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Analytical solutions of the Bogoliubov–de Gennes equations for excitations of a trapped Bose-Einstein-condensed gas

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We propose a method for finding analytical solutions of the Bogoliubov–de Gennes (BdG) equations for the low-lying collective excitations in a harmonically trapped Bose-Einstein condensate beyond the Thomas-Fermi limit. We first use a simple variational wave function for ground state to eliminate the divergence at the boundary layer of the condensate, which appears in the Thomas-Fermi approximation. We then solve the BdG equations analytically and obtain explicit and divergence-free expressions for the eigenvalues and eigenfunctions of the excitations for the traps with spherical and cylindrical symmetries. The solutions of the zero-energy mode of the BdG equations are also presented.

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I. INTRODUCTION

The study of elementary excitations is one of the main subjects in quantum many-body physics. Due to the remarkable experimental realization of Bose-Einstein condensation, much attention has been paid to the investigation on the elementary excitations in trapped, weakly interacting atomic gases [1]. Up to now the analytical works on the elementary excitations in Bose-Einstein condensates (BECs) are mainly based on Gross-Pitaevskii and Bogoliubov theories. At zero temperature a weakly interacting Bose-condensed gas can be well-described by the Gross-Pitaevskii equation (GPE) [2], which can be transformed into a hydrodynamic form in terms of the amplitude and phase of order parameter. However, this approach usually neglects the quantum fluctuations of the condensate. In the Bogoliubov theory [3] one makes a canonical transformation for the field operators to diagonalize the Hamiltonian of the system by using the eigenfunctions of Bogoliubov–de Gennes equations (BdGEs). The Bogoliubov theory is easy to generalize to finite temperature (e.g., the Hartree-Fock and Popov approaches) when the quantum fluctuations are taken into account properly [1]. The sum rule approach is easy to give the excitation spectrum [4]. The key in the equivalent GP and Bogoliubov theories [5] is to solve respectively the hydrodynamic equations and the BdGEs, which is for an analytical treatment very difficult for a trapped BEC. For a repulsive atomic interaction and a large particle number an explicit result on the collective excitations can be obtained under a Thomas-Fermi (TF) limit by using the fact that atomic interaction is dominant compared with the quantum pressure [6–9]. To include the contribution from the surface of the condensate, Fetter and Feder calculated the leading correction of the condensate wave function, condensate energy, and low-lying collective modes beyond the TF approximation [10].

It is necessary to develop a more useful technique to study the elementary excitations in trapped BECs beyond the TF limit. The reasons are as follows. (i) At the boundary layer of a condensate the Bogoliubov amplitudes vary sharply and hence the kinetic energy of both the condensate and the excitations cannot be neglected [11]. (ii) There appears a singular point in the solution of the Bogoliubov amplitudes at the boundary of the condensate [8] which makes the theory unsatisfactory. (iii) When considering the nonlinear interaction of the elementary excitations, one must calculate the interacting matrix elements of the excitations. The existence of the singular point in the Bogoliubov amplitudes results in a divergence for the matrix elements [12]. In this work we investigate the low-lying collective excitations in BECs with spherically and cylindrically symmetric trapping potentials and propose a method to solve the BdGEs for the excitations beyond the TF limit. By taking into the contribution from zero-point pressure and applying a variational principle to the ground state of the condensate with a Fetter-like variational ground state function [13], the divergence mentioned above is eliminated completely. Although not rigorous, this method can lead us to solve the BdGEs exactly. The eigenvalues and the corresponding eigenfunctions are obtained explicitly and they are valid in all regions of space. The solutions of the zero-energy mode of the BdGEs are also provided.

II. MODEL AND BdGEs

We start from the grand canonical Hamiltonian density of a trapped, dilute Bose gas with a weak repulsive interaction between atoms [1] $\hat{\mathcal{H}}=-(\hbar^2/2M)\nabla^2+V_{ext}(\mathbf{r})-\mu+(g/2)\hat{\Psi}^{\dagger}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)$, where $\hat{\Psi}(\mathbf{r},t)$ is the field operator, $g=4\pi\hbar^2a_{sc}/M$ is the atomic interaction constant with M the atomic mass and $a_{sc}(>0)$ the s-wave scattering length, and μ is the chemical potential of the system. The trapping potential is of the form $V_{ext}(\mathbf{r})=(1/2)M\omega_{\perp}^2(x^2+y^2+\lambda^2z^2)$. Making

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the Bogoliubov approximation and canonical transformation $\hat{\Psi}(\mathbf{r},t)=\psi_G(\mathbf{r})+\Sigma_{n=0}^{\infty}[u_n(\mathbf{r})\hat{c}_n(t)+v_n^*(\mathbf{r})\hat{c}_n^{\dagger}(t)]$ with boson commutation relations of the annihilation and creation operators $\hat{c}_n(t)$ and $\hat{c}_n^{\dagger}(t)$, the Hamiltonian of the system $\hat{H}=\int d^3\mathbf{r}\hat{\Psi}^{\dagger}(\mathbf{r},t)\hat{\mathcal{H}}\hat{\Psi}(\mathbf{r},t)$ at zero temperature becomes $\hat{H}=E_G+\frac{\alpha}{2}\hat{p}^2+\Sigma_{n>0}\varepsilon_n\hat{c}_n^{\dagger}\hat{c}_n+$ the nonlinear terms, where the condensed state wave function ψ_G satisfies the GPE

$$[-\hbar^2\nabla^2/(2M)+V_{ext}(\mathbf{r})-\mu+g|\psi_G|^2]\psi_G=0, \qquad (1)$$

where E_G is the energy of the condensate and $\hat{p}=(1/\sqrt{2})(\hat{c}_0^{\dagger}+\hat{c}_0)$ is its "momentum" operator [14,15]. The excitations generated in the condensate occupy the various normal-mode eigenstates that satisfy the linear BdGEs [1]

$$\hat{L}u_n(\mathbf{r})+g|\psi_G|^2v_n(\mathbf{r})=+\varepsilon_n u_n(\mathbf{r}), \qquad (2)$$

$$\hat{L}v_n(\mathbf{r})+g|\psi_G|^2u_n(\mathbf{r})=-\varepsilon_n v_n(\mathbf{r}), \qquad (3)$$

where $u_n(\mathbf{r})$ and $v_n(\mathbf{r})$ are the eigenfunctions of the excitations associated with the eigenvalues ε_n. The operator $\hat{L}$ is defined by $\hat{L}=-\hbar^2\nabla^2/(2M)+V_{ext}(\mathbf{r})-\mu+2g|\psi_G|^2$. The zero-energy ($\varepsilon_0=0$) mode u_0 and v_0 fulfill the equations [14,15]

$$\hat{L}u_0(\mathbf{r})+g|\psi_G|^2v_0(\mathbf{r})=(\alpha/2)[u_0(\mathbf{r})-v_0(\mathbf{r})], \qquad (4)$$

$$\hat{L}v_0(\mathbf{r})+g|\psi_G|^2u_0(\mathbf{r})=(\alpha/2)[u_0(\mathbf{r})-v_0(\mathbf{r})], \qquad (5)$$

with the parameter α yet to be determined. The eigenfunctions u_n and v_n constitute an orthonormal and complete set of functions with the relations $\int d^3\mathbf{r}[u_n^*(\mathbf{r})u_{n'}(\mathbf{r})-v_n^*(\mathbf{r})v_{n'}(\mathbf{r})]=\delta_{nn'}$, $\int d^3\mathbf{r}[u_n(\mathbf{r})v_{n'}(\mathbf{r})-u_{n'}(\mathbf{r})v_n(\mathbf{r})]=0$, $\Sigma_{n=0}^{\infty}[u_n(\mathbf{r})u_n^*(\mathbf{r}')-v_n(\mathbf{r})v_n^*(\mathbf{r}')]=\delta(\mathbf{r}-\mathbf{r}')$, and $\Sigma_{n=0}^{\infty}[u_n(\mathbf{r})v_n^*(\mathbf{r}')-v_n(\mathbf{r})u_n^*(\mathbf{r}')]=0$.

For convenience we rewrite the BdGEs in terms of suitable rescaled variables. We let $\overline{r}\equiv r/R_{\perp}$ and $\overline{\nabla}\equiv R_{\perp}\nabla$ with $R_{\perp}=\sqrt{2\mu/M\omega_{\perp}^2}$ the characteristic condensate radius. By introducing an important parameter $\zeta\equiv\hbar\omega_{\perp}/2\mu$ and a function $\sigma(\mathbf{r})\equiv-[\overline{\nabla}^2\psi_G(\mathbf{r})]/\psi_G(\mathbf{r})$, which is proportional to the zero-point pressure, the GPE (1) for the ground state becomes $|\psi_G(\mathbf{r})/\psi_G(0)|^2=1-\overline{r}^2-\zeta^2\sigma(\mathbf{r})$ and the coupled BdGEs (2) and (3) are equivalent to the single fourth-order ordinary differential equations for the functions $\varphi_{\pm}(\mathbf{r})\equiv u_n\pm v_n$ and the dimensionless eigenfrequency $\overline{\omega}_n\equiv\omega_n/\omega_{\perp}$ [8]:

$$-\overline{\nabla}^2(1-\overline{r}^2)\varphi_+-(1-\overline{r}^2)\sigma\varphi_++(1/2)\zeta^2 \times[\overline{\nabla}^4+3\overline{\nabla}^2\sigma+\sigma\overline{\nabla}^2+3\sigma^2]\varphi_+=2\overline{\omega}_n^2\varphi_+, \qquad (6)$$

$$-(1-\overline{r}^2)\overline{\nabla}^2\varphi_--(1-\overline{r}^2)\sigma\varphi_-+(1/2)\zeta^2 \times[\overline{\nabla}^4+\overline{\nabla}^2\sigma+3\sigma\overline{\nabla}^2+3\sigma^2]\varphi_-=2\overline{\omega}_n^2\varphi_-. \qquad (7)$$

The general BdGEs (6) and (7) are very difficult to solve analytically because they involve the high-order derivative $\overline{\nabla}^4$ and the trapping potential. As pointed out in [5] that, for the low-lying excitations, i.e., $\hbar\omega_{\perp}\leqslant\varepsilon_n\ll\mu$, substitution of the TF solution into the BdGEs yields an incorrect eigenvalue spectrum, since the kinetic energies (the zero-point pressure) of the condensate and the excitations are equally important (also see [8]). Hence we cannot omit the kinetic energy term in Eq. (1). Furthermore, the TF solution leads to a logarithmic divergence in the mean kinetic energy [10], which is proportional to $\langle\sigma(\mathbf{r})\rangle\sim\int_0^1 d\overline{r}\overline{r}^2(3-\overline{r}^2)/(1-\overline{r}^2)=4/3+B(0^+,3/2)=4/3-\ln 0^+$ with $B(a,b)$ the beta function.

For enough large N_0 (the particle number in the condensate) the repulsive interaction becomes important and the condensate expands to $R_{\perp}$ from $a_{ho}\equiv(\hbar/M\omega_{\perp})^{1/2}$, where a_{ho} is the characteristic oscillator length of the trapping potential, and $R_{\perp}/a_{ho}\simeq(15\lambda P)^{1/5}\gg1$ with $P\equiv N_0a_{sc}/a_{ho}$. We called the case $P\gg1$ (a large but finite product of N_0 and the ratio a_{sc}/a_{ho}) the TF regime and the case $P\rightarrow\infty$ the TF limit. In the TF limit the chemical potential has a simple form $\mu\simeq\frac{1}{2}\hbar\omega_{\perp}(15\lambda P)^{2/5}$. In usual TF approximation and for large finite P, the boundary layer is at the location of finite $r=R_{\perp}$. The TF solution leads $\sigma(\mathbf{r})$ as well as $\zeta^2\langle\sigma(\mathbf{r})\rangle$ to be singular and hence the calculation for the interacting matrix elements to diverge [8,12], as mentioned above. Note that for low-lying excitations and a large particle number one can have two limit processes, i.e., $\zeta^2\simeq(15\lambda P)^{-4/5}\rightarrow0$ and $\sigma(\mathbf{r})\rightarrow\infty$ (for $\overline{r}\rightarrow1$). Thus one can consider the situation in which the product $\zeta^2\langle\sigma(\mathbf{r})\rangle$ remains finite, relying on the exact expression for ψ_G beyond the TF limit. It is this important observation that provides us with the possibility to get divergence-free analytical solutions of the BdGEs (6) and (7) in the TF regime.

III. SOLUTIONS FOR $\lambda=1$

The explicit analytical solutions for Eqs. (6) and (7) can be obtained in three steps. The first and key step is to use a Fetter-like variational solution for the GPE (1) by introducing a single variation parameter q, i.e., we take $\psi_G=C_G(1-\overline{r}^2)^{(q+1)/2}\Theta(1-\overline{r})$, where C_G is a normalization constant. Note that the condensate radius $R_{\perp}$ is a parameter in the TF regime rather than a variational parameter as Fetter did in the whole regime of N [13]. The energy minimum condition gives q as a function of P shown below in Eq. (13) for $\lambda=1$. The asymptotic behavior is $q\sim P^{-2/5}\rightarrow0$ and $R_{\perp}=a_{ho}[4\lambda P/B(3/2,2+q)]^{1/5}\sim P^{1/5}\rightarrow\infty$ for $P\rightarrow\infty$. The second step is to set the parameter ζ^2 to be zero but substitute $\sigma=(1+q)[3-(q+2)\overline{r}^2]/(1-\overline{r}^2)^2$ in Eqs. (6) and (7), i.e., we solve the leading-order approximation solutions of Eqs. (6) and (7) by taking ζ^2 as a small expansion parameter. We get the solutions of the form $\varphi_{\pm}(\mathbf{r})=C_{\pm}(1-\overline{r}^2)^{(q\mp1)/2}\overline{r}^lP(\overline{r}^2)Y_{lm}(\theta,\varphi)$, where the radial function $P(x)$ with $x=\overline{r}^2$ satisfies a hypergeometric differential equation

$$2x(1-x)P''(x)+[2l+3-(2l+5+2q)x]P'(x) +[(\overline{\omega}_{n_rl}^{(0)})^2-l-lq]P(x)=0. \qquad (8)$$

The solutions of Eq. (8) are classical n_rth-order Jacobi polynomials $P_{n_r}^{(l+1/2,q)}(1-2x)$, which form an orthonormal function set in the interval $0\leqslant x\leqslant1$ with weight $x^{l+1/2}(1-x)^q$.

The normalization integration in the axial direction is $I_{n_r l} \equiv \int_0^1 dx x^{l+1/2}(1-x)^q P_{n_r l}^2(x)$. The eigenvalues of Eq. (8) are given by $(\overline{\omega}_{n_r l}^{(0)})^2 = (\overline{\omega}_{n_r l}^{TF})^2 + (2n_r+l)q$, where $(\overline{\omega}_{n_r l}^{TF})^2 = 2n_r^2 + 2n_r l + 3n_r + l$ is the result in the TF limit for $n=(n_r,l,m)$ modes. Since the normalization coefficients $C_\pm$ have the relation $C_+ = \zeta\overline{\omega}_{n_r l}^{(0)} C_-$, from Eqs. (6) and (7) we obtain the normalized solutions by use of the condition $\int d^3\mathbf{r}(u_{n_r l}^2 - v_{n_r l}^2) = 1$:

$$\varphi_\pm(\mathbf{r}) = [2/(I_{n_r l}R_\perp^3)]^{1/2}(\zeta\overline{\omega}_{n_r l}^{(0)})^{\pm 1/2}(1-\overline{r}^2)^{(q\mp 1)/2}\overline{r}^l \times P_{n_r l}(\overline{r}^2)Y_{lm}(\theta,\varphi). \tag{9}$$

Note that Eq. (8) is similar to that in [5,7,8] but the solutions $\varphi_\pm(\mathbf{r})$ are very different. The main difference is that there is a weight factor $(1-\overline{r}^2)^q$ for $q\neq 0$ in the TF regime and its limit recovers the result obtained by using the TF limit (i.e., for $q\to 0$). We stress that it is the use of the ground state wave function given above, which vanishes smoothly when $\overline{r}\to 1$, that makes not only $\zeta^2\langle\sigma(\mathbf{r})\rangle$ finite for $1\ll P<\infty$, but also the eigenvalues and eigenfunctions of the BdGEs can be obtained analytically.

The third step is to use a perturbation theory to calculate the contribution from the terms proportional to ζ^2 in Eqs. (6) and (7). Using the leading-order solution $\varphi_\pm(\mathbf{r})$ given above we obtain the first-order correction of the eigenfrequency

$$\overline{\omega}_{n_r l} = \frac{\overline{\omega}_{n_r l}^{(0)}}{2I_{n_r l}}\int_0^1 dx x^{l+1/2}P^2(x)\left\{(1-x)^q + (1-x)^{2q} + \zeta^2(1-x)^{q-2}\left[(\overline{\omega}_{n_r l}^{(0)})^2 - 3 - 2l - 2\frac{1-q}{1-x}\sqrt{x} + 4x\frac{P'(x)}{P(x)}\right]\right\}. \tag{10}$$

For the special modes of $n_r=0$ and $l\geqslant 1$, the above result is simplified into

$$\overline{\omega}_{0l} = \frac{\sqrt{l+lq}}{2B(l+3/2,q+1)}\{B(l+3/2,q+1) + B(l+3/2,2q+1) + \zeta^2[(lq-l-3)B(l+3/2,q-1) - 2(1-q)B(l+1,q-2)]\}. \tag{11}$$

Note that because of the existence of the weight factor $(1-x)^q$ the divergence in the integration (10) is eliminated completely. This point can also be seen clearly from Eq. (11) since $\overline{\omega}_{0l} = \sqrt{l} + \text{const}\ \zeta^2/q$ and $\zeta^2/q \sim P^{-2/5}\to 0$ in the TF limit. Although $B(l,q)$ has singularity $\sim 1/q$ at $q=0$, the result for the eigenspectrum and wave functions is divergence-free.

We now consider the solution u_0 and v_0 of the zero-energy mode for Eqs. (4) and (5). We let $\varphi_{0\pm}(\mathbf{r})\equiv u_0 \pm v_0$ and find that φ_{0-} is just ψ_G and φ_{0+} has the form $\varphi_{0+} = y^{(q+5)/2}W(y)$, where $y = 1-\overline{r}^2$ and $W(y)$ satisfies the equation

$$-2y^2(1-y)W'' + [(9+2q)y - 2(5+q)]yW' + (5+q)[3+q - (6+q)y]W + y^3W/\zeta^2 = \alpha C_G/(\mu\zeta^2). \tag{12}$$

We make the series expansion $W(y) = a_0\Sigma_{k=0}^\infty \tilde{a}_k y^k$, where a_0 is determined by the normalization condition $\int d^3\mathbf{r}(u_0^2 - v_0^2) = 1$. The single eigenvalue of Eq. (12) is found to be $\alpha = -\hbar\omega_\perp(9+4q-q^2)\zeta/[2\pi R_\perp^3 C_G^2 \Sigma_{k=0}^\infty \tilde{a}_k B(3/2,k+q+2)]$.

IV. SOLUTIONS FOR $\lambda\neq 1$

In a similar way we may solve the BdGEs (6) and (7) for the case of cylindrical symmetry, in which case we have $\overline{r}^2 = \overline{s}^2 + \lambda^2\overline{z}^2$ with $\overline{s}\equiv s/R_\perp$ and $\overline{z}\equiv z/R_\perp$. We use the trial wave function for the ground state $\psi_G = C_G\sqrt{\lambda}(1-\overline{r}^2)^{(q+1)/2}\Theta(1-\overline{r})$ and the variation parameter q is chosen by minimizing the energy, i.e., q satisfies the equation

$$\frac{2+\lambda^2}{B^{1/5}(3/2,2+q)(4\lambda P)^{4/5}} \times\left\{\frac{5}{2q^2} - 1 - \left(\frac{1}{q}+\frac{7}{5}+\frac{2}{5}q\right)\left[\psi(2+q) - \psi\left(\frac{7}{2}+q\right)\right]\right\} = C' - \frac{3}{(7/2+q)B(3/2,2+q)} + \frac{3}{5}\left[\psi(2+q) - \psi\left(\frac{7}{2}+q\right)\right] \times\left[\frac{2}{(7/2+q)B(3/2,2+q)} - C\right], \tag{13}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function, $C = B(1+q,7/2+q)/B(3/2,2+q)B(1+q,2+q)$ and C' is the derivative with respect to q. Here $q\sim P^{-2/5}\to 0$, $R_\perp \sim P^{1/5}\to\infty$, and $\zeta^2/q \sim P^{-2/5}\to 0$ at $P\to\infty$. Note that the condensate radius $R_\perp$ and $R_\parallel = R_\perp/\lambda$ have two parameters in the TF regime rather than two variational parameters as Fetter did in the whole regime of N [13]. The solutions of Eqs. (6) and (7) have the form $\varphi_\pm(\mathbf{r}) = C_\pm(1-\overline{r}^2)^{(q\mp 1)/2}\overline{s}^m P(\overline{s},\overline{z})e^{im\varphi}$, where the coupled axial and radial function $P(\overline{s},\overline{z})$ satisfies a two-dimensional differential equation

$$\left\{(1-\overline{s}^2-\lambda^2\overline{z}^2)\left[\frac{\partial^2}{\partial\overline{s}^2} + (1+2|m|)\frac{\partial}{\overline{s}\,\partial\overline{s}} + \frac{\partial^2}{\partial\overline{z}^2}\right] - 2(1+q) \times\left(\overline{s}\frac{\partial}{\partial\overline{s}} + \lambda^2\overline{z}\frac{\partial}{\partial\overline{z}}\right) + 2[(\overline{\omega}_{n_z n_s m}^{(0)})^2 - |m| - |m|q]\right\}P = 0. \tag{14}$$

The solutions of Eq. (14) are given by $P_{n_p}^{(2n_s)}(\overline{z},\overline{s}) = \Sigma_{k=0}^{n_p}\Sigma_{n=0}^{\text{int}[k/2]} b_{k,n}\overline{z}^{k-2n}\overline{s}^{2n}$, where the principal quantum number n_p and the zero-order eigenvalue $(\overline{\omega}_{n_z n_s m}^{(0)})^2$ are the solutions of a standard continued fraction equation [7]. The polynomials $P_{n_p}^{(2n_s)}(\overline{z},\overline{s})$ form an orthonormal function set in the interval $0\leqslant\overline{r}\leqslant 1$ with the weight $\overline{s}^m(1-\overline{r}^2)^q$. The normalization integral reads $I_{n_z n_s m}\equiv 2\int_0^1 \overline{s}d\overline{s}\int_0^{\sqrt{1-\overline{s}^2}/\lambda} d\overline{z}\overline{s}^{2m}(1-\overline{s}^2-\lambda^2\overline{z}^2)^q[P_{n_p}^{(2n_s)}(\overline{z},\overline{s})]^2$. The normalized eigenfunctions are

$$\varphi_{\pm}(\mathbf{r}) = \frac{(\zeta\overline{\omega}^{(0)}_{n_z n_s m})^{\pm 1/2}}{\sqrt{2\pi R_\perp^3 I_{n_z n_s m}}}(1-\overline{s}^2-\lambda^2\overline{z}^2)^{(q\mp 1)/2}\overline{s}^m P^{(2n_s)}_{n_p}(\overline{s,z})e^{im\varphi}. \tag{15}$$

The eigenvalues are determined by

$$\begin{aligned}\overline{\omega}_{n_z n_s m} = {} & \frac{\overline{\omega}^{(0)}_{n_z n_s m}}{2I_{n_z n_s m}}\int_0^1 \overline{s}d\overline{s}\int_0^{\sqrt{1-\overline{s}^2}/\lambda} d\overline{z}\overline{s}^{2m}P^2(\overline{z,s}) \\ & \times\Bigg\{(1-\overline{r}^2)^q+(1-\overline{r}^2)^{2q}+\zeta^2(1-\overline{r}^2)^{q-2} \\ & \times\Bigg[(\overline{\omega}^{(0)}_{n_z n_s m})^2-2-\lambda^2-2|m|-2\frac{1-q}{1-\overline{r}^2}(\overline{s}+\lambda^2\overline{z}) \\ & -\frac{2}{P(\overline{z,s})}\left(\overline{s}\frac{\partial}{\partial\overline{s}}P(\overline{z,s})+\overline{z}\frac{\partial}{\partial\overline{z}}P(\overline{z,s})\right)\Bigg]\Bigg\}.\end{aligned} \tag{16}$$

The zero-energy mode takes the form $\varphi_{0+}=(1-\overline{r}^2)^{(q+5)/2}W(\overline{z,s})$, where $W(\overline{z,s})$ satisfies the equation

$$\begin{aligned}\sqrt{\lambda}\frac{\alpha C_G}{\mu\zeta^2} = {} & \Bigg[-(1-\overline{r}^2)^2\overline{\nabla}^2+2(5+q)(1-\overline{r}^2)\left(\overline{s}\frac{\partial}{\partial\overline{s}}+\lambda^2\overline{z}\frac{\partial}{\partial\overline{z}}\right) \\ & +2(1-q)(2+\lambda^2)(1-\overline{r}^2)+(q^2-4q-9) \\ & \times(\overline{s}^2+\lambda^4\overline{z}^2)+\frac{2}{\zeta^2}(1-\overline{r}^2)^3\Bigg]W(\overline{z,s}).\end{aligned} \tag{17}$$

Its special solution gives $\alpha=(2c_0\mu/C_G\sqrt{\lambda})\{1+\zeta^2[(1-q)(2+\lambda^2)-(b_{2,0}+2b_{2,1})\tilde{c}_2]\}$, where c_0 is the normalization coefficient, and $\tilde{c}_2$ is the second-order coefficient of the $W(\overline{z,s})$ expanded in terms of $P^{(2n_s)}_{n_p}(\overline{z,s})$.

V. CONCLUSION

We have proposed a method for finding analytical solutions of the BdGEs for the low-lying collective excitations of a harmonically trapped Bose-condensed gas beyond the TF limit. The singularity at the boundary layer of the condensate has been eliminated by introducing a self-consistent variational parameter for the ground wave function of the condensate. We have solved the BdGEs analytically and obtained their eigenfrequencies and corresponding eigenfunctions, which are divergence-free and valid in all spatial regions. These general results cover also those obtained in the TF limit. In addition, we have also presented the solution of the zero-energy mode of the BdGEs. Because the region near $\overline{r}=1$ has been analytically treated, the formulas obtained in this work allow us to calculate the interacting matrix elements of the collective excitations, which will be given elsewhere.

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