PHYSICAL REVIEW A VOLUME 7, NUMBER 4 APRIL 1973

Radiative Corrections to the Theory of an Electron Gas*

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(Received 6 December 1971; revised manuscript received 5 October 1972)

Radiative corrections to the Bohm–Pines dispersion equation and the Pines–Schrieffer–Drummond coupled pair of quasilinear equations are examined in the light of the Weisskopf–Wigner theory of line broadening. The principal results are the natural broadening and the Lamb shift of the particle-wave resonance condition. The dimensionless width of the nonresonant or adiabatic interaction is of the order of $\alpha(c/v)$ for electron-plasmon interaction and $\alpha(v/c)$ for electron-photon interaction, where α is the fine-structure constant. The Lamb shift of the resonance condition $\omega-\vec{k}\cdot\vec{v}\approx 0$ is of the order of $q^2\omega^2/\pi\mu v^3$ for electron-plasmon interaction and $q^2\omega^2/\pi\mu c^2 v$ for electron-photon interaction. Here ω and $\vec{k}$ are the frequency and wave vector, respectively, of the photon or the plasmon and q, μ, and $\vec{v}$ are the charge, mass, and velocity of the electron, respectively.

I. INTRODUCTION

In this paper we are interested in examining the radiative corrections to the linear and quasilinear theories of a spinless electron gas. In particular, it is our aim to investigate the radiative corrections to the Bohm–Pines dispersion equation[1,2] and the Pines–Schrieffer–Drummond coupled pair of quasilinear equations.[3–9] We shall use the solutions for the particle orbits as the basis of our discussion and obtain the radiative corrections with the help of the Weisskopf–Wigner theory of the broadening.[10–13]

The Weisskopf–Wigner theory only takes account of the natural broadening due to radiation damping. However, there are other physical processes that can give rise to a broadening of the current $J(\vec{k},\omega)$ that is responsible for the emission and absorption, i.e., that can give rise to a broadening of the particle-wave resonance condition $\omega-\vec{k}\cdot\vec{v}\approx 0$. Some of these processes are (a) dispersion broadening (i.e., broadening due to the finite lifetime of a photon or a plasmon in a growing or decaying plasma wave), (b) velocity-space-diffusion broadening (i.e., broadening due to the perturbations in the particle orbits caused by the quasilinear velocity-space diffusion), (c) broadening due to particle trapping (i.e., stochastic Stark broadening due to the finite amplitude of the photon or the plasmon field), and (d) conventional collisional broadening. We will examine processes (a)–(d) briefly in the Appendix. Our primary concern in this paper, however, will be with the natural broadening of lines in the spectrum of waves in a plasma and for this purpose we will adapt the Weisskopf–Wigner theory to describe Čerenkov transitions in a plasma.

Let us first consider an electron of charge q and mass μ moving with a velocity $\vec{v}$ in a medium of dielectric coefficient D_0. It is well known that such an electron will emit and absorb transverse photons as well as the longitudinal polarization or Bohr waves. If it emits or absorbs a photon or a quantum of the Bohr waves of energy $\hbar\omega$ and momentum $\hbar\vec{k}$, then its final velocity $\vec{v}'=\vec{v}-\hbar\vec{k}/\mu$ for emission and $\vec{v}'=\vec{v}+\hbar\vec{k}/\mu$ for absorption, since the linear momentum must be conserved. Hence the mean velocity of the electron during the emission or the absorption process is $\frac{1}{2}(\vec{v}+\vec{v}')=\vec{v}\mp\hbar\vec{k}/2\mu$. Thus the mean current $J(\vec{r},t)$ that is responsible for emission or absorption is proportional to $\delta(\vec{r}-(\vec{v}\mp\hbar\vec{k}/2\mu)t)$. The transition probability j for the emission or the absorption is proportional to[14–16] $|J(\vec{k},\omega)|^2$, where $J(\vec{k},\omega)$ is the space-time Fourier transform of $J(\vec{r},t)$. Hence for emission or absorption the transition probability $j\propto\delta(\omega-\vec{k}\cdot(\vec{v}\mp\hbar\vec{k}/2\mu))$. This, as we shall see later, is the result in the Golden Rule approximation.[17] In this result we have neglected the effects of radiation damping.

However, because of the emission and absorption processes, the electron cannot stay indefinitely in any quantum state $|\vec{v}\rangle$. In emission it jumps down to a lower state and in absorption it jumps up to a higher state. If we now consider all the transitions between two given quantum states $|\vec{v}\rangle$ and $|\vec{v}'\rangle$, by Kirchhoff's law[10,18] the transition probability for absorption is exactly equal to the transition probability for induced or stimulated emission. Thus, there is no net depletion of the quantum states $|\vec{v}\rangle$ or $|\vec{v}'\rangle$ due to the combined effect of absorption and induced emission. However, the energetically higher state is being depleted by the spontaneous emission process. Hence the radiative lifetime $1/\gamma_{\vec{v}}$ of the quantum state $|\vec{v}\rangle$, say, is determined only by the total spontaneous emission probability from this state $|\vec{v}\rangle$, and does not depend on the induced emission and absorption processes. Thus the lifetime of any quantum state does not depend on the intensity of the radiation field. This result is quite different from that suggested by the correspondence principle since classically one might conclude that the radiation damping would be proportional to the intensity of the radiation field. Because of the Heisenberg uncertainty principle this is, however, not true in the quantum theory.[10–13,19,20]

Thus, if one takes account of the radiation damping, the current $J(\vec{r}, t)$ that is responsible for the emission and absorption is proportional to $\delta(\vec{r} - (\vec{v} \mp \hbar\vec{k}/2\mu)t) \exp[-\frac{1}{2}(\gamma_{\vec{v}} + \gamma_{\vec{v} \mp \hbar\vec{k}/\mu})t]$. Hence the transition probabilities j for emission or absorption should be proportional to

$$\frac{(\gamma_{\vec{v}} + \gamma_{\vec{v} \mp \hbar\vec{k}/\mu})/2\pi}{[\omega - \vec{k}\cdot(\vec{v} \mp \hbar\vec{k}/2\mu)]^2 + [(\gamma_{\vec{v}} + \gamma_{\vec{v} \mp \hbar\vec{k}/\mu})/2]^2} \quad .$$

This, as we shall see later, is the essential result of the Weisskopf–Wigner theory of line broadening. However, in the Weisskopf–Wigner theory[10,12] there is a small additional correction to this result arising from the radiative self-energy of the quantum states $|\vec{v}\rangle$ and $|\vec{v} \mp \hbar\vec{k}/\mu\rangle$.

We will begin with a brief review of the basic concepts and the general results of the time-dependent perturbation theory. We will then derive the retarded frequency and wave-vector-dependent linear dielectric coefficient appropriate for the description of the collective behavior of the electron gas under consideration. It will be seen that the dielectric coefficient can be obtained from a knowledge of the quantum-mechanical transition probabilities for the emission and absorption of the photons or the plasmons (that is, from a knowledge of the Einstein A and B coefficients). Throughout this paper we will treat the emission and the absorption of both the transverse photons and the longitudinal plasmons on a somewhat similar footing. That is to say that the method of analysis presented in this paper is applicable equally well both for the transverse photons and the longitudinal plasmons. Furthermore, the method of analysis presented in this paper dictates how one should apply the radiative corrections.

By making use of the principle of detailed balance, we then derive the coupled pair of quasilinear equations, one describing the time evolution of the photon or the plasmon distribution function and the other describing the time evolution of the electron distribution function. In the Golden Rule approximation, the equation for the time rate of change of the electron distribution function reduces, in the classical limit, to a Fokker–Planck equation[21,22] in which there appear the usual diffusion and dynamical friction terms. We will show that the Golden Rule approximation of the coupled pair of quasilinear equations which deal only with the effect of waves on resonant electrons conserves both the energy and momentum, contrary to the statement found in the literature.[23]

We then examine in detail the proper radiative corrections to the linear and the quasilinear theories of the spinless electron gas. Here we follow the Weisskopf–Wigner theory of line broadening. We will show that according to Weisskopf and Wigner the width of the nonresonant or the adiabatic interaction is determined by the reciprocal of the lifetime of the quantum state of the electron. It will be seen that this Weisskopf–Wigner result for the width of the nonresonant or the adiabatic interaction is consistent both with the law of causality and the Heisenberg uncertainty principle. Furthermore, we will evaluate explicitly the Lamb shift[24,25] of the particle-wave resonance condition (both for the photon and the plasmon emission and absorption processes). This Lamb shift arises from the self-energy of the quantum states of the electron.[10,26]

II. REVIEW OF BASIC CONCEPTS

The Hamiltonian $\mathcal{H}$ of an electron gas and the radiation field may be written

$$\mathcal{H} = H + H_{\text{int}} \quad , \tag{1}$$

where

$$H|n\rangle = E_n|n\rangle \tag{2}$$

and H_{int} is the interaction Hamiltonian that is responsible for the emission and absorption of photons and plasmons. According to the Golden Rule of time-dependent perturbation theory, the transition probability $j(f;i)$ from an initial state $|i\rangle$ of energy E_i to a final state $|f\rangle$ of energy E_f is given by[10]

$$j(f;i) = (2\pi/\hbar)|\langle f|H_{\text{int}}|i\rangle|^2\delta(E_f - E_i) \quad . \tag{3}$$

By making use of Eq. (3) and the principle of detailed balance, we will later derive both the Bohm–Pines dispersion equation and the Pines–Schrieffer–Drummond coupled pair of quasilinear equations. However, the right-hand side of Eq. (3) is only the first term of the series solution of the integral equation due to Heitler and Ma.[10] According to Heitler and Ma, the transition probability $j(f;i)$ is given by

$$j(f;i) = (2\pi/\hbar)|\langle f|T|i\rangle|^2\delta(E_f - E_i) \quad , \tag{4}$$

where

$$\langle f|T|i\rangle = \langle f|H_{\text{int}}|i\rangle + \sum_{f'\neq i} \frac{\langle f|H_{\text{int}}|f'\rangle\langle f'|T|i\rangle}{E_i - E_{f'}} \quad , \tag{5}$$

where $|f'\rangle$ is an intermediate state of energy $E_{f'}$. The series solution of the integral equation (5) may be written

$$\langle f|T|i\rangle = \langle f|H_{\text{int}}|i\rangle + \sum_{f'\neq i} \frac{\langle f|H_{\text{int}}|f'\rangle\langle f'|H_{\text{int}}|i\rangle}{E_i - E_{f'}}$$
$$+ \sum_{f'\neq i}\sum_{f''\neq i} \frac{\langle f|H_{\text{int}}|f'\rangle\langle f'|H_{\text{int}}|f''\rangle\langle f''|H_{\text{int}}|i\rangle}{(E_i - E_{f'})(E_i - E_{f''})}$$

$$+\cdots \quad , \tag{6}$$

where $|f''\rangle$ is another intermediate state of energy $E_{f''}$. The higher-order terms of Eqs. (5) and (6) will yield all the nonlinear processes (such as the nonlinear mode coupling, nonlinear Landau damping, and nonlinear decay interactions) of the theory of electron gas. However, in this paper we are concerned only with the linear and the quasilinear theories of the electron gas. In particular, our aim is to examine the radiative corrections to the linear and the quasilinear theories of electron gas in the light of the Weisskopf–Wigner theory of line broadening.

According to Weisskopf and Wigner, the transition probability $j(f;i)$ that takes account of the radiative corrections is given by[10]

$$j(f;i)=(2\pi/\hbar)\left|\langle f|H_{\rm int}|i\rangle\right|^2 \times\left(\frac{\hbar\gamma_{f,i}/2\pi}{(E_f-E_i-\Delta E_{f,i})^2+(\frac{1}{2}\hbar\gamma_{f,i})^2}\right) \quad , \tag{7}$$

where

$$\gamma_{f,i}=\gamma_f+\gamma_i \quad , \tag{8}$$

$$\Delta E_{f,i}=\Delta E_i-\Delta E_f \quad , \tag{9}$$

$$\gamma_n=\sum_{n'} j_{\rm SE}(n';n) \quad \text{such that} \quad E_{n'}<E_n \quad , \tag{10}$$

and

$$\Delta E_n=\sum_{n'\neq n}\frac{\langle n|H_{\rm int}|n'\rangle\langle n'|H_{\rm int}|n\rangle}{E_n-E_{n'}} \quad . \tag{11}$$

Here $j_{\rm SE}(n';n)$ is the transition probability for spontaneous emission of photons or plasmons from a state $|n\rangle$ to an energetically lower state $|n'\rangle$ and ΔE_n is the self-energy of the state $|n\rangle$. It is important to note that the spontaneous-emission probability $j_{\rm SE}$ is always greater than or equal to zero and is independent of the statistical population of the quantum states. Hence $\gamma_{f,i}\geq 0$ always, irrespective of whether the statistical population of the quantum states is normal or inverted. That is to say, the Weisskopf–Wigner transition probability of Eq. (7) satisfies the requirements of the causality principle (which states that the effect should not precede the cause) and always reduces to the Fermi Golden-Rule result of Eq. (3) when one neglects the radiative corrections, regardless of the statistical population of the quantum states.

III. THEORY IN GOLDEN RULE APPROXIMATION

We now consider the emission and absorption processes illustrated in Figs. 1 and 2. In the Golden Rule approximation, it is relatively easy to show that the transition probabilities for absorption j_A and emission j_E of a photon or a plasmon of momentum $\hbar\vec{k}$, energy $\hbar\omega$, and polarization vector $\vec{\epsilon}_k$ are given by[15–17,27,28]

$$j_A(\vec{v}';\vec{v})=N_k M\delta_{\vec{v}',\vec{v}+\hbar\vec{k}/\mu}\,\delta(\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)) \quad , \tag{12}$$

$$j_E(\vec{v};\vec{v}')=(N_k+1)M\delta_{\vec{v}',\vec{v}+\hbar\vec{k}/\mu}\,\delta(\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)) \quad , \tag{13}$$

$$j_A(\vec{v};\vec{v}'')=N_k M\delta_{\vec{v}'',\vec{v}-\hbar\vec{k}/\mu}\,\delta(\omega-\vec{k}\cdot(\vec{v}-\hbar\vec{k}/2\mu)) \quad , \tag{14}$$

and

$$j_E(\vec{v}'';\vec{v})=(N_k+1)M\delta_{\vec{v}'',\vec{v}-\hbar\vec{k}/\mu}\,\delta(\omega-\vec{k}\cdot(\vec{v}-\hbar\vec{k}/2\mu)) \quad , \tag{15}$$

where

$$M=\left(\frac{8\pi^2q^2}{L^3\hbar(\partial/\partial\omega)(\omega^2D_0)}\right)\times\begin{cases}|\vec{\epsilon}_k\cdot\vec{v}|^2 & \text{for transverse photons}\\ (\omega/k)^2 & \text{for plasmons}\end{cases} \quad . \tag{16}$$

Here L^3 is the volume, N_k represents the number of photons or plasmons, and the Kronecker δ's and the Dirac δ functions indicate the conservation of linear momentum and energy, respectively. Let $N_0F(\vec{v})$ represent the number of electrons per unit volume which are in the quantum state $|\vec{v}\rangle$. By applying the principle of detailed balance for the transition probabilities per unit volume of emission and absorption, we get

$$\frac{\partial(N_k/L^3)}{\partial t}=\int d\vec{v}\,N_0[F(\vec{v}')j_E(\vec{v};\vec{v}')-F(\vec{v})j_A(\vec{v}';\vec{v})] \quad , \tag{17}$$

$$\frac{\partial F(\vec{v})}{\partial t}=\sum_{\vec{k}}\sum_{\vec{\epsilon}_k}[F(\vec{v}')j_E(\vec{v};\vec{v}')-F(\vec{v})j_A(\vec{v}';\vec{v}) +F(\vec{v}'')j_A(\vec{v};\vec{v}'')-F(\vec{v})j_E(\vec{v}'';\vec{v})] \tag{18}$$

for a nondegenerate electron gas, and

$$\frac{\partial(N_k/L^3)}{\partial t}=\int d\vec{v}\,N_0\{F(\vec{v}')[1-F(\vec{v})]j_E(\vec{v};\vec{v}') -F(\vec{v})[1-F(\vec{v}')]j_A(\vec{v}';\vec{v})\} \quad , \tag{19}$$

$$\frac{\partial F(\vec{v})}{\partial t}=\sum_{\vec{k}}\sum_{\vec{\epsilon}_k}\{F(\vec{v}')[1-F(\vec{v})]j_E(\vec{v};\vec{v}') -F(\vec{v})[1-F(\vec{v}')]j_A(\vec{v}';\vec{v}) +F(\vec{v}'')[1-F(\vec{v})]j_A(\vec{v};\vec{v}'') -F(\vec{v})[1-F(\vec{v}'')]j_E(\vec{v}'';\vec{v})\} \tag{20}$$

for a degenerate electron gas. These are the coupled pair of quasilinear equations. In general, however, it is extremely difficult to solve the coupled pair of quasilinear equations, since at any instant of time the rate of change of the photon or the plasmon distribution function depends on the instantaneous value of the electron distribution function, and the rate of change of the electron distribution function, in turn, depends on the instantaneous value of the photon or the plasmon distribution function. Nevertheless, one can show that if the probability function $F(\vec{v})$ is normal,

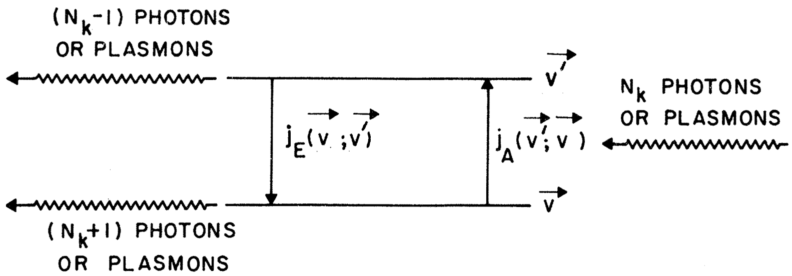

FIG. 1. Emission and absorption of a photon or a plasmon by an electron.

then the absorption will exceed the induced or stimulated emission and the coupled pair of quasilinear equations will always drive the system towards thermodynamic equilibrium. But if under certain circumstances the probability function $F(\vec{v})$ exhibits an inverted population of states, then the induced or stimulated emission will exceed the absorption and consequently the electron gas will exhibit maser action for photons or plasmons (that is, the familiar two-stream instability). For a system that exhibits maser action, if the photon or the plasmon density is sufficiently low so that one could neglect the higher-order Heitler-Ma terms of Eqs. (5) or (6) (that is, if one could neglect nonlinear mode coupling, nonlinear Landau damping, nonlinear decay interactions, etc.) then the coupled pair of quasilinear equations will drive the system not towards thermodynamic equilibrium but towards the Drummond-Pines quasilinear steady state.[4,29,30]

Although the evolution of the system (for both growing and damped waves) is always determined by the coupled pair of quasilinear equations,[31] it is physically instructive to examine the solution of Eq. (17) or (19) when $F(\vec{v})$ is very near its steady-state value. If we assume $N_k = 0$ at $t=0$, then the solution of Eq. (17) or (19) is of the form $N_k \propto 1 - e^{-2\Gamma_k t}$, where

$$2\Gamma_k = \int d\vec{v}\,(L^3 N_0 M)\delta(\omega - \vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)) \times [F(\vec{v}) - F(\vec{v}+\hbar\vec{k}/\mu)] \quad (21)$$

in the Golden Rule approximation. Since the dissipation or the absorption of energy by any system is represented by the anti-Hermitian part of the dielectric tensor appropriate for the description of the system, one can easily show[18,32] that the imaginary part of the dielectric coefficient $\mathrm{Im}D(\omega,\vec{k}) = -(D_0/\omega)2\Gamma_k$. Thus, on making use of the well-known Kramers-Kronig relations[22,33] we get

$$D(\omega,\vec{k}) = D_0 - \lim_{\gamma\to 0^+}\left(\frac{D_0}{\pi\omega}\right)\int d\vec{v}(L^3N_0M) \times \frac{F(\vec{v}) - F(\vec{v}+\hbar\vec{k}/\mu)}{\omega - \vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu) - i\gamma} \quad . \quad (22)$$

Equation (22) can be easily rewritten in the Bohm-Pines form[1,2,34]

$$D(\omega,\vec{k}) = D_0 - \lim_{\gamma\to 0^+}\left(\frac{D_0}{\pi\omega}\right)\int d\vec{v}\left(\frac{L^3N_0M\hbar k^2}{\mu}\right) \times \frac{F(\vec{v})}{(\omega - \vec{k}\cdot\vec{v} - i\gamma)^2 - (\hbar k^2/2\mu)^2} \quad . \quad (23)$$

Since the energy $\mathcal{E}_k$ in the electromagnetic wave or the plasma wave is $\mathcal{E}_k = N_k\hbar\omega$ and since by Taylor-series expansion

$$\psi(\vec{v}\pm\Delta\vec{v}) = [1 \pm (\Delta\vec{v}\cdot\vec{\nabla}_v) + \tfrac{1}{2}(\Delta\vec{v}\cdot\vec{\nabla}_v)(\Delta\vec{v}\cdot\vec{\nabla}_v) + \cdots]\psi(\vec{v}) \quad , \quad (24)$$

one can show that the classical limits (of the Golden Rule approximation) of the coupled pair of Eqs. (17) and (18) become

$$\frac{\partial\mathcal{E}_k}{\partial t} = \int d\vec{v}\,(L^3N_0M\hbar)\left(\frac{\mathcal{E}_k}{\mu}\,\delta(\omega - \vec{k}\cdot\vec{v})\vec{k}\cdot\vec{\nabla}_v F(\vec{v}) + \omega\delta(\omega - \vec{k}\cdot\vec{v})F(\vec{v})\right) \quad (25)$$

and

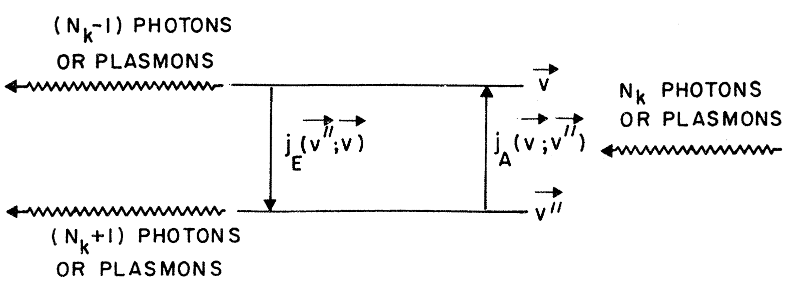

FIG. 2. Emission and absorption of a photon or a plasmon by an electron.

$$\frac{\partial F(\vec{v})}{\partial t} = \sum_{\vec{k}} \sum_{\vec{\epsilon}_k} \frac{M\hbar}{\mu\omega}\left(\frac{\mathcal{E}_k}{\mu}\vec{k}\cdot\vec{\nabla}_v[\delta(\omega-\vec{k}\cdot\vec{v})\vec{k}\cdot\vec{\nabla}_v F(\vec{v})] + \omega\vec{k}\cdot\vec{\nabla}_v[\delta(\omega-\vec{k}\cdot\vec{v})F(\vec{v})]\right) . \quad (26)$$

It is seen that Eq. (26) is a Fokker–Planck equation whose first and second terms represent a diffusion and dynamical friction, respectively. This is simply a manifestation of the fact that, owing to Čerenkov emission and absorption of photons or plasmons, the electrons must undergo a recoil in order to conserve both energy and momentum, and this recoil motion of the electrons is essentially a Brownian motion in velocity space. From the point of view of the conservation laws of energy and momentum the coupled pair of Eqs. (25) and (26) is self-consistent since

$$\sum_{\vec{k}} \sum_{\vec{\epsilon}_k} \frac{\partial \mathcal{E}_k}{\partial t} = -\int d\vec{v}\ \frac{L^3 N_0 \mu v^2}{2}\ \frac{\partial F(\vec{v})}{\partial t} \quad (27)$$

and

$$\sum_{\vec{k}} \sum_{\vec{\epsilon}_k} \frac{\vec{k}}{\omega}\frac{\partial \mathcal{E}_k}{\partial t} = -\int d\vec{v}\ (L^3 N_0 \mu \vec{v})\ \frac{\partial F(\vec{v})}{\partial t} . \quad (28)$$

Thus, the Golden Rule approximation of the coupled pair of quasilinear equations which deal only with the effects of waves on resonant electrons conserves both the energy and momentum, contrary to the statement found in the literature.[23] This result is hardly surprising and is simply a manifestation of the conservation laws implied by the Kronecker δ's and the Dirac δ functions of Eqs. (12)–(15).

The elements of the velocity-space diffusion tensor of Eq. (26) may be written

$$\mathfrak{D}_{ij} = \sum_{\vec{k}} \sum_{\vec{\epsilon}_k} \frac{k_i k_j M\hbar}{\mu^2 \omega}\ \mathcal{E}_k \delta(\omega - \vec{k}\cdot\vec{v}) . \quad (29)$$

Thus we see that in the Golden Rule approximation one gets only a resonant diffusion of Eq. (29) for both the damped and growing electromagnetic wave or the plasma wave. For damped waves this resonant diffusion is partly responsible for driving the system towards thermodynamic equilibrium, while for growing waves it is partly responsible for driving the system towards the Drummond–Pines quasilinear steady state.

In Sec. IV we will examine the generalization of the results of Eq. (29) in the light of Weisskopf–Wigner theory of line broadening.

IV. RADIATIVE CORRECTIONS

Our aim now is to examine the radiative corrections to the linear dielectric coefficient and the coupled pair of quasilinear equations. Here we will make use of the Weisskopf–Wigner improved solution of the equations of the perturbation theory. In the Golden Rule approximation, the transition probability $j(f;i)$ from an initial state $|i\rangle$ of energy E_i to a final state $|f\rangle$ of energy E_f is given by Eq. (3) and this approximation is valid only for a time t which is small compared with $\gamma_{f,i}^{-1}$, where $\gamma_{f,i}$ is given by Eqs. (8) and (10). Furthermore, in the Golden Rule approximation both the energy and the momentum are strictly conserved and consequently the electron-photon or the electron-plasmon interaction is resonant. The energy conservation is explicitly stated in Eq. (3) by the Dirac δ function, while the momentum conservation is implicit in the matrix element of the interaction Hamiltonian.

However, according to the Heisenberg uncertainty relation $(\delta E)(\delta t) \approx \hbar$. This uncertainty relation states that the energy of a system is only known with an accuracy δE if, for the measurement of the energy, a time δt is available. In our case the initial state $|i\rangle$ has a lifetime $1/\gamma_i$ and the final state $|f\rangle$ has a lifetime $1/\gamma_f$. Therefore, the energy of the initial state is only defined with an uncertainty $\delta E_i \approx \hbar\gamma_i$ and that of the final state is uncertain by $\delta E_f \approx \hbar\gamma_f$. Hence the frequency of the emitted or the absorbed line will have the breadth $\delta\omega_{f,i} \approx (\delta E_f + \delta E_i)/\hbar \approx \gamma_f + \gamma_i = \gamma_{f,i}$. This is then the natural width of the nonresonant or the adiabatic electron-photon or the electron-plasmon interaction. Furthermore, according to the second-order perturbation theory, the initial and final states should have the self-energy corrections ΔE_i and ΔE_f, respectively, due to transitions via intermediate states. This means a correction $\Delta\omega_{f,i} \approx -\Delta E_{f,i}/\hbar = (\Delta E_f - \Delta E_i)/\hbar$ to the frequency $\omega_{f,i} = (E_f - E_i)/\hbar$ of the emitted or absorbed line. This is the familiar Lamb shift. Thus, according to Weisskopf and Wigner the transition probability $j(f;i)$ that takes account of the radiative corrections is given by Eq. (7) and this improved formula for $j(f;i)$ is valid not only for times t which are small compared with $1/\gamma_{f,i}$ but also for times t comparable with $1/\gamma_{f,i}$.

From Eq. (15) we get the following transition probability for spontaneous emission j_{SE} of a photon or a plasmon of momentum $\hbar\vec{k}$, energy $\hbar\omega$, and polarization vector $\vec{\epsilon}_k$ by an electron in an initial state $|\vec{v}\rangle$:

$$j_{SE} = \sum_{\vec{v}''} j_{SE}(\vec{v}'';\vec{v}) = M\delta(\omega - \vec{k}\cdot(\vec{v} - \hbar\vec{k}/2\mu)) , \quad (30)$$

where M is given by Eq. (16). Hence the reciprocal of the radiative lifetime of the state $|\vec{v}\rangle$ is given by

$$\gamma_{\vec{v}} = \sum_{\vec{k}} \sum_{\vec{\epsilon}_k} M\delta(\omega - \vec{k}\cdot(\vec{v} - \hbar\vec{k}/2\mu)) \quad (31)$$

and the radiative self-energy of the state $|\vec{v}\rangle$ is given by

$$\Delta E_{\vec{v}} = \frac{-\hbar}{\pi} P \sum_{\vec{k}} \sum_{\vec{\epsilon}_k} \frac{M}{\omega - \vec{k}\cdot(\vec{v} - \hbar\vec{k}/2\mu)} \quad . \tag{32}$$

Here P denotes the principal value and

$$\sum_{\vec{k}} [\ ] \to \left(\frac{L}{2\pi}\right)^3 \int_0^\infty dk\, k^2 \int d\Omega [\ ] \ , \tag{33}$$

where $d\Omega$ is the element of solid angle. Let θ be the angle between the vectors $\vec{k}$ and $\vec{v}$. Then $\vec{k}\cdot\vec{v} = kv\cos\theta$ and $d\Omega = 2\pi \sin\theta\, d\theta$. From Eqs. (16), (31), and (33) for plasmons we get

$$\gamma_{\vec{v}} = \left(\frac{L}{2\pi}\right)^3 \int_0^\infty dk\, k^2 \int d\theta\, 2\pi \sin\theta \times \frac{8\pi^2 q^2}{L^2 \hbar\, \partial(\omega^2 D_0)/\partial\omega} \left(\frac{\omega}{k}\right)^2 \times \delta((\omega + \hbar k^2/2\mu) - kv\cos\theta)$$
$$= \left(\frac{q^2}{\hbar v}\right) \int_0^\infty dk \left(\frac{\omega}{k D_0}\right) \quad . \tag{34}$$

But for plasmons ω is approximately the plasma frequency $\omega_0 = (4\pi N_0 q^2/D_0\mu)^{1/2}$ and hence $\gamma_{\vec{v}}$ of Eq. (34) is logarithmically divergent both at small and large k. However, since $\omega \approx \omega_0 \approx \vec{k}\cdot\vec{v} = kv\cos\theta$, $k \gtrsim \omega_0/v$, and since the plasmon wavelength must be larger than the plasma Debye length $k \lesssim \omega_0/v_0$, where v_0 is the statistical average thermal velocity of the electron gas under consideration. Hence Eq. (34) becomes

$$\gamma_{\vec{v}} \approx q^2/\hbar v \int_{k_{\min}\approx\omega_0/v}^{k_{\max}\approx\omega_0/v_0} dk\, (\omega_0/kD_0) \quad \text{for } v \gtrsim v_0$$
$$\approx 0 \qquad\qquad \text{for } v < v_0 \ , \tag{35}$$

that is, for plasmon emission and absorption

$$\gamma_{\vec{v}} \approx \omega_0 (q^2/\hbar v D_0) \ln(v/v_0) \quad \text{for } v \gtrsim v_0$$
$$\approx 0 \qquad\qquad \text{for } v < v_0 \ . \tag{36}$$

It may be pointed out that by using a somewhat more rigorous analysis one can show that the improved form of Eq. (36) may be written[15]

$$\gamma_{\vec{v}} = \omega_0 (q^2/\hbar v D_0)[(v_0/v) K_1(v_0/v) K_0(v_0/v)] \tag{37}$$

for all v, where $K_\nu(x)$ is the modified Bessel function of order ν. For transverse photons

$$\sum_{\vec{\epsilon}_k} |\vec{\epsilon}_k \cdot \vec{v}|^2 = v^2 \sin^2\theta$$

and $k^2 dk = (\omega^2 D_0^{3/2}/c^3)\, d\omega$. Hence for photons from Eqs. (16), (31), and (33) we get

$$\gamma_{\vec{v}} = \left(\frac{L}{2\pi}\right)^3 \int d\omega\, \frac{\omega^2 D_0^{3/2}}{c^3} \int d\theta\, 2\pi\sin\theta \times \frac{8\pi^2 q^2}{L^3 \hbar\, \partial(\omega^2 D_0)/\partial\omega}\, v^2 \sin^2\theta \times \delta((\omega + \hbar k^2/2\mu) - kv\cos\theta)$$
$$= \frac{q^2}{\hbar c}\, \frac{v}{c}\, \bar{\omega} \quad , \tag{38}$$

where

$$\bar{\omega} = \int d\omega \left\{ 1 - \left(\frac{c^2}{D_0 v^2}\right) \left[1 + D_0 \left(\frac{\hbar\omega}{\mu c^2}\right) + \frac{1}{4} D_0^2 \left(\frac{\hbar\omega}{\mu c^2}\right)^2 \right] \right\} \ . \tag{39}$$

It may be noted that $(1/v)(\partial\gamma_{\vec{v}}/\partial\omega)\,d\omega$ is the Ginsburg expression[35] for the number of quanta with angular frequency between ω and $\omega + d\omega$ spontaneously emitted per unit distance by a particle of charge q and mass μ moving with a speed v through a dielectric of dielectric coefficient D_0. It might appear that $\bar{\omega}$ of Eq. (39) is divergent at large ω. This apparent divergence is not a physical reality, of course. Since the velocity of propagation for high frequencies in a dielectric approaches its vacuum value, $D_0(\omega)$ will become smaller than c^2/v^2 for sufficiently large values of ω, and $\bar{\omega}$ of Eq. (39) is finite.[36]

Let us now examine the radiative self-energy of the quantum state $|\vec{v}\rangle$. From Eqs. (16), (32), and (33) for plasmon emission and absorption we get

$$\Delta E_{\vec{v}} = -\left(\frac{\hbar}{\pi}\right) P \int_0^\infty dk \left(\frac{q^2\omega}{\hbar D_0 k v}\right) \int_1^{-1} d(\cos\theta) \times \frac{1}{\cos\theta - (\omega + \hbar k^2/2\mu)/kv}$$
$$= \left(\frac{q^2}{\hbar v}\right) P \int_0^\infty dk \left(\frac{\hbar\omega}{\pi k D_0}\right) \times \ln\left(\frac{|(\omega + \hbar k^2/2\mu) - kv|}{(\omega + \hbar k^2/2\mu) + kv}\right) \ . \tag{40}$$

For plasmons $\omega \approx \omega_0$, $k_{\max} \approx \omega_0/v_0$, $k_{\min} \approx \omega_0/v$, and following Bethe[25] it seems permissible to consider the logarithm as a constant (independent of k) in the first approximation. Hence Eq. (40) may be rewritten in the form

$$\Delta E_{\vec{v}} \approx (\hbar\omega_0) \left(\frac{q^2}{\pi\hbar v D_0}\right) \left\langle \ln\left(\frac{|(\omega_0 + \hbar k^2/2\mu) - kv|}{(\omega_0 + \hbar k^2/2\mu) + kv}\right) \right\rangle \times \int_{k_{\min}\approx\omega_0/v}^{k_{\max}\approx\omega_0/v_0} dk \left(\frac{1}{k}\right) ,$$

where the angular brackets refer to an average value. That is, for plasmon emission and absorption, the self-energy of the quantum state $|\vec{v}\rangle$ may be written

$$\Delta E_{\vec{v}} \approx (\hbar\omega_0) \left(\frac{q^2}{\pi\hbar v D_0}\right) \ln\left(\frac{v}{v_0}\right) \times \left\langle \ln\left(\frac{|(\omega_0 + \hbar k^2/2\mu) - kv|}{(\omega_0 + \hbar k^2/2\mu) + kv}\right) \right\rangle \ . \tag{41}$$

A rough estimate of the average value of the logarithm may be written

$$\left\langle \ln\left(\frac{|(\omega_0+\hbar k^2/2\mu)-kv|}{(\omega_0+\hbar k^2/2\mu)+kv}\right)\right\rangle \approx \tfrac{1}{2}\left[\ln(\ )_{k\approx k_{\min}}+\ln(\ )_{k\approx k_{\max}}\right]$$

$$=\frac{1}{2}\left[\ln\left(\frac{\hbar\omega_0}{\hbar\omega_0+4\mu v^2}\right)+\ln\left(\frac{|\hbar\omega_0-2\mu v_0^2[(v/v_0)-1]|}{\hbar\omega_0+2\mu v_0^2[(v/v_0)+1]}\right)\right] \quad . \quad (42)$$

For the emission and absorption of transverse photons, from Eqs. (16), (32), and (33) one finds, after a certain amount of algebra, that the radiative self-energy of the quantum state $|\vec{v}\rangle$ may be written

$$\Delta E_{\vec{v}}=\left(\frac{\hbar}{\pi}\right)P\int d\omega\,\frac{q^2v}{\hbar c^2}\times\int_{-1}^{1}\frac{d(\cos\theta)\,[1-\cos^2\theta]}{\cos\theta-(c/D_0^{1/2}v)[1+(\frac{1}{2}D_0)(\hbar\omega/\mu c^2)]}$$

$$=\hbar\bar{\omega}_1\left(\frac{q^2}{\pi\hbar c}\right)\frac{v}{c}\quad , \quad (43)$$

where

$$\bar{\omega}_1=\int d\omega\left(\left\{1-\left(\frac{c^2}{D_0v^2}\right)\left[1+D_0\left(\frac{\hbar\omega}{\mu c^2}\right)+\frac{1}{4}D_0^2\left(\frac{\hbar\omega}{\mu c^2}\right)^2\right]\right\}\times\ln\left(\frac{|(c/D_0^{1/2}v)[1+\frac{1}{2}D_0(\hbar\omega/\mu c^2)]-1|}{(c/D_0^{1/2}v)[1+\frac{1}{2}D_0(\hbar\omega/\mu c^2)]+1}\right)-2\left(\frac{c}{D_0^{1/2}v}\right)\left[1+\frac{1}{2}D_0\left(\frac{\hbar\omega}{\mu c^2}\right)\right]\right) \quad (44)$$

and we have used the relation $D_0=(ck/\omega)^2$.

Let us again consider the emission and absorption processes illustrated in Figs. 1 and 2. In the Weisskopf–Wigner approximation, that is, from Eq. (7), we get the following transition probabilities for absorption j_A and emission j_E of a photon or a plasmon of momentum $\hbar\vec{k}$, energy $\hbar\omega$, and polarization vector $\vec{\epsilon}_k$:

$$j_A(\vec{v}';\vec{v})=N_kM\delta_{\vec{v}',\vec{v}+\hbar\vec{k}/\mu}\times\left(\frac{\gamma_{\vec{v}',\vec{v}}/2\pi}{[\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)-\Delta E_{\vec{v},\vec{v}'}/\hbar]^2+(\frac{1}{2}\gamma_{\vec{v}',\vec{v}})^2}\right), \quad (45)$$

$$j_E(\vec{v};\vec{v}')=(N_k+1)M\delta_{\vec{v}',\vec{v}+\hbar\vec{k}/\mu}\times\left(\frac{\gamma_{\vec{v}',\vec{v}}/2\pi}{[\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)-\Delta E_{\vec{v},\vec{v}'}/\hbar]^2+(\frac{1}{2}\gamma_{\vec{v}',\vec{v}})^2}\right), \quad (46)$$

$$j_A(\vec{v};\vec{v}'')=N_kM\delta_{\vec{v}'',\vec{v}-\hbar\vec{k}/\mu}\times\left(\frac{\gamma_{\vec{v}'',\vec{v}}/2\pi}{[\omega-\vec{k}\cdot(\vec{v}-\hbar\vec{k}/2\mu)-\Delta E_{\vec{v}'',\vec{v}}/\hbar]^2+(\frac{1}{2}\gamma_{\vec{v}'',\vec{v}})^2}\right), \quad (47)$$

and

$$j_E(\vec{v}'';\vec{v})=(N_k+1)M\delta_{\vec{v}'',\vec{v}-\hbar\vec{k}/\mu}\times\left(\frac{\gamma_{\vec{v}'',\vec{v}}/2\pi}{[\omega-\vec{k}\cdot(\vec{v}-\hbar\vec{k}/2\mu)-\Delta E_{\vec{v}'',\vec{v}}/\hbar]^2+(\frac{1}{2}\gamma_{\vec{v}'',\vec{v}})^2}\right), \quad (48)$$

where M is given by Eq. (16). It should be noted that the frequency shift in both Eqs. (45) and (46) is the same both in magnitude and sign. Similarly the Lamb shift in Eqs. (47) and (48) is the same both in magnitude and sign. Thus we see that the Weisskopf–Wigner approximation is consistent with the Kirchhoff radiation law.

On making use of Eqs. (45) and (46) in Eqs. (17) and (19), one sees readily that in the Weisskopf–Wigner approximation one has to replace the Dirac δ function $\delta(\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu))$ in Eq. (21) by the factor

$$\frac{\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}/2\pi}{[\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)-\Delta E_{\vec{v},\vec{v}+\hbar\vec{k}/\mu}/\hbar]^2+(\frac{1}{2}\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}})^2}\,.$$

It is seen from Eq. (21) that in the Golden Rule approximation the Landau growth or damping of a plasmon or a photon is resonant. That is to say, a plasmon or a photon will have a damping or a growth only from the resonant electrons whose velocity $\vec{v}$ satisfies the condition $\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)=0$. However, in the Weisskopf–Wigner approximation, there exists a nonresonant or adiabatic electron-plasmon or electron-photon interaction due to the finite (but positive definite) value of $\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}\approx 2\gamma_{\vec{v}}$. Hence, there exists a nonresonant or adiabatic contribution to the growth or damping of a plasmon or a photon. This is a direct consequence of the Heisenberg uncertainty principle. Furthermore, in the limit $\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}\to 0^+$, the growth or the damping of a plasmon or a photon is different from zero only when $\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)-\Delta E_{\vec{v},\vec{v}+\hbar\vec{k}/\mu}/\hbar=0$, that is, if the energy, including any displacements of the energy levels caused by the interaction with radiation, is conserved. However, it should be emphasized that one is never allowed to take the noncausal limit $\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}\to 0^-$, since $\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}\geq 0$ always. That is to say if the statistical population of the quantum states is normal then the plasmon or the photon is always damped, while if the population is inverted then the plasmon or the photon is always growing (in accordance with the causality principle).

In this Weisskopf–Wigner approximation, the retarded-frequency- and wave-vector-dependent dielectric coefficient may be written

$$D(\omega,\vec{k})=D_0-\frac{D_0}{\pi\omega}\int d\vec{v}\,\frac{L^3N_0M[F(\vec{v})-F(\vec{v}+\hbar\vec{k}/\mu)]}{[\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)-\Delta E_{\vec{v},\vec{v}+\hbar\vec{k}/\mu}/\hbar]-i(\frac{1}{2}\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}})}\quad . \quad (49)$$

Equation (49) can be rewritten in the form

$$D(\omega,\vec{k})=D_0-\frac{D_0}{\pi\omega}\int d\vec{v}\,\frac{(L^3N_0M)F(\vec{v})}{y_1y_2}\left[\left(\frac{\hbar k^2}{\mu}\right)+\left(\frac{\Delta E_{\vec{v}+\hbar\vec{k}/\mu}+\Delta E_{\vec{v}-\hbar\vec{k}/\mu}}{\hbar}\right)-\frac{i}{2}\left(\gamma_{\vec{v}-\hbar\vec{k}/\mu}-\gamma_{\vec{v}+\hbar\vec{k}/\mu}\right)\right]\quad, \tag{50}$$

where

$$y_1=\omega-\vec{k}\cdot\vec{v}-\Delta E_{\vec{v},\vec{v}+\hbar\vec{k}/\mu}/\hbar-\tfrac{1}{2}i\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}-\hbar k^2/2\mu \tag{51}$$

and

$$y_2=\omega-\vec{k}\cdot\vec{v}-\Delta E_{\vec{v}-\hbar\vec{k}/\mu,\vec{v}}/\hbar-\tfrac{1}{2}i\gamma_{\vec{v},\vec{v}-\hbar\vec{k}/\mu}+\hbar k^2/2\mu\ . \tag{52}$$

Here we have made use of Eqs. (8) and (9). By Taylor-series expansion we get

$$\Delta E_{\vec{v}\pm\hbar\vec{k}/\mu}=[1\pm(\hbar/\mu)(\vec{k}\cdot\vec{\nabla}_v)+\tfrac{1}{2}(\hbar/\mu)^2(\vec{k}\cdot\vec{\nabla}_v)(\vec{k}\cdot\vec{\nabla}_v)+\cdots]\Delta E_{\vec{v}} \tag{53}$$

and

$$\gamma_{\vec{v}\pm\hbar\vec{k}/\mu}=[1\pm(\hbar/\mu)(\vec{k}\cdot\vec{\nabla}_v)+\tfrac{1}{2}(\hbar/\mu)^2(\vec{k}\cdot\vec{\nabla}_v)(\vec{k}\cdot\vec{\nabla}_v)+\cdots]\gamma_{\vec{v}}\ . \tag{54}$$

On making use of Eqs. (8), (9), (53), and (54), the approximate form of Eq. (50) may be written

$$D(\omega,\vec{k})\approx D_0-\frac{D_0}{\pi\omega}\int d\vec{v}\,\frac{(L^3N_0M)F(\vec{v})[\hbar k^2/\mu+2\Delta E_{\vec{v}}/\hbar+i(\hbar/\mu)(\vec{k}\cdot\vec{\nabla}_v)\gamma_{\vec{v}}]}{[\omega-\vec{k}\cdot\vec{v}-(1/\mu)(\vec{k}\cdot\vec{\nabla}_v)\Delta E_{\vec{v}}-i\gamma_{\vec{v}}]^2-(\hbar k^2/2\mu)^2}\ . \tag{55}$$

Here we have retained only the leading nonzero term in the Taylor-series expansions of Eqs. (53) and (54). This approximate result of Eq. (55) is of the Bohm–Pines form and is to be compared with the result of Eq. (23). However, the result of Eq. (49) is much more complete and is the full result in the Weisskopf–Wigner approximation. We should perhaps emphasize that $\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}\geq 0$ always,[37] regardless of the sign of Γ_k.

Let us now examine the quasilinear theory of the electron gas. In the Weisskopf–Wigner approximation, by making use of Eqs. (45)–(48) in Eqs. (17) and (18) one gets the coupled pair of quasilinear equations for a nondegenerate electron gas, while the use of Eqs. (45)–(48) in Eqs. (19) and (20) yields the coupled pair of quasilinear equations for a degenerate electron gas. Thus, in the Weisskopf–Wigner approximation the coupled pair of quasilinear Eqs. (17) and (18), for example, may be written

$$\frac{\partial N_k}{\partial t}=\int d\vec{v}\,L^3N_0M\left(\frac{\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}/2\pi}{[\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)-(\Delta E_{\vec{v},\vec{v}+\hbar\vec{k}/\mu}/\hbar)]^2+(\frac{1}{2}\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}})^2}[(N_k+1)F(\vec{v}+\hbar\vec{k}/\mu)-N_kF(\vec{v})]\right) \tag{56}$$

and

$$\frac{\partial F(\vec{v})}{\partial t}=\sum_{\vec{k}}\sum_{\vec{\epsilon}_k}M\left[\left(\frac{\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}}/2\pi}{[\omega-\vec{k}\cdot(\vec{v}+\hbar\vec{k}/2\mu)-(\Delta E_{\vec{v},\vec{v}+\hbar\vec{k}/\mu}/\hbar)]^2+(\frac{1}{2}\gamma_{\vec{v}+\hbar\vec{k}/\mu,\vec{v}})^2}\right)[(N_k+1)F(\vec{v}+\hbar\vec{k}/\mu)-N_kF(\vec{v})]\right.$$
$$\left.+\left(\frac{\gamma_{\vec{v}-\hbar\vec{k}/\mu,\vec{v}}/2\pi}{[\omega-\vec{k}\cdot(\vec{v}-\hbar\vec{k}/2\mu)-(\Delta E_{\vec{v}-\hbar\vec{k}/\mu,\vec{v}}/\hbar)]^2+(\frac{1}{2}\gamma_{\vec{v}-\hbar\vec{k}/\mu,\vec{v}})^2}\right)[N_kF(\vec{v}-\hbar\vec{k}/\mu)-(N_k+1)F(\vec{v})]\right]\ , \tag{57}$$

respectively. Similarly one can easily write down the Weisskopf–Wigner approximation of the coupled pair of quasilinear Eqs. (19) and (20) appropriate for the degenerate electron gas.

It is interesting and physically instructive to examine the quasiclassical limit of the coupled pair of quasilinear Eqs. (56) and (57). By quasiclassical limit we mean the limit in which $\hbar\to 0$ keeping the dimensionless fine-structure constant $\alpha=q^2/\hbar c$ finite. The approximate value of α is $\frac{1}{137}$. The fine-structure constant α is the electron-electromagnetic field coupling constant and the radiative corrections are generally obtained as a power-series expansion in α. If one blindly takes the classical limit by letting $\hbar\to 0$, then $\alpha\to\infty$ and hence all the radiative corrections will be infinite.

On making use of Eqs. (8), (9), (24), (53), and (54), we find, after a certain amount of algebra, that the quasiclassical limits of the coupled pair of Eqs. (56) and (57) become

$$\frac{\partial \mathcal{E}_k}{\partial t} \approx \int d\vec{v}\, L^3 N_0 M\hbar \left(\frac{\mathcal{E}_k}{\mu} \chi(\omega, \vec{k}, \vec{v}) \vec{k} \cdot \vec{\nabla}_v F(\vec{v}) + \omega \chi(\omega, \vec{k}, \vec{v}) F(\vec{v}) \right) \quad (56a)$$

and

$$\frac{\partial F(\vec{v})}{\partial t} \approx \sum_{\vec{k}} \sum_{\vec{\epsilon}_k} \frac{M\hbar}{\mu\omega} \left(\frac{\mathcal{E}_k}{\mu} \vec{k} \cdot \vec{\nabla}_v [\chi(\omega, \vec{k}, \vec{v}) \vec{k} \cdot \vec{\nabla}_v F(\vec{v})] + \omega \vec{k} \cdot \vec{\nabla}_v [\chi(\omega, \vec{k}, \vec{v}) F(\vec{v})] \right) \quad , \quad (57a)$$

respectively, where

$$\chi(\omega, \vec{k}, \vec{v}) \approx \frac{\gamma_{\vec{v}}/\pi}{[(\omega - \vec{k}\cdot\vec{v}) - (1/\mu)(\vec{k}\cdot\vec{\nabla}_v)\Delta E_{\vec{v}}]^2 + \gamma_{\vec{v}}^2} \quad . \quad (58)$$

Here we have retained only the leading nonzero term in the Taylor-series expansions of Eqs. (53) and (54). In this approximation, it is interesting to note that Eq. (57a) is a Fokker–Planck equation whose first and second terms represent a diffusion and dynamical friction, respectively. Here again one can examine the self-consistency of this coupled pair of Eqs. (56a) and (57a) for the conservation laws of energy and momentum. On making use of Eqs. (56a) and (57a), it is relatively easy to show that Eq. (28) is satisfied, and further

$$\sum_{\vec{k}} \sum_{\vec{\epsilon}_k} \frac{\partial \mathcal{E}_k}{\partial t} = - \int d\vec{v} \left[L^3 N_0 \left(\frac{\mu v^2}{2} + \Delta E_{\vec{v}} \right) \right] \frac{\partial F(\vec{v})}{\partial t} \quad , \quad (59)$$

since within the limits allowed by the Heisenberg uncertainty principle $\omega - \vec{k}\cdot\vec{v} - (1/\mu)(\vec{k}\cdot\vec{\nabla}_v)\Delta E_{\vec{v}} \approx 0$. To prove these results one simply has to integrate the right-hand sides of Eqs. (28) and (59) once by parts and make use of the fact that $F(\vec{v})$ vanishes at $\vec{v} = \pm\infty$. The left-hand sides of Eqs. (28) and (59) represent the average rate of gain of the wave momentum and the wave energy, respectively, and the right-hand sides of these equations represent the average rate of loss of the momentum and the energy (including the self-energy), respectively, of the $L^3 N_0$ electrons (both the resonant and the nonresonant electrons) inside the box of volume L^3.

The elements of the velocity-space diffusion tensor of Eq. (57a) may be written[38]

$$\mathfrak{D}_{ij} = \sum_{\vec{k}} \sum_{\vec{\epsilon}_k} \frac{k_i k_j M\hbar}{\mu^2 \omega} \times \mathcal{E}_k \left(\frac{\gamma_{\vec{v}}/\pi}{[(\omega - \vec{k}\cdot\vec{v}) - (1/\mu)(\vec{k}\cdot\vec{\nabla}_v)\Delta E_{\vec{v}}]^2 + \gamma_{\vec{v}}^2} \right) . \quad (60)$$

Thus, according to Weisskopf and Wigner, the result of Eq. (60) is the proper generalization of the Golden Rule result of Eq. (29). We see then that the width of the nonresonant or adiabatic electron-photon or electron-plasmon interaction is determined by $\gamma_{\vec{v}}$ (the reciprocal of the lifetime of the quantum state $|\vec{v}\rangle$). Additional sources of line broadening are discussed in the Appendix. It is also seen from Eq. (60) that the usual particle-wave resonance condition $\omega - \vec{k}\cdot\vec{v} \approx 0$ is modified by the Lamb shift to yield $\omega - \vec{k}\cdot\vec{v} - (1/\mu) \times (\vec{k}\cdot\vec{\nabla}_v)\Delta E_{\vec{v}} \approx 0$.

Let us now examine the order of magnitude of both the width of the nonresonant or adiabatic electron-plasmon or electron-photon interaction and the Lamb shift of the particle-wave resonance condition. From Eqs. (36) and (38) it is readily seen that the dimensionless width of the nonresonant interaction is given by[39]

$$\gamma_{\vec{v}}/\omega_0 \approx \alpha(c/vD_0)\ln(v/v_0) \quad \text{for } v > v_0$$
$$\approx 0 \qquad \text{for } v < v_0 \quad (61)$$

for electron-plasmon interaction, and

$$\gamma_{\vec{v}}/\bar{\omega} \approx \alpha(v/c) \quad (62)$$

for electron-photon interaction. The fine-structure constant $\alpha = q^2/\hbar c \approx \frac{1}{137}$. Since

$$\vec{k}\cdot\vec{\nabla}_v = \left(\vec{k}\cdot\frac{\vec{v}}{v}\right)\frac{\partial}{\partial v} \approx \frac{\omega}{v}\frac{\partial}{\partial v}$$

from Eqs. (41) and (43), we find that the Lamb shift of the particle-wave resonance condition is given by

$$-\frac{1}{\mu}(\vec{k}\cdot\vec{\nabla}_v)\Delta E_{\vec{v}} \approx -\frac{q^2\omega_0^2}{\pi\mu D_0 v^3}\left[1 - \ln\left(\frac{v}{v_0}\right)\right] \times \left\langle \ln\left(\frac{|\omega_0 + \hbar k^2/2\mu - kv|}{(\omega_0 + \hbar k^2/2\mu) + kv}\right)\right\rangle \quad (63)$$

for plasmon emission and absorption and

$$-\frac{1}{\mu}(\vec{k}\cdot\vec{\nabla}_v)\Delta E_{\vec{v}} \approx -\left(\frac{q^2\omega\bar{\omega}_1}{\pi\mu c^2 v}\right)\left[1 + \left(\frac{v}{\bar{\omega}_1}\right)\frac{\partial\bar{\omega}_1}{\partial v}\right] \quad (64)$$

for photon emission and absorption.

We have seen that $\gamma_{\vec{v}}$ is the total spontaneous emission probability from the quantum state $|\vec{v}\rangle$. Hence, according to the Heisenberg uncertainty principle, $1/\gamma_{\vec{v}}$ is the lifetime of the quantum state $|\vec{v}\rangle$. However, $1/\gamma_{\vec{v}}$ is not the radiative slowing-down time of the particle. Since the total rate of spontaneous emission of energy from this state is $\hbar\omega\gamma_{\vec{v}}$, we can define a radiative slowing-down time $\tau_{\vec{v}}$ for the particle of velocity $\vec{v}$ by the relation[40]

$$\hbar\omega\gamma_{\vec{v}} = \frac{\partial(\frac{1}{2}\mu\vec{v}^2)}{\partial t} = \mu\vec{v}\cdot\left(\frac{\partial\vec{v}}{\partial t}\right) \approx \mu\vec{v}\cdot\left(\frac{\vec{v}}{\tau_{\vec{v}}}\right) \quad ; \quad (65)$$

hence

$$1/\tau_{\vec{v}} \approx (\hbar\omega/\mu v^2)\gamma_{\vec{v}} \quad . \quad (66)$$

For most cases of practical interest $\hbar\omega/\mu v^2 \ll 1$ and hence $\tau_{\vec{v}} \gg \gamma_{\vec{v}}^{-1}$, that is, the radiative slowing-down time of the particle of velocity $\vec{v}$ is very much larger than the radiative lifetime of the quantum state $|\vec{v}\rangle$ of the particle. This, of course, is a necessary criterion for the validity of the entire analysis presented in this paper.

V. THEORETICAL LIMITATIONS AND EXPERIMENTAL IMPLICATIONS

For plasmons (longitudinal waves, $v_\phi \ll c$) the quantum perturbation theory is meaningful if and only if $(\gamma_{\vec{v}}/\omega_0)_{\max} \ll 1$; that is, if and only if $v_0/c \gg \alpha/2.7$, since it can be easily shown from Eq. (61) that the maximum value $(\gamma_{\vec{v}}/\omega_0)_{\max} \approx \alpha c/2.7 v_0$. This means that any result of quantum perturbation theory of plasmons (i.e., the Golden Rule approximation, the Weisskopf–Wigner approximation, and the entire Heitler–Ma expansion) is meaningful if and only if the temperature of the electron gas is very much greater than 3.7 eV. However, for photon emission and absorption (transverse waves, $v_\phi \approx c$) it is seen from Eq. (62) that the results of the quantum perturbation theory are meaningful for plasmas of any temperature. This intrinsic difference between the plasmon and the photon processes in an electron gas [for example, the difference in the results of Eqs. (61) and (62)] can easily be traced back to the corresponding difference in the square of the interaction matrix element M of Eq. (16). In an approximate sense, the dimensionless expansion parameter in quantum electrodynamics is the appropriate fine-structure constant $q^2/\hbar v_\phi = \alpha c/v_\phi$, and the quantum perturbation theory is meaningful if and only if this coupling constant is very much less than 1.

We go now to look at the experimental implications of this analysis. The broadening will affect the plasma dielectric function, the velocity-space diffusion, and the wave energy spectrum. We discuss these in turn.

With respect to the effect of $\gamma_{\vec{v}}$ on the dielectric function, we note from Eq. (61) that for the bulk of plasma electrons (i.e., $v \lesssim v_0$) $\gamma_{\vec{v}} \approx 0$. This means that $\mathrm{Re}D(\omega, \vec{k})$ is essentially the same both in the Golden Rule approximation and in the Weisskopf–Wigner approximation. If $\partial F/\partial v$ does not change appreciably in the velocity interval $(\omega - \gamma_{\vec{v}})/k \lesssim v \lesssim (\omega + \gamma_{\vec{v}})/k$, then $\mathrm{Im}D(\omega, \vec{k})$ is also essentially the same both in the Golden Rule approximation and in the Weisskopf–Wigner approximation. However, if $\partial F/\partial v$ does change appreciably in this velocity interval, then the result for $\mathrm{Im}D(\omega, \vec{k})$ in the Weisskopf–Wigner approximation will be different from that in the Golden Rule approximation. Since the damping or growth rate Γ_k of the plasma waves is $\Gamma_k = -(\omega/2D_0)\,\mathrm{Im}D(\omega, \vec{k})$, it appears that with suitably chosen distributions $F(v)$ one may be able to experimentally verify the predictions of the Weisskopf–Wigner theory of line broadening.

We observe here with regard to the choice, in Eq. (35), of $k_{\max} \approx \lambda_D^{-1}$, that this assumption is quite reasonable for the problem discussed in this paper, i.e., the approach to equilibrium of electrons and the thermally excited plasmons. As the system approaches thermodynamic equilibrium, by the equipartition theorem the energy $\mathcal{E}_k$ will approach the value $\kappa T = \mu v_0^2$ for every $k \lesssim k_{\max} \approx \omega_0/v_0$. According to Weisskopf and Wigner the approach to equilibrium is governed by the coupled pair of Eqs. (56a) and (57a). However, in a particular experiment where particle diffusion is important, the range of k may be more restricted. For example, let us now consider a collisionless beam-plasma system. This system is, of course, initially unstable, but if the beam-electron density is sufficiently small, it is a weak instability. The beam electrons will raise the level of plasma oscillations (i.e., the value of $\mathcal{E}_k$) for values of k in the restricted range $k_{\min} \approx \omega_0/v_D \lesssim k \lesssim k_{\max} \approx \omega_0/(v_D - v_B)$, where v_D is the beam velocity and v_B is the velocity spread of the beam. Thus, for the beam electrons, Eq. (34) yields

$$\begin{aligned}\gamma_B/\omega_0 &\approx (q^2/\hbar v_D D_0)\ln(k_{\max}/k_{\min}) \\ &\approx (q^2/\hbar v_D D_0)\ln[v_D/(v_D - v_B)] \\ &\approx (q^2/\hbar v_D D_0)(v_B/v_D) \quad \text{for } v_B \ll v_D \ . \qquad (67)\end{aligned}$$

The velocity-space diffusion of the beam electrons depends now on γ_B, $\mathfrak{D}_B \sim \sum_{\vec{k}} (\mathcal{E}_k \gamma_B/\pi)[(\omega - \vec{k}\cdot\vec{v})^2 + \gamma_B^2]^{-1}$, and the frequency width of the unstable wave spectrum due to radiation damping will also depend on γ_B,

$$\Delta\omega/\omega_0 \approx \gamma_B/\omega_0 \ . \qquad (68)$$

If the initially unstable beam-plasma system is long enough so that the unstable plasma waves can undergo a sufficient number of e-foldings, it will eventually reach the Drummond–Pines quasilinear steady state. At the quasilinear steady state $\partial F/\partial v \approx 0$ and $\partial(\mathfrak{D}\partial F/\partial v)/\partial v \approx 0$ in the range of initially unstable phase velocities. That is, in the quasilinear steady state the initially unstable waves are now neither growing nor damped and the beam electrons no longer diffuse in velocity space. Consequently, one would expect the dispersion broadening and the velocity-space diffusion broadening to vanish in the quasilinear steady state. Furthermore, if, in the quasilinear steady state, $\mathcal{E}_{k\approx\omega_0/v_D}$ is sufficiently small so that one could neglect the broadening due to particle trapping, then for a homogeneous system (a plasma in which the broadening due the spatial variations in the plasma frequency is zero) the frequency width of the wave spectrum at quasilinear steady state should be given by Eq. (68). Hence it appears that one should be able to experimentally verify the Weisskopf–Wigner theory by examining the frequency spectrum of waves for the quasilinear steady state of an initially weakly unstable beam-plasma system. For example, for the experimental conditions of Fig. 4 of Arunasalam, Heald, and Sinnis (Ref. 30), we find that $\gamma_B/\omega_0 \approx 5.2\%$. Hence the expected Weisskopf–Wigner full width is $2\gamma_B/\omega_0 \approx 10.4\%$. However, the experimentally measured fractional full width was only 5.5%. We do not understand this factor-of-2 discrepancy between the predictions of Weisskopf–Wigner theory of line broadening and the experimental results of Arunasalam, Heald, and Sinnis (Ref. 30).

ACKNOWLEDGMENTS

It is a pleasure to express my thanks to Professor T. H. Stix, Professor J. M. Dawson, and Professor M. N. Rosenbluth and to E. A. Williams for useful discussions.

APPENDIX

If we choose the z axis along the particle velocity $\vec{v}$, one can show from Eq. (29) (after somewhat lengthy algebra) that, in the Golden Rule approximation, the nonzero elements of the velocity-space diffusion tensor are given by

$$\mathfrak{D}_{zz} = (q^2\omega_0^2/\mu^2 D_0 v^3) I_{-1} \ ,$$
$$\mathfrak{D}_{xx} = \mathfrak{D}_{yy} = (q^2\omega_0^2/2\mu^2 D_0 v^3)[(v^2/\omega_0^2)I_1 - I_{-1}] \ , \tag{A1}$$

where

$$I_n = \int_{k_{\min}\approx\omega_0/v}^{k_{\max}\approx\omega_0/v_0} dk\, k^n \mathcal{E}_k \tag{A2}$$

for plasmons, and

$$\mathfrak{D}_{zz} = \int d\omega\, \frac{q^2\omega\mathcal{E}_\omega}{\mu^2 c^2 v}\left(1 - \frac{c^2}{D_0 v^2}\right) \ ,$$
$$\mathfrak{D}_{xx} = \mathfrak{D}_{yy} = \int d\omega\, \frac{q^2\omega\mathcal{E}_\omega}{2\mu^2 c^2 v}\,\frac{D_0 v^2}{c^2}\left(1 - \frac{c^2}{D_0 v^2}\right)^2 \tag{A3}$$

for photons. It is interesting to note that for plasmons $\mathfrak{D}_{zz}$ of Eq. (A1) will agree with the corresponding result found in Ref. 8 if we write $I_{-1} \approx \mathcal{E}_{k=\omega_0/v}$. From Eq. (60), one can show that the leading term of the contribution $\Delta_1\mathfrak{D}_{ij}$ to $\mathfrak{D}_{ij}$ of Eqs. (A1) and (A3) due to the nonresonant electrons (i.e., the contribution from $\gamma_{\vec{v}}^{-}$) is given by

$$\Delta_1\mathfrak{D}_{zz} = \Delta_1\mathfrak{D}_{xx} = \Delta_1\mathfrak{D}_{yy} \approx (2\gamma_{\vec{v}}^{-} q^2/3\pi\mu^2 D_0\omega_0^2) I_2 \tag{A4}$$

for plasmons [i.e., for Eqs. (A1)] and

$$2(\Delta_1\mathfrak{D}_{zz}) = \Delta_1\mathfrak{D}_{xx} = \Delta_1\mathfrak{D}_{yy} \approx \int d\omega\, \mathcal{E}_\omega\, \frac{8D_0^{3/2} q^2 v^2 \gamma_{\vec{v}}^{-}}{15\pi\mu^2 c^5} \tag{A5}$$

for photons [i.e., for Eqs. (A3)]. Similarly, one can show that the leading term of the Lamb-shift contribution $\Delta_2\mathfrak{D}_{ij}$ to the $\mathfrak{D}_{ij}$ of Eqs. (A1) and (A3) is given by

$$\Delta_2\mathfrak{D}_{ij} \approx \frac{1}{\mu}\left(\frac{\partial \Delta E_{\vec{v}}^{-}}{\partial v}\right)\left(\frac{\partial \mathfrak{D}_{ij}}{\partial v}\right) \ . \tag{A6}$$

Let us now make a rough estimate of the higher-order Heitler–Ma terms (that is, the terms corresponding to nonlinear mode coupling, nonlinear Landau damping, etc.). We shall do this only for the case of plasmons, and for this purpose we will assume that the Golden Rule approximation results are of order unity. The Heitler–Ma expansion corresponds to the multipole expansion of $e^{i\vec{k}\cdot\vec{r}_1} = e^{i\vec{k}\cdot\vec{v}_1/\omega}$, where $\vec{r}_1$, and $\vec{v}_1$ are the amplitudes of the position and velocity oscillations of the electron in the existing wave field. The dipole approximation $e^{i\vec{k}\cdot\vec{v}_1/\omega} \approx 1$ corresponds to the Golden Rule approximation. Thus the matrix element of the next Heitler–Ma term is of order $i\vec{k}\cdot\vec{v}_1/\omega$ and the corresponding transition probabilities are of order $|i\vec{k}\cdot\vec{v}_1/\omega|^2$. One can easily show that

$$\beta = \sum_{\vec{k}} \left| \frac{i\vec{k}\cdot\vec{v}_1}{\omega} \right|^2 \leq \sum_k \frac{\mathcal{E}_k}{L^3 N_0 \mu v_0^2} = \beta_{\max} \tag{A7}$$

for the case of plasmons, since $\omega \approx \omega_0 \geq k v_0$. Since I_2 is of order $k^3 I_{-1} \approx \omega_0^3 I_{-1}/v^3$, we find from Eqs. (A1) and (A4) that $\Delta_1\mathfrak{D}_{zz}/\mathfrak{D}_{zz}$ is of order $\gamma_{\vec{v}}^{-}/\omega_0$. Hence the radiative corrections will dominate over the higher-order Heitler–Ma terms if $\gamma_{\vec{v}}^{-}/\omega_0 > \beta_{\max}$. However, the[41] Golden Rule approximation (and, indeed, the entire perturbation-expansion procedure) is meaningful if and only if $\gamma_{\vec{v}}^{-}/\omega_0 < 1$.

Thus far nothing has been said about the effects of particle trapping. This is a highly nonlinear process and does not seem to fall within the framework of the Heitler–Ma formalism. The effect of particle trapping in the periodic potential of a single plasma wave is to replace the free-particle wave function by the appropriate Bloch wave function and, as a consequence, lead to a Stark splitting of the energy levels of the particle in the box.[42] For a stochastic distribution of plasma waves this will lead to a Stark broadening of the energy levels, and this broadening is of order $\beta^{1/2}$. Because of the Heisenberg uncertainty principle this will give rise to a broadening of the particle-wave resonance condition, and this is of order $\beta^{1/2}$.

The diffusion-dependent broadening found in Ref. 20 can to some extent be looked upon as an iterative solution of the coupled pair of nonlinear equations for a turbulent plasma. That is, having first arrived at the coupled pair of quasilinear equations in the Golden Rule approximation, one may assume that the "new oscillating test-charge density" $T(\vec{k}, \omega, \mathfrak{D})$ in a turbulent plasma is the space-time Fourier transform of the solution of the equation $[(\partial/\partial t) + \vec{v}\cdot\vec{\nabla}_r - k^{-2}\vec{k}\cdot\vec{\nabla}_v\mathfrak{D}\vec{k}\cdot\vec{\nabla}_v]T = 0$. The "new oscillating current density" $J(\vec{k}, \omega, \mathfrak{D})$ that is responsible for the emission and absorption in a turbulent plasma is proportional to $T(\vec{k}, \omega, \mathfrak{D})$. Thus the "new transition probabilities" j will be broadened in accordance with the results found in

Ref. 20, and the principle of detailed balance will then yield the "new coupled pair of rate equations." But, if one examines the solution of the energy balance equation (i.e., the equation for $\partial\mathcal{E}_k/\partial t$), one will find that as time progresses the wave spectrum will be broadened if the waves are damped and will narrow down if the waves are growing, since usually Γ_k is a peaked function of k and ω. This broadening or the narrowing of the wave spectrum does not affect the oscillating test-charge density directly, since the coupled pair of nonlinear equations are the rate equations at any given instant of time. However, in the sense of an ensemble average, this type of broadening or narrowing of the wave spectrum (for damped or growing waves, respectively) can affect the oscillating test-charge density via $\mathfrak{D}$, as found in Ref. 20. The oscillating test-charge density determines the spontaneous emission probability, which in turn determines the lifetime of the quantum states.

Finally, we may note that Fukai and Harris[43] have examined in detail a broadening that depends on Γ_k. The essential result of Fukai and Harris is the replacement of $\delta(\omega - \vec{k}\cdot\vec{v})$ by the factor $(|\Gamma_k|/\pi)[(\omega - \vec{k}\cdot\vec{v})^2 + \Gamma_k^2]^{-1}$. This result is of course consistent with the principle of causality. Stix[44] has shown that within the framework of the quasilinear approximation the most general result is obtained by replacing $\gamma_{\vec{v}}$ in Eq. (58), for example, by $\gamma_{\vec{v}} + |\Gamma_k|$. This result is physically reasonable since the most general broadening of the particle-wave resonance condition must depend on both $\gamma_{\vec{v}}$ (the reciprocal of the lifetime of the particle state $|\vec{v}\rangle$) and $|\Gamma_k|$ (the reciprocal of the lifetime of the wave, i.e., the reciprocal of the e-folding time of the wave) and since $\gamma_{\vec{v}} + |\Gamma_k|$ is the reciprocal of the lifetime of the particle-wave state $|N_k, \vec{v}\rangle$.

In the resonant approximation of the classical quasilinear theory one generally treats the case of real ω and the resonant diffusion $\mathfrak{D} \propto \sum_{\vec{k}} |E_k(t)|^2 \delta \times (\omega - \vec{k}\cdot\vec{v})$. However, if $E_k(t) = E_k e^{-i(\omega_k \pm i\Gamma_k)t}$, then

$$E_k(\omega) = (2\pi)^{-1/2} \int_{-T}^{T} E_k e^{-i(\omega_k \pm i\Gamma_k)t} e^{i\omega t}\, dt \quad ;$$

hence

$$E_k^*(\omega)E_k(\omega) = (E_k^2/2\pi)[(\omega - \omega_k)^2 + \Gamma_k^2]^{-1} e^{|2\Gamma_k T|} \quad ,$$

but

$$\langle |E_k(t)|^2 \rangle = \frac{E_k^2}{2T}\int_{-T}^{T} e^{|2\Gamma_k t|}\, dt = \frac{E_k^2}{|4\Gamma_k T|} e^{|2\Gamma_k T|} \quad .$$

Thus

$$E_k^*(\omega)E_k(\omega) = \frac{4T}{2\pi} \langle |E_k(t)|^2 \rangle |\Gamma_k| [(\omega - \omega_k)^2 + \Gamma_k^2]^{-1} \quad . \tag{A8}$$

In the nonresonant approximation $\omega = \omega_k \pm i\Gamma_k$, and $E_k(\omega)$ is a wave packet in ω for every k. Thus in this approximation the diffusion

$$\mathfrak{D} \propto \sum_{\vec{k}} \frac{1}{2T}\int_{-\infty}^{\infty} d\omega\, \frac{4T}{2\pi} \langle |E_k(t)|^2 \rangle \frac{|\Gamma_k|}{(\omega - \omega_k)^2 + \Gamma_k^2} \times \delta(\omega - \vec{k}\cdot\vec{v})$$

$$= \sum_{\vec{k}} \langle |E_k(t)|^2 \rangle \frac{|\Gamma_k|}{\pi} [(\omega_k - \vec{k}\cdot\vec{v})^2 + \Gamma_k^2]^{-1} \; . \tag{A9}$$

This derivation, due to Stix,[44] gives essentially the same result as that of Fukai and Harris. From Eqs. (58) and (A8) it follows that the diffusion due to the combined effect of natural broadening and dispersion broadening is

$$\mathfrak{D} \propto \sum_{\vec{k}} \langle |E_k(t)|^2 \rangle \int_{-\infty}^{\infty} d\omega\, \frac{|\Gamma_k|/\pi}{(\omega - \omega_k)^2 + \Gamma_k^2}\, \frac{\gamma_{\vec{v}}/\pi}{(\omega - \omega_1)^2 + \gamma_{\vec{v}}^2}$$

$$= \sum_{\vec{k}} \langle |E_k(t)|^2 \rangle \frac{\gamma_{\vec{v}} + |\Gamma_k|}{\pi} [(\omega_k - \omega_1)^2 + (\gamma_{\vec{v}} + |\Gamma_k|)^2]^{-1} , \tag{A10}$$

where $\omega_1 = \vec{k}\cdot\vec{v} + (1/\mu)(\vec{k}\cdot\vec{\nabla}_v)\Delta E_{\vec{v}}$. In carrying out the integration over ω in Eq. (A10) it is convenient to decompose the integrand by partial fractions and thus rewrite this integral as the sum of four integrals each of which has a simple pole.

In an electron gas of finite density an electron moving with a velocity $\vec{v}$ undergoes collisions with neighboring electrons. The effect of these collisions on the line breadth can be described as follows: If the number of effective collisions per second is $\nu_{\vec{v}}$ the lifetime of the state $|\vec{v}\rangle$ will be shortened. The total number of transitions per second (radiative plus collisions) is now equal to $\gamma_{\vec{v}} + \nu_{\vec{v}}$. According to the Heisenberg uncertainty relation the breadth of the level $|\vec{v}\rangle$ will therefore be $\delta E_{\vec{v}} = \hbar(\gamma_{\vec{v}} + \nu_{\vec{v}})$. The line emitted has the same intensity distribution as the natural line Eq. (58), the only[10,18,45] difference being that $\gamma_{\vec{v}}$ has to be replaced by $\gamma_{\vec{v}} + \nu_{\vec{v}}$.

*Work supported by U.S. Atomic Energy Commission under Contract No. AT(30-1)-1238.

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