Exact analytical scattering lengths for a class of long-range potentials with r^{-4} asymptotics

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Exact analytical expressions for scattering lengths for potentials with four-parameter long-range tails of the form $-Ar^{-2}(r+\rho)^{-2}-Br^{-\nu}(r+\rho)^{\nu-4}$ are presented. Contributions due to a potential core, with the latter not specified explicitly, are taken into account through a short-range scattering length and a core radius. For $\nu\neq 2$ and $B\neq 0$ the derived expressions contain the Bessel functions; for $\nu=2$ or $B=0$ they contain elementary functions.

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I. INTRODUCTION

When a charged particle collides with a neutral polarizable target, at low energies their interaction may be satisfactorily described by using suitably chosen long-range central model potentials which for large separation distances r are attractive and fall as r^{-4}. Such model potentials are called *polarization potentials*. At very low energies, the most important parameter characterizing the collision process is a scattering length. Exact or approximate analytical expressions for scattering lengths are available for several polarization potentials [1–16].

It is the purpose of this Brief Report to show that exact analytical expressions for the scattering length may be found for a four-parameter polarization potential which beyond some radial distance (core radius) r_s is of the form

$$V(r) = -\frac{A}{r^2(r+\rho)^2} - \frac{B}{r^{\nu}(r+\rho)^{4-\nu}} \quad (r \geqslant r_s), \tag{1.1}$$

with A, B, ν real and ρ real non-negative. In our considerations, we shall not specify the explicit form of the potential core $V(r)H(r_s-r)$, where $H(r_s-r)$ is the Heaviside unit step function. We shall be assuming only that the core is nonabsorptive and that a due short-range scattering length a_s is known.

II. METHOD

For any polarization potential $V(r)$ a physically acceptable solution to the zero-energy s-wave radial Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2 f(r)}{dr^2} + V(r)f(r) = 0 \tag{2.1}$$

has the asymptotic form

$$f(r) \overset{r\to\infty}{\sim} \text{const} \times [r-a], \tag{2.2}$$

where a is the scattering length [17]. It is evident from Eq. (2.2) that

$$a = \lim_{r\to\infty} a(r), \tag{2.3}$$

where the function $a(r)$ is defined in terms of the logarithmic derivative of $f(r)$,

$$L(r) = \frac{f'(r)}{f(r)} \tag{2.4}$$

(here and hereafter the prime at a function denotes its derivative with respect to an argument), as

$$a(r) = r - \frac{1}{L(r)}. \tag{2.5}$$

Assume now that the polarization potential $V(r)$ is such that for $r \geqslant r_s$ two linearly independent particular solutions $f_1(r)$ and $f_2(r)$ to the Schrödinger equation (2.1) are known. Then it holds that

$$f(r) = c_1 f_1(r) + c_2 f_2(r) \quad (r \geqslant r_s), \tag{2.6}$$

where c_1 and c_2 are constants, and consequently

$$L(r) = \frac{f_1'(r) + b f_2'(r)}{f_1(r) + b f_2(r)} \quad (r \geqslant r_s), \tag{2.7}$$

where $b=c_2/c_1$. Setting in Eq. (2.7) $r=r_s$ and solving for b yields

$$b = -\frac{f_1'(r_s) - L(r_s)f_1(r_s)}{f_2'(r_s) - L(r_s)f_2(r_s)}. \tag{2.8}$$

Since $L(r_s)$ is related to a short-range scattering length $a_s \equiv a(r_s)$ through

$$a_s = r_s - \frac{1}{L(r_s)}, \tag{2.9}$$

in terms of a_s Eq. (2.8) reads

$$b = -\frac{(r_s-a_s)f_1'(r_s) - f_1(r_s)}{(r_s-a_s)f_2'(r_s) - f_2(r_s)}. \tag{2.10}$$

Combining Eqs. (2.5), (2.7), and (2.10), leads to

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$$a(r) = r - \frac{(r_s - a_s)[f_1(r)f_2'(r_s) - f_2(r)f_1'(r_s)] - [f_1(r)f_2(r_s) - f_2(r)f_1(r_s)]}{(r_s - a_s)[f_1'(r)f_2'(r_s) - f_2'(r)f_1'(r_s)] - [f_1'(r)f_2(r_s) - f_2'(r)f_1(r_s)]} \quad (r \geqslant r_s). \tag{2.11}$$

Once explicit forms of the particular solutions $f_1(r)$ and $f_2(r)$ are known, making the limiting passage $r \to \infty$ in Eq. (2.11) yields [cf. Eq. (2.3)] the scattering length a for the polarization potential $V(r)$, expressed in terms of the core radius r_s and the short-range scattering length a_s.

III. SCATTERING LENGTHS FOR POTENTIALS WITH THE TAILS (1.1)

If we define

$$\alpha = \frac{2m}{\hbar^2\rho^2}A, \quad \beta = \frac{2m}{\hbar^2\rho^2}B, \tag{3.1}$$

in the region $r \geqslant r_s$ the zero-energy radial Schrödinger equation (2.1) with the potential (1.1) may be written as

$$\frac{d^2 f(r)}{dr^2} + \frac{\alpha\rho^2}{r^2(r+\rho)^2}f(r) + \frac{\beta\rho^2}{r^\nu(r+\rho)^{4-\nu}}f(r) = 0 \quad (r \geqslant r_s). \tag{3.2}$$

The following simultaneous transformations of the variable and function,

$$\xi = \frac{r}{r+\rho}, \tag{3.3}$$

$$f(r) = rF(\xi), \tag{3.4}$$

convert Eq. (3.2) into

$$\xi^2 \frac{d^2F(\xi)}{d\xi^2} + 2\xi\frac{dF(\xi)}{d\xi} + [\alpha + \beta\xi^{2-\nu}]F(\xi) = 0. \tag{3.5}$$

Below we shall exploit the fact that solutions to Eq. (3.5) are known in analytical forms [18,19].

A. Case $B \neq 0$ and $\nu \neq 2$

When the constraints $\beta \neq 0$ (i.e., $B \neq 0$) and $\nu \neq 2$ are satisfied simultaneously, two linearly independent solutions to Eq. (3.5) may be expressed in terms of the Bessel and Neumann functions in the following way:

$$F_1(\xi) = \xi^{-1/2}J_\mu(\zeta), \quad F_2(\xi) = \xi^{-1/2}Y_\mu(\zeta), \tag{3.6}$$

where

$$\zeta = \zeta_\infty \xi^{1-\nu/2}, \tag{3.7}$$

with

$$\zeta_\infty = \frac{2\beta^{1/2}}{|\nu - 2|} \tag{3.8}$$

and

$$\mu = \frac{2\eta}{|\nu - 2|}, \tag{3.9}$$

with

$$\eta = \left(\frac{1}{4} - \alpha\right)^{1/2}. \tag{3.10}$$

Consequently, if in Eq. (2.11) we choose

$$f_1(r) = r\xi^{-1/2}J_\mu(\zeta), \quad f_2(r) = r\xi^{-1/2}Y_\mu(\zeta), \tag{3.11}$$

exploit the recurrence relation

$$\zeta Z_\mu'(\zeta) + \mu Z_\mu(\zeta) = \zeta Z_{\mu-1}(\zeta) \tag{3.12}$$

obeyed both by $J_\mu(\zeta)$ and $Y_\mu(\zeta)$, and pass subsequently to the limit $r \to \infty$, we obtain

$$a = \rho\left[\operatorname{sgn}(\nu-2)(\eta - \beta^{1/2}D) - \frac{1}{2}\right], \tag{3.13}$$

where

$$D = \frac{J_{\mu-1}(\zeta_\infty) + bY_{\mu-1}(\zeta_\infty)}{J_\mu(\zeta_\infty) + bY_\mu(\zeta_\infty)}, \tag{3.14}$$

with

$$b = -\frac{\{[\mu(\nu-2)-1]\rho(r_s - a_s) - 2a_s(r_s+\rho)\}J_\mu(\zeta_s) - (\nu-2)\rho(r_s - a_s)\zeta_s J_{\mu-1}(\zeta_s)}{\{[\mu(\nu-2)-1]\rho(r_s - a_s) - 2a_s(r_s+\rho)\}Y_\mu(\zeta_s) - (\nu-2)\rho(r_s - a_s)\zeta_s Y_{\mu-1}(\zeta_s)}, \tag{3.15}$$

$$\zeta_s = \zeta_\infty \xi_s^{1-\nu/2}, \quad \xi_s = \frac{r_s}{r_s+\rho}. \tag{3.16}$$

The result (3.13)–(3.16) simplifies considerably when α and ν are such that $\mu = 1/2$. Then it holds

$$J_{\pm 1/2}(\zeta) = \left(\frac{2}{\pi\zeta}\right)^{1/2} \times \begin{cases} \sin\zeta, \\ \cos\zeta, \end{cases} \tag{3.17}$$

$$Y_{\pm 1/2}(\zeta) = \left(\frac{2}{\pi\zeta}\right)^{1/2} \times \begin{cases} -\cos\zeta, \\ \sin\zeta, \end{cases} \tag{3.18}$$

and Eq. (3.13) reduces to

$$a = \frac{1}{4}\rho[(\nu-4) - 4\,\mathrm{sgn}(\nu-2)\beta^{1/2}\cot(\zeta_\infty - \zeta_s + \phi)], \tag{3.19}$$

where

$$\phi = \arctan\frac{2(\nu-2)\zeta_s\rho(r_s-a_s)}{\nu\rho(r_s-a_s) - 4r_s(a_s+\rho)}. \tag{3.20}$$

In particular, the condition $\mu=1/2$ is satisfied when $\alpha=0$ (i.e., $A=0$) and $\nu=4$; i.e., when the potential tail (1.1) is the inverse-fourth-power potential:

$$V(r) = -\frac{B}{r^4} \quad (r \geqslant r_s). \tag{3.21}$$

In this case Eq. (3.19) simplifies to

$$a = \rho\beta^{1/2}\frac{a_s r_s - \beta^{1/2}\rho(r_s-a_s)\tan(\beta^{1/2}\rho/r_s)}{\beta^{1/2}\rho(r_s-a_s) + a_s r_s\tan(\beta^{1/2}\rho/r_s)}, \tag{3.22}$$

which agrees with our earlier findings [6,7,11].

B. Cases $B=0$ and $\nu=2$

It remains to consider the case $B=0$ and the case $\nu=2$. Since it is evident from Eq. (1.1) that these two cases are essentially equivalent, below we shall consider in details only the case $B=0$.

For $B=0$ we have $\beta=0$ and Eq. (3.5) becomes

$$\xi^2\frac{d^2F(\xi)}{d\xi^2} + 2\xi\frac{dF(\xi)}{d\xi} + \alpha F(\xi) = 0. \tag{3.23}$$

This is the Euler equation and for $\alpha \neq 1/4$ its two linearly independent solutions may be chosen as

$$F_1(\xi) = \xi^{-1/2}\sinh(\eta\ln\xi), \tag{3.24}$$

$$F_2(\xi) = \xi^{-1/2}\cosh(\eta\ln\xi), \tag{3.25}$$

with η defined in Eq. (3.10). Hence, we have

$$f_1(r) = r\xi^{-1/2}\sinh(\eta\ln\xi), \tag{3.26}$$

$$f_2(r) = r\xi^{-1/2}\cosh(\eta\ln\xi), \tag{3.27}$$

and substitution of Eqs. (3.26) and (3.27) into Eq. (2.11), followed by performing the limiting passage $r\to\infty$, yields the scattering length in the form

$$a = \frac{1}{2}\rho\frac{4\eta a_s(r_s+\rho) + [(2a_s r_s + a_s\rho + r_s\rho) - 4\eta^2\rho(r_s-a_s)]\tanh(\eta\ln\xi_s)}{2\eta\rho(r_s-a_s) - (2a_s r_s + a_s\rho + r_s\rho)\tanh(\eta\ln\xi_s)}. \tag{3.28}$$

Formula (3.28) is valid independently of whether η is real or imaginary. Still, for η imaginary, which corresponds to $\alpha > 1/4$, it is suitable to rewrite Eq. (3.28) as

$$a = \frac{1}{2}\rho\frac{4|\eta|a_s(r_s+\rho) + [(2a_s r_s + a_s\rho + r_s\rho) + 4|\eta|^2\rho(r_s-a_s)]\tan(|\eta|\ln\xi_s)}{2|\eta|\rho(r_s-a_s) - (2a_s r_s + a_s\rho + r_s\rho)\tan(|\eta|\ln\xi_s)}. \tag{3.29}$$

The scattering length for the case $\alpha=1/4$, which corresponds to $\eta=0$, may be found either by observing that then linearly independent solutions to Eq. (3.23) are

$$F_1(\xi) = \xi^{-1/2}, \quad F_2(\xi) = \xi^{-1/2}\ln\xi, \tag{3.30}$$

or by making the limiting passage $\eta\to 0$ in Eq. (3.28). Whatever procedure is adopted, at its end one arrives at the following expression for the scattering length:

$$a = \frac{1}{2}\rho\frac{4a_s(r_s+\rho) + (2a_s r_s + a_s\rho + r_s\rho)\ln\xi_s}{2\rho(r_s-a_s) - (2a_s r_s + a_s\rho + r_s\rho)\ln\xi_s}. \tag{3.31}$$

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