\partial\Omega'}\,\frac{\Gamma^2/\Lambda^2}{[(\Omega-\Omega')^2+\Gamma^2]} \times \sin k_0(z-l)\, e^{-i(\Omega' t+\phi)} \quad , \tag{B4}$$

$$\rho = \frac{\partial N}{\partial\Omega'} = \frac{L}{c\pi} \quad . \tag{B5}$$

Because of the factor $e^{-i\Omega' t}$, $(t>0)$ we may close the contour in the lower half of the complex Ω' plane, and obtain

$$E(z,\ t) = (l|E_0|\Gamma\Lambda^2/c)\sin k_0(z-l)\, e^{-i(\Omega t+\varphi)-\Gamma t} \quad . \tag{B6}$$

Using the expressions (A21) and (A9) for Γ and Λ, we finally obtain Eq. (2.12).

APPENDIX C: ROOTS OF EQ. (5.10)

In this appendix we investigate the roots of Eq. (5.10). Introducing a new variable y,

$$y = p + \gamma \quad , \tag{C1}$$

and noting $\Gamma/\Delta\Omega \gg 1$, we can rewrite (5.10) as

$$S = [i\alpha M^2\Gamma/(\Gamma^2 - y^2)][\cot(i\pi y/\Delta\Omega) + iy/\Gamma] = 1 \ , \tag{C2}$$

where we used the approximation

$$\cot(\pm i|x|) \simeq \mp i \ \text{ for } |x| \gg 1 \quad . \tag{C3}$$

Let us first seek the root y_0 for which

$$|\mathrm{Re}(y_0)| \gg \Delta\Omega \ . \tag{C4}$$

Recalling the condition (4.12), (C2) can be now simplified, using (C3), to

$$S = [i\alpha M^2\Gamma/(\Gamma^2 - y^2)](\mp i + iy/\Gamma) = 1 \quad ,$$

which yields

$$y = \alpha M^2 \pm \Gamma \quad , \tag{C5}$$

where the sign in front of Γ should be chosen opposite to the sign of $\mathrm{Re}(y_0)$ assumed in (C4).

Checking the self-consistency of the assumption (C4) and the roots (C5), we find that there is one root which satisfies (C4), if $\alpha M^2 - \Gamma > 0$, but there is none if otherwise.

Let us now seek the remaining roots for which $\mathrm{Re}(Y_0) \lesssim \Delta\Omega$ by writing

$$y = i(n\Delta\Omega + \Delta_n) + G_n = in\Delta\Omega + \Delta p_n \quad . \tag{C6}$$

In view of the assumption (C4) and the consequence (C5), we conclude

$$|g_n| \lesssim \Delta\Omega \quad . \tag{C7}$$

In addition we may choose

$$-\tfrac{1}{2}\Delta\Omega \leq \Delta_n < \tfrac{1}{2}\Delta\Omega \tag{C8}$$

without loss of generality. Conditions (C7) and (C8) guarantee us that

$$|g_n| \ll \Gamma \ \text{ and } \ |\Delta_n| \ll \Gamma \ \text{ or } \ |\Delta p_n| \ll \Gamma \ , \tag{C9}$$

and hence (C2) is now simplified to

$$S = \frac{i\alpha M^2\Gamma}{\Gamma^2 + (n\Delta\Omega)^2}\left[\cot\left(\frac{\Delta p_n \pi}{\Delta\Omega}\right) - \frac{n\Delta\Omega}{\Gamma}\right] = 1 \quad , \tag{C10}$$

i.e.,

$$\Delta p_n = ig_n - \Delta_n = \frac{\Delta\Omega}{\pi}\cot^{-1}\left(\frac{n\Delta\Omega}{\Gamma} - \frac{i[\Gamma^2 + (n\Delta\Omega)^2]}{\alpha M^2\Gamma}\right) \ . \tag{C11}$$

The right-hand side of this equation can be separated, after tedious but elementary calculation, into real and imaginary parts, and we find that for small $n(n\Delta\Omega \ll \Gamma)$:

$$\Delta_n \simeq -\tfrac{1}{2}\Delta\Omega \ , \quad g_n \sim \Delta\Omega \ ; \tag{C12}$$

and for large $n(n\Delta\Omega \gtrsim \Gamma)$:

$$\Delta p_n \simeq 0 \ . \tag{C13}$$

---

*Based on the thesis submitted by R. Lang to the Massachusetts Institute of Technology in partial fulfillment of requirements for the Ph.D. degree.

†Work supported in part by the U. S. Air Force Office of Scientific Research, and in part by Kirtland Air Force Base Laser Laboratory, Air Force.

‡Alfred P. Sloan Fellow.

§Work supported in part by the National Aeronautics and

Space Administration and in part by the U. S. Air Force Office of Scientific Research.

[1]W. E. Lamb, Jr., Phys. Rev. 134, A1429 (1964).

[2]M. O. Scully and W. E. Lamb, Jr., Phys. Rev. **159**, 208 (1967).

[3]For recent reviews see M. O. Scully, in *Proceedings of the International School of Physics, "Enrico Fermi", Course XLI*, edited by R. J. Glauber (Academic, New York, 1969); W. Weidlich and H. Haken, in *Proceedings of the International School of Physics, "Enrico Fermi", Course XLI,* edited by R. J. Glauber (Academic, New York, 1969); M. Lax, in *Brandeis University Summer Institute in Theoretical Physics, 1966 Lectures*, edited by M. Chretien, S. Deser, and E. P. Gross (Gordon and Breach, New York, 1968), Vol. 2.

[4]The problem of light propagation has been considered from a multimode point of view in an interesting series of papers by R. Graham and H. Haken [Z. Phys. **213,** 420 (1968); Z. Phys. **234,** 193 (1970); Z. Phys. **235,** 166 (1970); Z. Phys. **237,** 31 (1970)] and V. Ernst and P. Stehle [Phys. Rev. **176,** 1456 (1968)]. However, the analysis of many-cavity modes (corresponding to one Fox–Li mode) in a lasing medium has not been treated previously. The two problems are fundamentally different. For the treatment of the propagation problem, see also F. T. Arecchi and R. Bonifacio, IEEE J. Quantum Electron. QE-1, 139 (1965); N. G. Basov, R. V. Ambertsmuyan, V. S. Zuev, P. G. Kryuko, and V. S. Letokhov, Zh. Eksp. Teor. Fiz. **50,** 23 (1966) [Sov. Phys.-JETP **23,** 16 (1966)]; S. L. McCall and E. L. Hahn, Phys. Rev. Lett. **18,** 908 (1967); G. L. Lamb, Jr., Phys. Rev. Lett. **25,** 181 (1967); J. A. Armstrong and E. Courtens, IEEE J. Quantum Electron. **4,** 411 (1968); IEEE J. Quantum Electron. **5,** 249 (1969); F. A. Hopf and M. O. Scully, Phys. Rev. **179,** 399 (1969); A. Icsevgi and W. E. Lamb, Jr., Phys. Rev. **185,** 517 (1969).

[5](a) A. G. Fox and T. Li, Bell Syst. Tech. J. **40,** 453 (1961). (b) We note that the present work treats the field as classical. As is well known, the semiclassical theory in Ref. 1 implies a δ-function line shape at steady state. In a future paper [Lang and Scully (unpublished)], we will consider the problem of a quantized multimode laser, which leads to a small but finite linewidth.

[6](a) The gain-narrowing argument for the laser linewidth is inadequate for a number of reasons, for example, the nonlinearity of the problem. (b) For further discussion of this point, see M. Scully, in The Proceedings of The Third Rochester Conference on Coherence and Quantum Optics (unpublished).

[7]Strictly speaking, we should include a factor $\epsilon(z)/\epsilon_0$ as in Eq. (A23) of Appendix A, but the contribution at $z=0$ from the δ function to this integral turns out to be negligible, because the thickness parameter η is very small compared to l. Hence we replace this factor by unity.

[8]P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York), p. 467.

[9]In fact, we obtain a more complicated expression for $G(A)$ owing to the standing-wave nature of the mode functions. However, if we replace $\sin^2 k_0 z$ by 1/2 we obtain the stated result. For the more complete treatment, see Sec. XVI in Ref. 1.

[10]See, for example, Sec. XIV of Ref. 1.