Representation of the SO(3) group by a maximally entangled state

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(Received 12 March 2004; published 11 June 2004)

A representation of the SO(3) group is mapped into a maximally entangled two qubit state according to the literature. To show the evolution of the entangled state, a model is set up on an maximally entangled electron pair, two electrons of which pass independently through a rotating magnetic field. It is found that the evolution path of the entangled state in the SO(3) sphere breaks an odd or even number of times, corresponding to the double connectedness of the SO(3) group. An odd number of breaks leads to an additional π phase to the entangled state, but an even number of breaks does not. A scheme to trace the evolution of the entangled state is proposed by means of entangled photon pairs and Kerr medium, allowing observation of the additional π phase.

DOI: 10.1103/PhysRevA.69.064301 PACS number(s): 03.67.Mn, 03.65.Vf

It is well known that when the spin of a spin-$\frac{1}{2}$ particle rotates for a whole cycle on the Bloch sphere the wave-function of the particle changes a phase of π. This π phase has been observed in several experiments [1,2]. This property is commonly attributed to the topological property, i.e., the double connectedness of the SO(3) group. The path on the manifold of the SO(3) group is categorized into two classes under a continuous deformation, one of which leads to a change of π in phase to the wave function, the other of which does not. However, it was argued by Milman and Mosseri [3] that this π phase may be shared by the multi-connectedness of both SO(3) and SO(2) groups. They also argued that, in general, the π phase is partly geometric and partly dynamic. Only in the extreme case that the spin precesses on the xy plane in the Bloch sphere is the π phase fully geometric. Especially, the π phase still exists when the spin initially points to the same direction of the magnetic field, where there is no rotation at all. In such case the π phase is fully dynamic. Therefore, this π phase may not be directly related to the SO(3) group.

Milman and Mosseri found a one-to-one correspondence between the representation of the SO(3) group and the evolution of a maximally entangled state of a two-qubit system (MES) [3]. They adopted a discontinuously changing magnetic field, which suddenly jumps from one direction to another. In the present paper a rotating magnetic field is used to drive the evolution of a MES. A clearer formalism is presented for the trajectory in SO(3).

A MES finds great application to quantum communication and quantum computation techniques, and also to the study of fundamental problems, e.g., nonlocality, of quantum mechanics [4–6]. Much attention has been paid to MESs in recent years. It is interesting that MES can be applied to the representation of the SO(3) group.

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I. MAPPING BETWEEN A MES AND SO(3)

A two-qubit maximally entangled state (MES) of a two-state system can be written as [3]

$$|(\alpha,\beta)\rangle = \frac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|01\rangle - \beta^*|10\rangle + \alpha^*|11\rangle), \qquad (1)$$

where the coefficients α and β are normalized to unity:

$$\alpha\alpha^* + \beta\beta^* = 1. \qquad (2)$$

It is seen that a MES is defined by a pair of complex numbers (α,β). To visualize a MES, α and β can be parametrized to

$$\alpha = \cos\frac{a}{2} - ik_z \sin\frac{a}{2}, \qquad (3)$$

$$\beta = -(k_y + ik_x)\sin\frac{a}{2}, \qquad (4)$$

where $(k_x,k_y,k_z)=\mathbf{k}$ is a unit vector, and a is an angle between 0 and π. Hence a MES can also be written as $|\Psi(\mathbf{k},a)\rangle$ in the parameter space. It is easy to check that $|\Psi(\mathbf{k},\pi+a)\rangle = -|\Psi(-\mathbf{k},\pi-a)\rangle$. That is, $(\mathbf{k},\pi+a)$ and $(\mathbf{k},\pi-a)$ correspond to the same state except for a global phase factor. This is just the case of the double-valued representation of the SO(3) group, which is written as

$$D^{1/2}(\mathbf{k},a) = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \qquad (5)$$

corresponding to a rotation $R(\mathbf{k},a)$ in real space to a two-state particle. Although $R(\mathbf{k},\pi+a)$ and $R(-\mathbf{k},\pi-a)$ are the same rotation, one has $D^{1/2}(\mathbf{k},\pi+a) = -D^{1/2}(-\mathbf{k},\pi-a)$. Therefore, there is a one-to-one correspondence between the two-qubit MES and the double-valued representation of

SO(3). In fact, any MES can be constructed by a rotation from an initial MES, e.g.,

$$D_1|(1,0)\rangle = |(\alpha,\beta)\rangle,$$
$$D_1 \equiv D_1^{1/2}(\mathbf{k},a), \tag{6}$$

where D_1 operates on the first particle. If a rotation operates on the second particle, one has

$$D_2|(1,0)\rangle = |(\alpha,-\beta^*)\rangle,$$
$$D_2 \equiv D_2^{1/2}(\mathbf{k},a). \tag{7}$$

One could define a SO(3) sphere with diameter π filled by vectors $a\mathbf{k}=(ak_x,ak_y,ak_z)$. Due to (6) a MES corresponds to a point in the SO(3) sphere, and an evolution of the MES corresponds to a trajectory connecting two points. The initial state $|(1,0)\rangle$ locates at the center of the SO(3) sphere.

II. A MODEL HAMILTONIAN

Consider an electron in a rotating magnetic field $\mathbf{B}(t)=B(\sin\theta\cos\omega t,\sin\theta\sin\omega t,\cos\theta)$, where θ is the angle between the field and the z-axis, and ω is the rotating frequency of the field. The Hamiltonian of the electron is given by

$$H(t) = \hat{\sigma}\cdot\mathbf{B}(t)=B\begin{pmatrix}\cos\theta & \sin\theta e^{-i\omega t}\\ \sin\theta e^{i\omega t} & -\cos\theta\end{pmatrix}. \tag{8}$$

The two exact solutions of the time-dependent Schrödinger equation are given by

$$|\psi_\pm(t)\rangle = \begin{pmatrix}a_\pm e^{-i\omega t/2}\\ b_\pm e^{i\omega t/2}\end{pmatrix}e^{\mp i\omega_0 t}, \tag{9}$$

corresponding to energy eigenvalues $\hbar\omega_0$ and $-\hbar\omega_0$, respectively, where

$$b_\pm = a_\pm\frac{\hbar(\omega\pm 2\omega_0)-2B\cos\theta}{2B\sin\theta}, \tag{10}$$

$$\omega_0 = \frac{1}{2\hbar}\sqrt{(\hbar\omega)^2-4B\hbar\omega\cos\theta+4B^2}, \tag{11}$$

where the values of $a_\pm$ can be determined by normalization of solutions.

Now consider an initial state $|(1,0)\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$ of two electrons, where $|0\rangle=|\psi_+(0)\rangle, |1\rangle=\psi_-(0)\rangle$. Suppose the first electron travels through a rotating magnetic field, then the system of two electrons evolutes in the form

$$|(1,0)\rangle \to [|\psi_+(t)\psi_+(0)\rangle + |\psi_-(t)\psi_-(0)\rangle]/\sqrt{(2)} \tag{12}$$

$$=|(\alpha,\beta)\rangle = D_1(\omega t,\omega_0)|(1,0)\rangle, \tag{13}$$

where new arguments have been assigned to the group element for convenience, and

$$\alpha = \left[\cos\frac{\omega t}{2} + i\sin\frac{\omega t}{2}\frac{\hbar\omega-2B\cos\theta}{2\hbar\omega_0}\right]e^{-i\omega_0 t}, \tag{14}$$

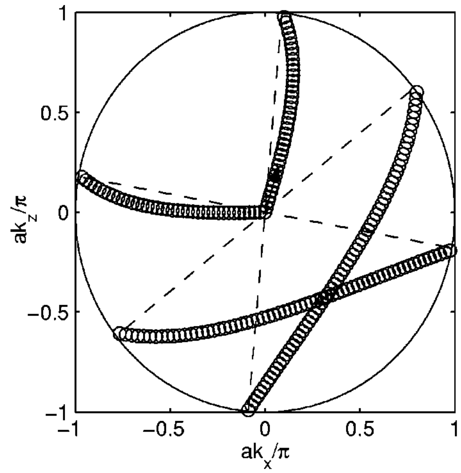

FIG. 1. Closed trajectory in the SO(3) sphere with $\theta=\pi/5, B=1.3603$. The arrow stands for the beginning state and direction of evolution.

$$\beta = i\sin\frac{\omega t}{2}\frac{a_+}{a_-}\frac{\hbar(\omega-2\omega_0)-2B\cos\theta}{2\hbar\omega_0}e^{i\omega_0 t}. \tag{15}$$

It is seen that a rotating magnetic field leads to an evolution of a MES through a continuous trajectory in the SO(3) sphere. Therefore, a rotating magnetic field is equivalent to a three dimensional rotation in real space to the MES.

It is not surprising that when $\omega t=2\pi, \omega_0=n\omega, n=$integers, the initial state acquires an additional phase of π, i.e.,

$$D_1(2\pi,\omega)|(1,0)\rangle = -|(1,0)\rangle. \tag{16}$$

The amazing property is that the above operation can be allocated to two particles of the initial state, i.e.,

$$D_1(\pi,n\omega)D_2(\pi,n\omega)|(1,0)\rangle = -|(1,0)\rangle. \tag{17}$$

In general, one does not have such a property for another initial state. If $\omega_0=(n+1/2)\omega, n=$integers, one will have

$$D_1(\pi,\omega_0)D_2(\pi,\omega_0)|(1,0)\rangle = |(1,0)\rangle, \tag{18}$$

acquiring no additional phase. Hence, one has a choice for the additional phase through selecting the value of ω_0.

Now we can trace the following evolution:

$$|(1,0)\rangle \to [|\psi_+(t)\psi_+(t)\rangle + |\psi_-(t)\psi_-(t)\rangle]/\sqrt{(2)} \tag{19}$$

$$=D_1(\omega t,\omega_0)D_2(\omega t,\omega_0)|(1,0)\rangle. \tag{20}$$

Under the choice $\omega_0=n\omega$, or$(n+1/2)\omega, n=$integers, this evolution makes a closed trajectory in the SO(3) sphere. An example is shown in Fig. 1, where parameters θ and B are set to meet $\omega_0=\omega$. The final time is $t=\pi/\omega$, that is, both magnetic fields of the two electrons rotate half a cycle. It is seen that this trajectory breaks three times on the surface of the sphere. It is known that two ends of a diameter of the sphere correspond to the same rotation but the group element, (5), changes its sign. In this case, through a whole trajectory, the MES acquires an additional phase of π.

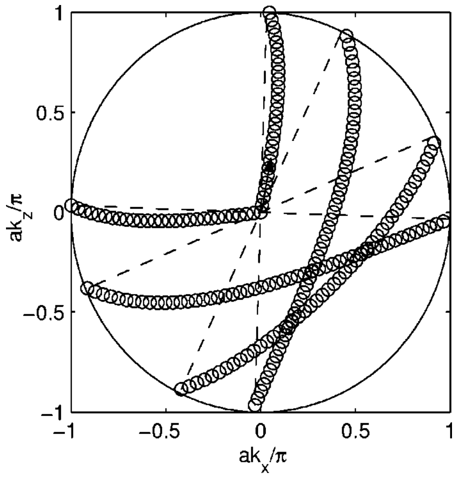

FIG. 2. Closed trajectory in the SO(3) sphere with $\theta=\pi/5$, $B=1.8754$. The arrow stands for the beginning state and direction of evolution.

With proper parameters, one can have closed trajectories with even numbers of breaks, corresponding to a change of 2π in phase. An example is shown in Fig. 2. Hence, one has two classes of trajectories, one of which has an odd number of breaks on the surface of the SO(3) sphere and the other which has an even number of breaks, corresponding to the two classes of the double connectedness of the SO(3) group. This is the case that Milman and Mosseri considered [3], whereas their trajectories are hardly possible to be realized, since their magnetic field has to jump through a few discrete points in the parameter space.

It can easily checked that the closed trajectories A-B-F-D-A and A-B-F-$\bar{E}$-$\bar{A}$ and other ones that Milman and Mosseri considered [3] belong to the two simplest classes of trajectories which have 0 or 1 breaks, respectively, in the SO(3) sphere. Therefore, the present work extends their model to include a great number of closed trajectories with even or odd numbers of breaks.

III. REALIZATION OF THE π PHASE BY ENTANGLED PHOTON PAIRS

A entangled photon pair emerging from a double refraction crystal [7] can be in one of four Bell states $|\Phi^+\rangle = (|H_a H_b\rangle + V_a V_b\rangle)/\sqrt{2}$, where $|H\rangle$ and $|V\rangle$ denote a horizontally polarized photon state and a vertically polarized one, respectively. These two entangled photons separate with each other after emission, and then pass through two negative Kerr media P1 and P3, and two positive Kerr media P2 and P4, as seen in Fig. 3. The Kerr media are modulated by electric fields, so that their optical axes are in directions as shown in the lower part in Fig. 3. P1 and P3 point to the same direction, say the z-axis, and the directions of P2 and P4 can be adjusted by changing the directions of their electric fields.

P1 and P3 change the relative phase between $|H\rangle$ and $|V\rangle$, working as the following matrix,

$$U_1 = \begin{pmatrix} e^{-i\phi_1/2} & 0 \\ 0 & e^{i\phi_1/2} \end{pmatrix}, \tag{21}$$

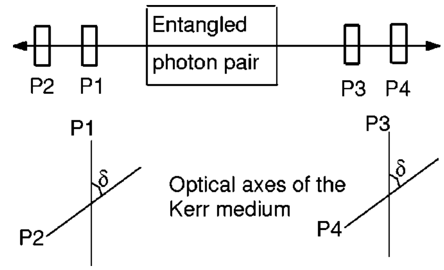

FIG. 3. Scheme to produce the π phase. P1, P2, P3, and P4 are Kerr media. The optical axes are given by the directions of the electric fields on the Kerr media.

$$\phi_1 = \frac{2\pi}{\lambda}(n_{e1} - n_{o1})d_1 = \frac{2\pi}{\lambda}k_1 d_1 E_1^2, \tag{22}$$

where E_1 is the electric field applied to Kerr media P1 and P3. Since the optical exes of P2 and P4 take an angle of δ with the z-axis, they work as the following matrix,

$$U_2 = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}, \tag{23}$$

$$A = \cos\frac{\phi_2}{2} + i\,\sin\frac{\phi_2}{2}\cos 2\delta, \tag{24}$$

$$B = i\,\sin\frac{\phi_2}{2}\sin 2\delta, \tag{25}$$

$$\phi_2 = \frac{2\pi}{\lambda}(n_{o2} - n_{e2})d_2 = \frac{2\pi}{\lambda}k_2 d_2 E_2^2, \tag{26}$$

where E_2 is the electric field applied to P3 and P4.

It is seen that the combination, U_2U_1, is equivalent to a rotating magnetic field. By comparing Eqs. (13)–(15) and Eqs. (21), (24), and (25) one finds correspondence $\phi_1 \sim \omega_0 t, \phi_2 \sim \omega t, \cos 2\delta \sim (\hbar\omega - 2B\cos\theta)/2\hbar\omega_0$. Hence, the evolution in Eq. (19), as shown in Figs. 1 and 2, can be exactly traced by varying the electric fields E_1 and E_2 on the Kerr media. With proper values of electric fields such that $\phi_1 = n\phi_2$, one may obtain an additional π phase, or zero additional phase if $\phi_1 = (n+\frac{1}{2})\phi_2$.

The π phase can be easily observed by various interference experiments. For example, according to the scheme described by Milman and Mosseri [3], one arm of the entangled photon pair can be transformed into a Mach-Zender interferometer. The two wave plates in that scheme are replaced by combinations of Kerr media P1 and P2 and that of P3 and P4.

In summary, the present paper sets up a representation for the SO(3) group by maximally entangled two-qubit states. The evolution of the entangled states showed the double connectedness of the SO(3) group. In the SO(3) sphere the evolution path breaks an odd or even number of times. An odd number of breaks causes an additional π phase to the entangled state, but an even number of breaks does not. The additional π phase can be observed by interference experiments of entangled photon pairs.

ACKNOWLEDGMENTS

The author acknowledges Shidong Liang for helpful discussions and financial support from the Science and Technology Project of Guangzhou 2001-2-095-01.

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