Glauber. But, from a practical point of view, we must take into account the fact that $g_l(t)$ is *measurable* only for l of the order of a few units or at best of a few tens, and for a non-negligible intensity of radiation. With these restrictions we have found that large fluctuations will be reached only for m_0 of order smaller than l^2. This result means that the atomic system must be prepared in a state very near the excited state. For example, if the excitation can be described by an angle θ, m_0 is given by the relation

$$m_0 = \tfrac{1}{2} N (\cos\theta + 1) \ .$$

Obviously, in the limit of large N, m_0 will be a great number even if θ is very near π. For $\theta = \pi - \epsilon$, then $m_0 \approx N \tfrac{1}{4} \epsilon^2$. Therefore, unless the system is almost completely excited, it will exhibit in the present conditions of technology a classical behavior.

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Heating of a Plasma by Multiphoton Inverse Bremsstrahlung*

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The heating of plasma electrons in a laser beam by the inverse Bremsstrahlung process is considered from a quantum-mechanical viewpoint. A kinetic equation is derived and the change in kinetic energy of the electrons is calculated. Effective collision frequencies are found for the two cases of weak field and strong field and are compared to the classical results. It is found that the Coulomb logarithm, which appears in the classical expressions for the effective collision frequency, does not appear in the strong-field expression for effective collision frequency of the present work.

I. INTRODUCTION

The inverse Bremsstrahlung process is believed to play an important role in the breakdown and heating of a gas by a laser beam.[1] The exact nature of this process in intense laser beams has never been quite clear. From a classical viewpoint, an electron in an oscillatory electric field of frequency ω and intensity E_0 has time-average kinetic energy $e^2E_0^2/2m\omega^2$. If the electron makes a collision between the times of application and removal of the oscillatory field, then an energy of this order may be retained by the electron when the field is removed; otherwise, the energy is returned to the field. The rate of change of the kinetic energy of the electron is given by[2]

$$\frac{d\epsilon}{dt} = \frac{e^2E_0^2}{2m\omega^2}\, \nu_{\text{eff}} \ , \qquad (1)$$

where ν_{eff} is an effective collision frequency. Silin[3] has used a classical kinetic equation to find expressions for ν_{eff} for the two cases of weak and strong field. The Coulomb logarithm occurs in both expressions.

From a quantum-mechanical viewpoint, the electrons can gain energy only in units of $\hbar\omega$ and it is not clear that Eq. (1) and the classical argument which preceded it have any validity. However, Zel'Dovich and Raĭzer have shown[4] that Eq. (1) is valid when only one-photon processes are considered and $\epsilon \gg \hbar\omega$. One of the aims of this paper is to clarify the relationship between the classical and quantum-mechanical viewpoints.

To treat processes in which a large number of photons is absorbed or emitted by the usual perturbation theory requires a prohibitively high order of perturbation theory. However, when there is a large number of photons in the same state, then it is valid to treat the electromagnetic field classically. The electrons are treated quantum mechanically and nonrelativistically. The inverse Bremsstrahlung process is treated using first-order perturbation theory in a manner similar to other authors.[5,6] The unperturbed electron states are taken to be the solutions to the Schrödinger equation for an electron in the field of a classical electromagnetic wave. The field of a nucleus is treated as a perturbation. Transition probabilities are calculated between the unperturbed electron states.

We begin with a brief derivation of the transition probabilities. A kinetic equation for the electrons is then written using the transition probabilities.

After taking the classical limit, the rate of change of the kinetic energy of the electrons is calculated and comparison with the classical equation (1) is made. Expressions for effective collision frequency are found for the two cases of weak and strong field. The weak-field expression agrees exactly with Silin's[3] weak-field expression. The strong-field expression agrees closely with Silin's[3] strong-field expression except that the Coulomb logarithm does not occur in the present calculation. The Coulomb logarithm normally enters because it is necessary to cut off a divergent integral. We find that in our strong-field expression the integrals are convergent so that no cutoff is necessary.

II. TRANSITION PROBABILITIES

We assume a circularly polarized electromagnetic wave propagating in the z direction. The spacial dependence of the wave is neglected (dipole approximation). The vector potential is

$$\vec{A}(t) = A_0(\vec{e}_x \cos\omega t + \vec{e}_y \sin\omega t) \ . \tag{2}$$

The time-dependent Schrödinger equation

$$-\frac{\hbar}{i}\,\frac{\partial\psi(\vec{x},t)}{\partial t} = \frac{1}{2m}\left|\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}(t)\right|^2 \psi(\vec{x},t)$$

has the solution (normalized in a box of unit volume)

$$\psi(\vec{x},t) = \exp\left(\frac{i}{\hbar}\vec{p}\cdot\vec{x} - \frac{i}{2m\hbar}\int_0^t \left|\vec{p} - \frac{e}{c}\vec{A}(t')\right|^2 dt'\right) .$$

Using first-order perturbation theory and Eq. (2), the transition-probability amplitude for a transition from a state 1 with momentum $\vec{p}_1$ to a state 2 with momentum $\vec{p}_2$ is found to be

$$\begin{aligned} a(1\to 2) &= -\frac{i}{\hbar}\int\int_{-T/2}^{T/2} \psi_2^*(\vec{x},t)\, V(\vec{x})\, \psi_1(\vec{x},t)\, d^3x\, dt \\ &= -\frac{i}{\hbar}\, \overline{V}\left(\frac{\vec{p}_2 - \vec{p}_1}{\hbar}\right) \int_{-T/2}^{T/2} dt \\ &\qquad \times \exp\left[\frac{i}{\hbar}\left(\Omega t - \frac{\lambda}{\omega}\sin(\omega t - \delta)\right)\right] , \end{aligned}$$

where $V(\vec{x})$ is the potential of the perturbing nucleus, $\overline{V}[(\vec{p}_2 - \vec{p}_1)/\hbar]$ is the Fourier transform of $V(\vec{x})$, and

$$\Omega = (p_2^2 - p_1^2)/2m \ ,$$

$$\lambda = (eE_0/m\omega)\Delta p_\perp \ ,$$

$$\delta = \tan^{-1}(\Delta p_y/\Delta p_x) \ .$$

Here $\Delta\vec{p} = \vec{p}_2 - \vec{p}_1$ and the subscript $\perp$ refers to the direction perpendicular to the direction of propagation of the wave (the z direction). The transition probability per unit time is

$$\frac{|a(1\to 2)|^2}{T} = \frac{2\pi}{\hbar}\left|\overline{V}\left(\frac{\vec{p}_2 - \vec{p}_1}{\hbar}\right)\right|^2 \times \sum_{n=-\infty}^{\infty} J_n^2\left(\frac{\lambda}{\hbar\omega}\right)\delta(\Omega - n\hbar\omega) \ , \tag{3}$$

where J_n is the Bessel function of order n. Positive values for n correspond to the absorption of n photons and negative values to the emission of $|n|$ photons.

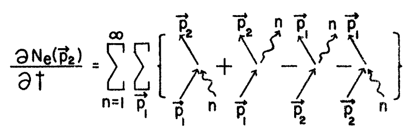
III. KINETIC EQUATION

From Eq. (3), we see that the transition probability per unit time for the transition from state 1 to state 2 with the absorption ($n > 0$) or emission ($n < 0$) of $|n|$ photons is

$$T(n, \vec{p}_1 - \vec{p}_2) = \frac{2\pi}{\hbar}\left|\overline{V}\left(\frac{\vec{p}_2 - \vec{p}_1}{\hbar}\right)\right|^2 J_n^2\left(\frac{\lambda}{\hbar\omega}\right)\delta(\Omega - n\hbar\omega) \ . \tag{4}$$

The δ function implies that the energy of the electron-photon system is conserved. The momentum of the electron-photon system is not conserved (the nucleus may carry off some momentum), and a sum over momentum will appear in the kinetic equation for the electrons.

The change in $N_e(\vec{p}_2)$, the number of electrons with momentum $\vec{p}_2$, may be written schematically as



(5)

The processes in which an electron with momentum $\vec{p}_2$ is destroyed are subtracted from the processes in which an electron with momentum $\vec{p}_2$ is created. This difference gives the increase in $N_e(\vec{p}_2)$. The schematic equation may be converted to a mathematical equation by replacing the diagrams in Eq. (5) by the transition probability per unit time for the processes given by Eq. (4). This mathematical equation is

$$\begin{aligned} \frac{\partial N_e(\vec{p}_2)}{\partial t} = \sum_{n=1}^{\infty}\sum_{\vec{p}_1} \{ & T(n, \vec{p}_1 - \vec{p}_2) N_e(\vec{p}_1)[1 - N_e(\vec{p}_2)] \\ & + T(-n, \vec{p}_1 - \vec{p}_2) N_e(\vec{p}_1)[1 - N_e(\vec{p}_2)] \\ & - T(-n, \vec{p}_2 - \vec{p}_1) N_e(\vec{p}_2)[1 - N_e(\vec{p}_1)] \\ & - T(n, \vec{p}_2 - \vec{p}_1) N_e(\vec{p}_2)[1 - N_e(\vec{p}_1)]\} \ . \end{aligned} \tag{6}$$

In Eq. (6), $[1 - N_e(\vec{p})]$ is the square of the matrix element of the fermion creation operator and $N_e(\vec{p})$ is the square of the matrix element of the fermion destruction operator. These factors appear in the transition probabilities when the electrons are treated using second quantized theory rather than the first quantized theory used in Sec. II.

We assume that the electrons are far from degeneracy so that $N_e(\vec{p}) \ll 1$. We take $V(\vec{x})$ to be the Coulomb potential. Then

$$\left|\bar{V}\left(\frac{\vec{p}_2-\vec{p}_1}{\hbar}\right)\right|^2=\left(\frac{4\pi Ze^2\hbar^2}{|\vec{p}_2-\vec{p}_1|^2}\right)^2, \tag{7}$$

and it is readily shown that $T(n, \vec{p}_1-\vec{p}_2)=T(-n, \vec{p}_2-\vec{p}_1)$. Equation (6) becomes

$$\frac{\partial N_e(\vec{p}_2)}{\partial t}=\sum_{\substack{n=-\infty\\ n\neq 0}}\sum_{\vec{p}_1} T(n,\vec{p}_1-\vec{p}_2)[N_e(\vec{p}_1)-N_e(\vec{p}_2)]\,. \tag{8}$$

We let the volume of the box in which the system is normalized become infinite so that the sum over $\vec{p}_1$ becomes an integral. A Maxwell distribution is assumed for the electrons. Using Eqs. (4) and (7), Eq. (8) becomes

$$\frac{\partial f_e(\vec{v}_2)}{\partial t}=\frac{4Z^2e^4N}{m}\left(\frac{m}{2\pi KT}\right)^{3/2}\int d^3v_1 \times\frac{e^{-mv_1^2/2kT}-e^{-mv_2^2/2kT}}{|\vec{v}_2-\vec{v}_1|^4} \times\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} J_n^2\left(\frac{\lambda}{\hbar\omega}\right)\delta(\Omega-n\hbar\omega)\,, \tag{9}$$

where $f_e(\vec{v})$ is the electron distribution function and N is the electron density. Equation (9) is the kinetic equation for the electrons.

IV. EFFECTIVE COLLISION FREQUENCY

Expressions for the effective collision frequency will now be found for the two cases of weak and strong field.

For the weak-field case, $\lambda\ll\hbar\omega$ and the square of the Bessel function is approximately

$$J_n^2(\lambda/\hbar\omega)\cong[1/(n!)^2](\lambda/2\hbar\omega)^{2|n|}\,. \tag{10}$$

Using Eq. (10) and retaining only the first two terms of the sum, Eq. (9) becomes

$$\frac{\partial f_e(\vec{v}_2)}{\partial t}=\frac{4Z^2e^4N}{m}\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-mv_2^2/2kT} \times\int d^3v_1\,\frac{(\lambda/2\hbar\omega)^2}{|\vec{v}_2-\vec{v}_1|^4}[(e^{-\hbar\omega/kT}-1)\delta(\Omega+\hbar\omega) +(e^{\hbar\omega/kT}-1)\delta(\Omega-\hbar\omega)]\,. \tag{11}$$

It is apparent from Eq. (11) that only single-photon processes are significant for the weak-field case.

Upon taking the classical limit $\hbar\to 0$, Eq. (11) becomes

$$\frac{\partial f_e(\vec{v}_2)}{\partial t}=\frac{8Z^2e^4N}{m^2}\left(\frac{m}{2\pi kT}\right)^{3/2}e^{-mv_2^2/2kT}\left(\frac{eE_0}{2\omega kT}\right)^2 \times\int d\phi_1\sin\theta_1\,d\theta_1\,v_1^2dv_1\,\frac{(\Delta v_\perp)^2}{|\vec{v}_2-\vec{v}_1|^4}\,\delta(v_2^2-v_1^2)\,. \tag{12}$$

Since $v_1=v_2$, we may write $|\vec{v}_2-\vec{v}_1|=2v_1\sin\frac{1}{2}\theta$, where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$ (the scattering angle). The major contribution to the integral over θ_1 comes from small scattering angles. We therefore assume θ is a small constant angle and take it outside the integral over θ_1. The Z_1 axis is chosen to be along the direction of propagation of the electromagnetic waves so that $\Delta v_\perp=|\vec{v}_2-\vec{v}_1|\sin\theta_1$. After evaluation of the integrals, Eq. (12) becomes

$$\frac{\partial f_e(\vec{v}_2)}{\partial t}=\frac{8\pi Z^2e^4N}{3m^2}\left(\frac{m}{2\pi kT}\right)^{3/2}\frac{e^{-mv_2^2/2kT}}{v_2} \times\left(\frac{eE_0}{2\omega kT}\right)^2\frac{1}{(\sin\frac{1}{2}\theta)^2}\,. \tag{13}$$

The change in average kinetic energy of the electrons is

$$\frac{d\langle\epsilon\rangle}{dt}=\int d^3v_2\,\frac{mv_2^2}{2}\,\frac{\partial f_e(\vec{v}_2)}{\partial t}\,. \tag{14}$$

Substituting Eq. (13) into Eq. (14) and choosing the Z_2 axis so that $\theta_2=\theta$, we find that

$$\frac{d\langle\epsilon\rangle}{dt}=\frac{e^2E_0^2}{2m\omega^2}\,\frac{4}{3}\,\frac{(2\pi)^{1/2}Z^2e^4N}{m^2v_T^3}L\,, \tag{15}$$

where $v_T\equiv(kT/m)^{1/2}$ is the thermal velocity of the electrons and L is the Coulomb logarithm defined by

$$L=\int_{\theta_{\min}}^{\pi}\cot\tfrac{1}{2}\theta\,d\theta\,. \tag{15a}$$

The angle $\theta_{\min}$ is the lower cutoff of the scattering angle, usually taken to correspond to an impact parameter equal to the Debye length.

Comparing Eq. (15) to Eq. (1), the effective collision frequency for the weak-field case is found to be

$$\nu_{\text{eff}}=\tfrac{4}{3}[(2\pi)^{1/2}Z^2e^4NL/m^2v_T^3]\,. \tag{16}$$

Equation (16) agrees exactly with Silin's expression[3] for the effective collision frequency for the weak-field case.

For the strong-field case, $\lambda\gg\hbar\omega$ and the argument of the Bessel function of Eq. (9) is large. Let us consider the expression

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} J_n^2\left(\frac{\lambda}{\hbar\omega}\right)\delta(\Omega-n\hbar\omega)=J_{\Omega/\hbar\omega}^2\left(\frac{\lambda}{\hbar\omega}\right)\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\delta(\Omega-n\hbar\omega) \tag{17}$$

as a function of Ω. For $\lambda\gg\hbar\omega$, $J_{\Omega/\hbar\omega}^2(\lambda/\hbar\omega)$ is a function which has maximum values near $\Omega=\pm\lambda$. We write approximately

$$J_{\Omega/\hbar\omega}^2\left(\frac{\lambda}{\hbar\omega}\right)\sum_{\substack{n=-\infty\\ n\neq\infty}}^{\infty}\delta(\Omega-n\hbar\omega)\cong\tfrac{1}{2}[\delta(\Omega-\lambda)+\delta(\Omega+\lambda)]\,. \tag{18}$$

The factor $\frac{1}{2}$ in Eq. (18) may be verified by integrating both sides of the equation over Ω.

Substituting Eq. (18) into Eq. (9), we have

$$\frac{\partial f_e(\vec{v}_2)}{\partial t} = \frac{2Z^2e^4N}{m}\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv_2^2/2kT} \times \int d^3v_1 \frac{1}{|\vec{v}_2-\vec{v}_1|^4}\left[(e^{\lambda/kT}-1)\,\delta(\Omega-\lambda) + (e^{-\lambda/kT}-1)\,\delta(\Omega+\lambda)\right] . \quad (19)$$

The first term in the brackets in Eq. (19) corresponds to the absorption of $s \equiv (eE_0\Delta v_\perp/\hbar\omega^2)$ photons and the second term to the emission of s photons. It is apparent from Eq. (4) that $T(n, \vec{p}_1-\vec{p}_2)=0$ for $\Delta p_\perp = 0$. Most photons are absorbed or emitted nearly perpendicular to the direction of propagation of the electromagnetic wave. Using $\frac{1}{2}m(\Delta v_\perp)^2 \cong s\hbar\omega$, we find that

$$s \cong \left(\frac{2e^2E_0^2}{m\omega^2}\right)\bigg/\hbar\omega .$$

For the strong-field case, multiphoton absorption and emission are dominant over single-photon processes, and the number s of photons absorbed or emitted is the same order of magnitude as the ratio of the classical oscillatory energy of the electrons to the photon energy.

We now assume the temperature to be low so that $\lambda \gg kT$. Then Eq. (19) becomes

$$\frac{\partial f_e(\vec{v}_2)}{\partial t} = \frac{2Z^2e^4N}{m}\left(\frac{m}{2\pi kT}\right)^{3/2} \times \int d^3v_1 \frac{e^{-mv_1^2/2kT}}{|\vec{v}_2-\vec{v}_1|^4}\,\delta(\Omega-\lambda) . \quad (20)$$

For low temperature, the contribution to the kinetic equation of processes in which protons are emitted is negligible compared to the contribution of processes in which photons are absorbed.

From the δ function of Eq. (20), we have $v_2 \gg v_1$ for the strong-field case. Then, $\Delta v_\perp \cong v_{2\perp}$ and $|\vec{v}_2-\vec{v}_1| \cong v_2$. Equation (20) becomes

$$\frac{\partial f_e(\vec{v}_2)}{\partial t} = \frac{2Z^2e^4N}{m}\left[\delta\left(\frac{mv_2^2}{2}-\frac{eE_0v_{2\perp}}{\omega}\right)\bigg/v_2^4\right] . \quad (21)$$

Substituting Eq. (21) into Eq. (14), we again find Eq. (1) with the effective collision frequency given by

$$\nu_{\text{eff}} = 8\pi^2NZ^2em\omega^3/E_0^3 . \quad (22)$$

This collision frequency is the same as that found by Silin [derived from Eq. (5-6) of Ref. 3] except that Silin's collision frequency contains an additional factor of L given by Eq. (15a).

The relative magnitudes of the electron velocities before and after scattering determine the occurrence of the Coulomb logarithm in our expressions for effective collision frequency. The integrand of the kinetic equation (9) contains a factor $|\vec{v}_2-\vec{v}_1|^{-4}$. In the weak-field case, the scattered electron absorbs or emits a single photon. The electron's speed is not altered significantly $(v_1 \cong v_2)$ and therefore

$$|\vec{v}_2-\vec{v}_1|^{-4} \cong (2v_1\sin\tfrac{1}{2}\theta)^{-4} ,$$

where θ is the scattering angle. The integral over θ is divergent for small scattering angles and the Coulomb-logarithm results. In the strong-field case, the scattered electron absorbs a large number of photons. The electron's speed is altered significantly $(v_1 \ll v_2)$ and therefore $|\vec{v}_2-\vec{v}_1|^{-4} \cong v_2^{-4}$. The integral over θ is convergent and the Coulomb logarithm does not occur.

It follows from the above analysis that the heating of electrons by multiphoton processes may be important if $2e^2E_0^2/m\omega^2 \gg \hbar\omega$. For a neodymium laser, this corresponds to a laser intensity of $I \gg 10^{12}$ W/cm^2. Equation (22) indicates that the rate of energy absorption by the multiphoton inverse Bremsstrahlung process is proportional to $I^{-1/2}$. This result agrees with an analysis by Bethe[7] in which the Rutherford cross section is used to calculate the energy absorbed by an electron from a strong electromagnetic field. It has been proposed that plasma be heated to thermonuclear temperature by the rapid absorption of electromagnetic energy from a laser beam. In a very strong electromagnetic field, the rate of energy absorption by the multiphoton inverse Bremsstrahlung process is slow and other processes, such as the excitation of collective instabilities, are required for rapid energy absorption. However, collective instabilities do not develop until at least several picoseconds[8] after initiation of the strong electromagnetic field. For intense laser beams of picosecond duration or less, collective instabilities do not develop during the period of the laser pulse, and the multiphoton inverse Bremsstrahlung process may be the dominant heating mechanism.

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