Quantum error correction against correlated noise

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We consider quantum error correction against correlated noise using simple and concatenated Calderbank-Shor-Steane codes as well as n-qubit repetition codes. We characterize the performance of various codes by means of the fidelity following the error correction of a single logical qubit in a quantum register. For concatenated codes we find a threshold in the single-qubit error rate below which the encoded qubit is perfectly protected. The threshold depends on the correlation strength of the noise and goes to zero for perfect correlation. Finally, we concatenate the traditional error correcting codes with a decoherence free subspace and evaluate the performance over the whole range from uncorrelated noise to perfectly correlated noise.

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I. INTRODUCTION

It is clear by now that the operation of a large-scale quantum computer will require some form of protection from decoherence arising from the interaction with a noisy environment, imperfect logical gates, and noisy control fields. There are three methods known to provide such protection—quantum error correcting codes [1–8], decoherence free subspaces [9–13], and dynamical suppression of decoherence [14]. The quantum error correcting codes have been developed under the assumption that errors affect different qubits independently. On the other hand, decoherence free subspaces arise due to the symmetric coupling of the environment to all the qubits when errors are perfectly correlated. In a physical realization of a quantum information processor the sources of noise will generally have a finite correlation length and time leading to partially correlated errors affecting qubits at different locations and different times. In this paper we focus on spatially correlated errors and investigate the performance of error correcting codes and decoherence free subspaces in both simple and concatenated forms for arbitrary correlations of the environment.

This work is a preliminary characterization of the effects of correlated noise using the simplest possible model for the environment and quantum information processor. In particular, we restrict our attention to a single logical qubit in a quantum register. Furthermore, we assume that the process of syndrome extraction and error correction proceeds perfectly—i.e., without introducing any additional errors. The environment is modeled as a set of fluctuating classical fields. Despite these simplifications we are able to assess the effect of partially correlated errors on the performance of error correcting codes and decoherence free subspaces. Our results—an enhanced probability for multiple-qubit errors—can be applied in the context of a fault-tolerant error correction protocol in a straightforward manner, providing a lower bound on the enhancement of the failure probability due to the correlations.

The layout of the paper is as follows. In Sec. II we investigate several error correcting codes in their simple (unconcatenated) form. We consider concatenation of the three-bit repetition code, the $[\![7,1,3]\!]$ Hamming code, and the $[\![23,1,7]\!]$ Golay code in Sec. III, where the notation $[\![n,k,d]\!]$ indicates k logical qubits encoded into n physical qubits using a distance d code. In Sec. IV we consider concatenating a decoherence free subspace with the three-bit repetition code and we conclude in Sec. V.

II. SIMPLE CALDERBANK-SHOR-STEANE CODES

A. Physical model

In order to explore the effects of correlated noise in quantum error correction we consider the simplest possible example. We consider a quantum register consisting of a single logical qubit in an arbitrary state encoded into n physical qubits by means of an error correcting code. We assume that the encoding proceeds perfectly. The physical qubits interact with a noisy environment through the interaction Hamiltonian

$$H_I = g\hbar \sum_{i=1}^{n} \mathcal{E}_i(t)\sigma_{i+} + \mathcal{E}_i^*(t)\sigma_{i-}, \tag{1}$$

where the fields $\mathcal{E}_i(t)=E_i(t)\exp[i\phi(t)]$ are a set of Gaussian random variables characterized by the correlations $\langle \mathcal{E}_i^*(t)\mathcal{E}_j(t+\tau)\rangle$. In order to analyze repetition codes we restrict the phase of the noise to $\phi_i(t)=0$ while for general Calderbank-Shor-Steane (CSS) codes we do not restrict the phase.

The use of classical noise fields simplifies the calculations but should not be viewed as a fundamental limitation of our treatment. It is quite likely that in a physical implementation of a quantum computer there will be external electric or magnetic fields used as control fields which would essentially be classical in nature. Any fluctuations of these fields would be well represented in our treatment. In addition, a quantum computer will always be subject to a bath of electromagnetic fields modes in either a thermal state or the vacuum state. In Appendix A we show that our results are unchanged for an environment of quantized field modes.

B. Calculation of the fidelity

We characterize the performance of the various error correcting codes by means of the fidelity of the density operator representing the quantum register after a single application of error correction. The fidelity is defined as

$$F^2 = \langle \psi(0)|\rho_{\mathrm{ec}}(t)|\psi(0)\rangle, \tag{2}$$

where the density operator ρ_{ec} represents the state following the evolution of the register and the application of the error correction protocol. The density operator is necessary to represent the system when we wish to consider an ensemble of many different realizations of the classical noise fields.

In our model we consider only one application of the error correction protocol, so the density operator has the form

$$\rho_{\mathrm{ec}}(t) = \hat{C}(\rho(t)), \tag{3}$$

where the notation $\mathcal{C}(\cdot)$ represents the application of the error correction protocol and the density operator prior to error correction is

$$\rho(t) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} U_k(t,0)|\psi(0)\rangle\langle\psi(0)|U_k^{\dagger}(t,0). \tag{4}$$

The unitary time evolution operator $U_k(t,0)$ represents the time evolution from time 0 to time t for the kth realization of the noise fields.

The error correction operation $\hat{\mathcal{C}}(\cdot)$ has the form $\sum_{j=0}^{t} P_j \cdot P_j^{\dagger}$ where P_j is a projection corresponding to the measurement of the jth error syndrome and the application of a unitary operator correcting the error indicated by the measured syndrome. The fidelity in this model can quite generally be written

$$F^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{j=1}^{t} |\langle\psi(0)||P_j U_k(t,0)|\psi(0)|^2. \tag{5}$$

C. Three-qubit repetition code

In general, an exact expression for the density operator may be unattainable, particularly for larger error correcting codes. We can, however, find the leading-order (in single-qubit error probability) correction to the fidelity for many codes. To motivate this we explicitly consider the three-qubit repetition code which can protect a single logical qubit against a single-bit-flip error or a single-bit phase error depending on the choice of the encoding. We consider an encoding

$$|0\rangle_L = |000\rangle, \tag{6a}$$

$$|1\rangle_L = |111\rangle, \tag{6b}$$

which protects against a bit flip on one of the physical qubits. The time evolution operator for a given realization k of the noise is

$$U_k(t,0) = \prod_{i=1}^{3} U_{ki}(t,0), \tag{7}$$

where

$$U_{ki}(t,0) = \cos \Omega_{ki}t - i \sin \Omega_{ki}t\sigma_{ix} \tag{8}$$

is the time evolution for the ith physical qubit and

$$\Omega_{ki} = gE_{ki}, \tag{9}$$

where we have taken the noise fields to be constant in time for each realization. This code can correct three different single-qubit errors as well as the trivial null-error condition. There are four projectors which correspond to the outcomes of the error correction protocol:

$$P_0 = (|000\rangle\langle 000| + |111\rangle\langle 111|), \tag{10a}$$

$$P_1 = \sigma_{1x}(|100\rangle\langle 100| + |011\rangle\langle 011|), \tag{10b}$$

$$P_2 = \sigma_{2x}(|010\rangle\langle 010| + |101\rangle\langle 101|), \tag{10c}$$

$$P_3 = \sigma_{3x}(|001\rangle\langle 001| + |110\rangle\langle 110|). \tag{10d}$$

Physically these projectors are realized by the appropriate series of interactions between the data qubits and ancilla qubits, the measurement of the ancilla qubits to determine the error syndrome, and subsequent application of a bit-flip operator to correct the error. Here we assume that this process occurs perfectly—i.e., without introducing further errors—and effectively instantaneously on the time scale of the system evolution.

The projectors in this case have the property that

$$\langle\psi(0)|P_j U_k(t,0)|\psi(0)\rangle = \langle\psi(0)|\sigma_{jx}U_k(t,0)|\psi(0)\rangle, \tag{11}$$

where we define $\sigma_{0x} \equiv I$. Noting this property and the structure of the most general initial state

$$|\psi(0)\rangle = \alpha|000\rangle + \beta|111\rangle \tag{12}$$

for the data qubits, it is clear that the only contributions to the fidelity are those terms with zero ($\sigma_{jx}^2 = I$) or three bit flips ($\sigma_{1x}\sigma_{2x}\sigma_{3x}$) including the flip applied during error correction. Then the fidelity is

$$\begin{aligned} F^2 = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\Bigg\{ &|\cos \Omega_{k1}t \cos \Omega_{k2}t \cos \Omega_{k3}t \\ &+ iA \sin \Omega_{k1}t \sin \Omega_{k2}t \sin \Omega_{k3}t|^2 \\ &+ \sum_{j=1}^{3} |-i \cos \Omega_{kp}t \cos \Omega_{kq}t \sin \Omega_{kj}t \\ &- A \cos \Omega_{kj}t \sin \Omega_{kp}t \sin \Omega_{kq}t|^2 \Bigg\}, \end{aligned} \tag{13}$$

where $p \neq q \neq j$ and $A = \alpha\beta^* + \alpha^*\beta$. Multiplying terms and taking the average over the realizations of noise fields yields

$$\begin{aligned} F^2 = \Big\langle &\cos^2\Omega_1 t \cos^2\Omega_2 t \cos^2\Omega_3 t \\ &+ A^2 \sin^2\Omega_1 t \sin^2\Omega_2 t \sin^2\Omega_3 t \\ &+ \sum_{j=1}^{3} [\cos^2\Omega_p t \cos^2\Omega_q t \sin^2\Omega_j t \end{aligned}$$

$$+ A^2 \cos^2\Omega_j t \sin^2\Omega_p t \sin^2\Omega_q t] \Big\rangle, \quad (14)$$

and noting that for $A^2=1$ this is $\Pi_{i=1}^3(\cos^2\Omega_i t+\sin^2\Omega_i t)=1$, we find

$$F^2 = 1 - \gamma\langle \sin^2\Omega_1 t \sin^2\Omega_2 t \sin^2\Omega_3 t + \cos^2\Omega_1 t \sin^2\Omega_2 t \sin^2\Omega_3 t + \sin^2\Omega_1 t \cos^2\Omega_2 t \sin^3\Omega_3 t + \sin^2\Omega_1 t \sin^2\Omega_2 t \sin^2\Omega_3 t\rangle, \quad (15)$$

where $\gamma=1-A^2$. To lowest order in time this is

$$F^2 = 1 - \gamma(gt)^4(\langle E_2^2E_3^2\rangle + \langle E_1^2E_3^2\rangle + \langle E_1^2E_2^2\rangle), \quad (16)$$

which is simply unity minus the probabilities for the lowest-order noncorrectable errors to occur.

D. Larger error correcting codes

We generalize the above result for larger error correcting codes. The assumption that the noise fields can be represented by Gaussian random variables ensures that all moments of the fields can be derived from the second-order moment $\langle E_i(t)E_j(t+\tau)\rangle$. We further assume that the spatial and temporal correlations factorize as $\langle E_i(t)E_j(t+\tau)\rangle = \langle E_iE_j\rangle f(\tau)$. In addition we note that the single-qubit error probability in this model is $p=(gh(t)\delta)^2$ where $\langle E_i^2\rangle=\delta^2$ and

$$h(t) = \int_0^t dt_1 \int_0^t dt_2 f(|t_1-t_2|). \quad (17)$$

We consider two different CSS codes—the well-known $[\![7,1,3]\!]$ Hamming code and a $[\![23,1,7]\!]$ Golay code which correct all one- and three-qubit errors, respectively. These codes are particularly simple in that the set of lowest-order uncorrectable errors is exactly the set of all $[(d+1)/2]$-bit X errors and $[(d+1)/2]$-bit Z errors (where d, the code distance, equals 3 and 7 for the 7-qubit Hamming and 23-qubit Golay code, respectively). The fidelities for these codes are

$$F^2_{[\![7,1,3]\!]} = 1 - p^2 \sum_{\{i_1,i_2\}} \langle E_{i_1}^2E_{i_2}^2\rangle \quad (18)$$

and

$$F^2_{[\![23,1,7]\!]} = 1 - p^4 \sum_{\{i_1,i_2,i_3,i_4\}} \langle E_{i_1}^2E_{i_2}^2E_{i_3}^2E_{i_4}^2\rangle, \quad (19)$$

respectively. (A more careful calculation yields terms of the form $\langle E_{i_1}E_{i_2}E_{i_3}E_{i_4}\rangle$ for the $[\![7,1,3]\!]$ code. See Appendix B for details.) For independent noise the field correlations factorize. This will not generally be valid, and correlations in the noise will enhance the failure probability and decrease the fidelity. In fact, we will show in Sec. III that strong correlations can completely destroy the ability of a code to protect a quantum computer from decoherence.

E. Results

We now make a specific choice for the form of the noise field correlations and demonstrate the effect of the correlations on the performance of these codes. As we stated above we assume the noise fields are represented by Gaussian random variables with spatial correlations of the form

$$\langle E_iE_j\rangle = e^{-|x_i-x_j|/l_0}, \quad (20)$$

where l_0 is the correlation length. Higher-order correlations are calculated by means of the Gaussian moment theorem. While this is a convenient choice for the correlations the general features presented here are not dependent on the specific form. Any type of positive correlation in the noise affecting different qubits will enhance the failure probability over that for independent noise. Furthermore, classical noise fields are representative of a thermal environment for both zero and finite temperature. The fidelity of a quantum register interacting with a reservoir of quantized field modes in the vacuum state is calculated in Appendix A.

We consider a regular linear array of qubits so that the field correlations can be expressed in terms of powers of $x=\exp(-s)$ where s is the spacing of the qubits in units of the correlation length. Then the fidelity for a code correcting $r-1$ errors can be written

$$F^2 = 1 - p^r C(x), \quad (21)$$

where $C(x)$ is a polynomial (with positive coefficients) which depends on the particular code. The independent error results are equivalent to taking $x=0$. The effect of the correlations is to enhance the rate at which uncorrectable errors are produced. This is easily seen for the three-qubit repetition code where evaluating the correlations in Eq. (16) yields $C_3(x)=3+4x^2+2x^4$. In the limit of perfect correlations ($x=1$) the failure rate is tripled over the rate for independent errors. From Eq. (18), the $[\![7,1,3]\!]$ code has

$$C_{[\![7,1,3]\!]} = 21 + 12x^2 + 10x^4 + 8x^6 + 6x^8 + 4x^{10} + 2x^{12}, \quad (22)$$

which also triples the failure rate for perfect correlations. For larger codes the enhancement is more pronounced. For example, the failure rate for the $[\![23,1,7]\!]$ Golay code is increased by a factor of 105 for perfect correlation.

In practice the enhanced failure rate in the presence of correlated noise does not necessarily have a large impact on the feasibility of carrying out error correction. Even the maximal enhancement needs only to be offset by a decrease in p by a factor of $3^{1/2}$ and $105^{1/4}$ for the Hamming code and Golay code, respectively. This requires the error correction to be applied at more frequent intervals but is unlikely to prove fatal to quantum error correction. On the other hand, one of the authors has shown that the minimum energy requirement per logical gate scales with the threshold as $p_{\mathrm{th}}^{-5/2}$ [15]. From this point of view the reduction in the threshold required to protect against correlated noise (to be exhibited in the following section) could represent a significant restriction on the physical implementations of quantum error correction.

The enhancement $[C(x)/C(0)]$ of the failure rates for the three codes is plotted in Fig. 1. While the $[\![7,1,3]\!]$ code has the same failure rate as the repetition code at $x=0$ and $x=1$, it is has a weaker dependence on the correlations for inter-

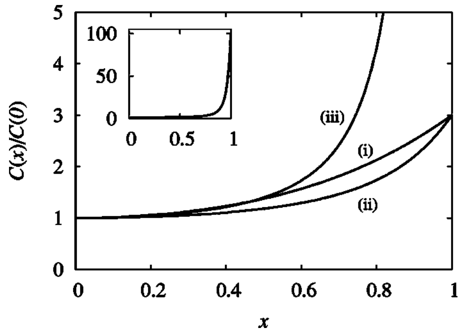

FIG. 1. The enhancement of the failure rate for the three-qubit repetition code (i), the $[\![7,1,3]\!]$ code (ii), and the $[\![23,1,7]\!]$ code (iii). The full curve for the $[\![23,1,7]\!]$ code is plotted in the inset.

mediate values. However, the larger Golay code which corrects up to three-qubit errors has a stronger dependence on x. We expect this will be a general feature of larger correcting codes. They will be more sensitive to correlations because the rate for r-bit errors which are necessary to cause a failure will comparatively be enhanced more strongly.

The fidelity, including the p dependence, is plotted in Fig. 2 for all three codes considered. Note that these surfaces are approximations to the fidelity because we include only the lowest order in p. They do, however, include all contributions in x to this order in p. The $F^2=0$ portions of the graph are imposed by hand wherever our formula gives $F^2<0$. In reality the fidelity must go smoothly to zero as p increases to 1. The enhancement of the failure rate seen in Fig. 1 is reflected in the x dependence of the fidelity with the $[\![23,1,7]\!]$ code showing the sharpest dependence. Comparing Figs.

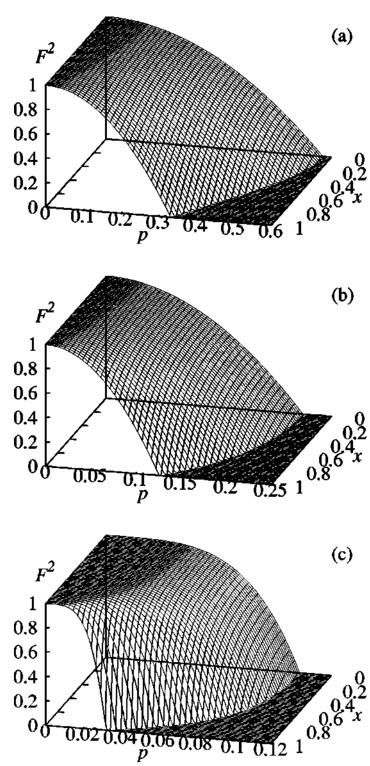

FIG. 2. The fidelity of the three-qubit repetition code (a), the $[\![7,1,3]\!]$ code (b), and the $[\![23,1,7]\!]$ code (c).

2(b) and 2(c) it is interesting to note that the fidelity falls to zero more quickly in p for the larger code while for small p the fidelity is flatter.

III. THRESHOLD RESULTS FOR CONCATENATED CODES

There is a well-known threshold result for fault-tolerant quantum computing [16–18]. The threshold can be derived by concatenating Calderbank-Shor-Steane codes many times while taking account of the scale-up in the number of fundamental operations which are needed to implement logic gates acting on encoded qubits in a fault-tolerant manner. Here we encode a single qubit by concatenating CSS codes an arbitrary number of times. For simplicity use the same code at each level of concatenation. Using an $[\![n,1,d]\!]$ code, at the kth level of concatenation we have n^k physical qubits with all errors on $[(d+1)/2]^k-1$ or fewer qubits corrected. Our approach does not include the scale-up in the number of operations which are needed to extract the error syndrome and apply the error correction because we assume these are carried out perfectly.

The error correction protocol is more complicated for concatenated codes because the errors must be corrected at each level of the concatenation in order to get the maximum protection. That is, we first correct errors on each block of n physical qubits at the lowest level. We then move up to the next level and apply error correction, now treating each block of n qubits as a single logical qubit. Repeating this procedure at each level of concatenation allows any error of $[(d+1)/2]^k-1$ or fewer physical qubits to be corrected.

We should note that we have presented an extremely idealized version of the error correction protocol for concatenated codes. In practice we are always physically interacting with the lowest level of the concatenation and the logic gates between blocks must be implemented in a fault-tolerant manner. This is feasible, but it takes many physical gates to implement a single logic gate and each gate introduces another opportunity for errors to occur. In a practical realization of error correction the procedure we have described is unlikely to be the optimal procedure [5]. Nonetheless, our procedure gives an estimate which represents a lower bound on the effect of correlations on the failure rate for concatenated codes.

Just as for fault-tolerant quantum computing, we find a threshold when we concatenate to arbitrary levels. That is, there is a single-qubit error rate p_{th} such that the fidelity, in the limit as the concatenation depth goes to infinity, becomes

$$F^2=\begin{cases}1, & p<p_{\mathrm{th}},\\ 0, & p>p_{\mathrm{th}},\end{cases} \tag{23}$$

where $p_{\mathrm{th}}\to p_{\mathrm{th}}(x)$ when correlated noise is allowed. For a code $\mathcal{C}$ correcting $t-1=(d-1)/2$ errors concatenated k times the fidelity is

$$F^2_{C,k}=1-p^{t^k}\sum\langle E^2_{i_1}\cdots E^2_{i_{t^k}}\rangle, \tag{24}$$

where the sum accounts for all the t^k-bit errors which cause a failure. As k is increased it quickly becomes unmanageable to directly evaluate this sum. However, the structure of the code allows us to evaluate the sum approximately. Consider working from the top level of concatenation down to the bottom. In order to have a failure (error at the top level) we must have an error in t blocks at the level below and there are $\binom{n}{t}$ ways to choose those blocks. This repeats through the hierarchy of levels so that going down to the $(k-j)$th level there are

$$\binom{n}{t}^{\sum_{i=0}^{k-j} t^i} = \binom{n}{t}^{t^{k-j}-1} \tag{25}$$

ways to choose blocks with errors which propogate up to cause a failure. There remain j levels of the concatenation for which we have performed the sum. We approximate the fidelity as

$$F_{C,k}^2 = 1 - p^{t^k}\binom{n}{t}^{t^{k-j}-1}\left[\sum_{t^j \text{ bits}} \langle E_{i_1}^2 \cdots E_{i_{t^j}}^2\rangle\right]^{t^{k-j}}, \tag{26}$$

where the sum is more manageable but we have assumed that correlations factorize into blocks of n^j qubits. Obviously for $j=0$ we neglect correlations altogether. However, we are able to carry out the sums for $j=1$, 2, and 3 and evaluate to some approximation the contributions to the fidelity from the correlations.

It is now straightforward to derive a threshold condition from the fidelity. Writing the fidelity as

$$F_{C,k}^2 = 1 - \binom{n}{t}^{-1}(pB)^{t^k}, \tag{27}$$

it is clear that for $pB<1$ the fidelity goes to unity and for $pB>1$ it goes to zero as k is increased. This yields a threshold curve

$$p_{\text{th}} = \left[\binom{n}{t}\sum_{t^j \text{bits}} \langle E_{i_1}^2 \cdots E_{i_{t^j}}^2\rangle\right]^{-t^{-j}}, \tag{28}$$

which approaches the true threshold as $j\to\infty$.

We first consider the threshold for the three-qubit repetition code. This code is small enough that we are able to calculate the correlations explicitly for $j=1$, 2, or 3. In other words we include the correlations within a block of 3, 9, or 27 qubits, respectively, while neglecting correlations between qubits in different blocks. For the $[\![7,1,3]\!]$ code we are able to find curves for $j=1$ and 2 and for the $[\![23,1,7]\!]$ code we are only able to calculate for $j=1$.

The threshold curves are plotted in Fig. 3. The presence of correlations in the noise lowers the threshold value below which the code perfectly protects the logical qubit from errors. The curves show a typical reduction of the threshold by a factor of 2 for perfect correlations. However, the $j=3$ curve for the three-qubit code hints that there may be a larger effect when more correlations are included. In particular, as we shall show below, the true curve must go to zero as $x\to 1$.

Another code which exhibits a type of threshold behavior is the n-bit repetition code. This is analogous to the three-bit code discussed previously, protecting a single logical qubit against either bit-flip or phase-flip errors on $(n-1)/2$ or fewer qubits. Note that this code makes more efficient use of qubits than the concatenated three-bit code. The fidelity for the n-bit code to lowest order in p has the form

$$F^2 = 1 - p^{(n+1)/2}\sum \langle E_{i_1}^2 \cdots E_{i_{(n+1)/2}}^2\rangle, \tag{29}$$

where the sum includes all $[(n+1)/2]$-bit errors. In order to see that this code has a thresholdlike behavior as n is increased we define a set of curves

$$p_n(x) = [B_n(x)]^{-2/(n-1)}, \tag{30}$$

where the fidelity of the n-bit code is equal to the fidelity of a single qubit. Here $B_n(x)$ results from the summation over $[(n+1)/2]$-bit errors. These curves, shown in Fig. 4 up to

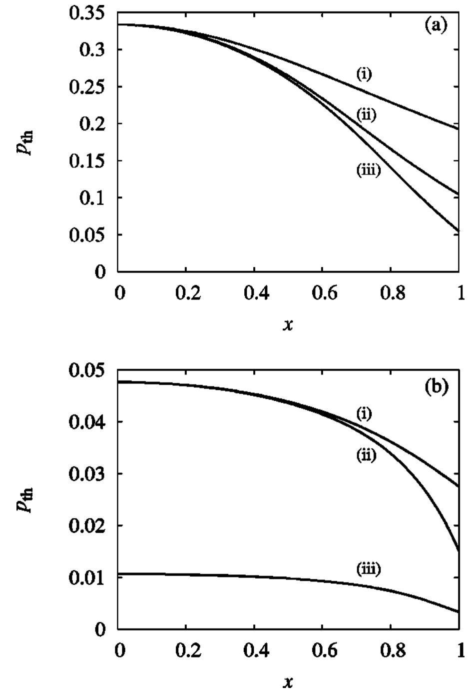

FIG. 3. The threshold curve for the three qubit code (a) for $j=1$ (i), 2 (ii), and 3 (iii). The threshold curves (b) for the $[\![7,1,3]\!]$ code for $j=1$ (i) and 2 (ii) and for the $[\![23,1,7]\!]$ code for $j=1$ (iii).

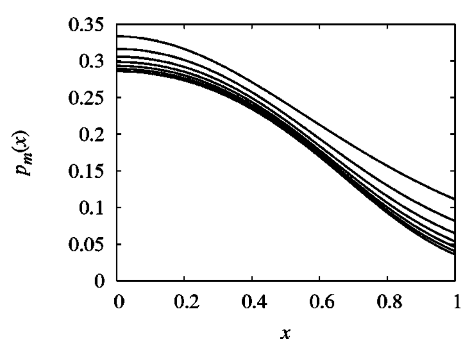

FIG. 4. A set of curves where the fidelity of the n-bit repetition code is equal to the fidelity of a single qubit. Shown for $n=3,5,\ldots,15$, with $n=3$ the uppermost curve and $n=15$ the lowermost curve.

$n=15$, shift steadily downward as n increases. They are asymptotically approaching a threshold curve as $n\to\infty$ with the end points $p_{\text{th}}(0)=1/4$ and $p_{\text{th}}(1)=0$.

These curves can be better understood in the following way. For any point below one of these curves, the total failure probability is reduced if the logical qubit is encoded in the corresponding number of physical qubits. For points between two curves, the failure probability is reduced for the upper curve and enhanced for the lower curve. Physically, even though the probability of n qubits failing decreases with n, the number of combinations increases with n. These two competing effects account for the structure of these curves with increasing n.

We can calculate the correlations exactly for $x=1$. The fidelity for a concatenated $[\![n,1,d]\!]$ code in this case is

$$F_{C,k}^2 = 1 - p^{t^k}\binom{n}{t}^{t^{k-1}}(2t^k-1)!!\,, \tag{31}$$

which leads to $p_{\text{th}}\to 0$ as $x\to 1$ because of the double factorial of t^k. This suggests that perfect correlations completely destroy the ability of the error correcting code to protect against errors. To put this another way, we can make the threshold arbitrarily small by taking x arbitrarily close to 1. For finite x we estimate that, for the repetition code, the threshold may be decreased by an order of magnitude or more if $x\gtrsim 0.8$, which corresponds to a qubit separation of the order of 0.22 coherence lengths.

A different strategy for error correction in the presence of strongly correlated noise is already known—that of decoherent free subspaces (DFS's). This is the subject of the next section.

IV. CSS CODES CONCATENATED WITH A DFS

In this section we explore the possibility that concatenating the error correcting codes of the previous section with a decoherence free subspace provides good protection against partially correlated noise. It should be noted that this has been considered previously [10], although without an explicit noise model.

We first consider encoding our logical qubit with a simple decoherence free subspace. For simplicity we restrict our attention to bit-flip noise. In this case there is a decoherence free subspace of two qubits defined as

$$|0\rangle_L = \frac{1}{2}(|0\rangle + |1\rangle)(|1\rangle - |0\rangle), \tag{32a}$$

$$|1\rangle_L = \frac{1}{2}(|0\rangle - |1\rangle)(|1\rangle + |0\rangle), \tag{32b}$$

where the first (second) term in parentheses refers to the first (second) physical qubit. For perfectly correlated noise this encoding will provide perfect protection of the logical qubit. For partially correlated noise we need the following properties of this DFS:

$$\sigma_{1x}|0\rangle_L = |0\rangle_L, \tag{33a}$$

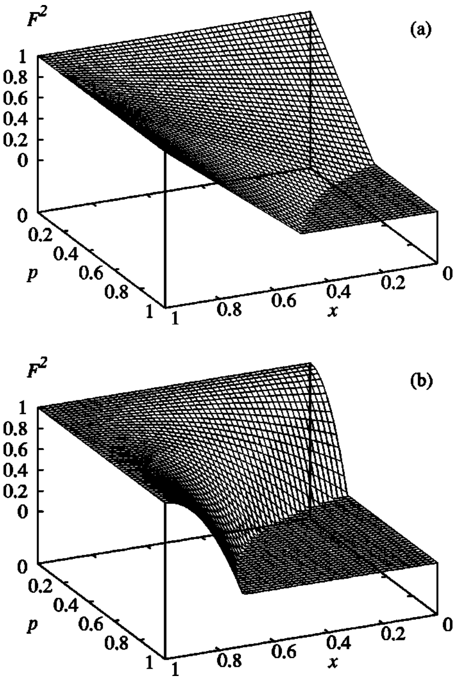

FIG. 5. The fidelity of the bare DFS (a) and the three-bit code concatenated with the DFS (b).

$$\sigma_{2x}|0\rangle_L = -|0\rangle_L, \tag{33b}$$

$$\sigma_{1x}|1\rangle_L = -|1\rangle_L, \tag{33c}$$

$$\sigma_{2x}|1\rangle_L = |1\rangle_L. \tag{33d}$$

A bit-flip error on one physical qubit acts as a phase error on the logical qubit. The fidelity of the bare DFS is

$$F_{\text{DFS}}^2 = 1 - 2p(1-x). \tag{34}$$

Since the repetition code can correct either bit flips or phase errors, it is straighforward to generalize the analysis of the previous section to account for one additional level of concatenation using this DFS to encode each physical qubit. The fidelity can be found by making the replacements

$$\langle E_iE_j\rangle \to 2x^{2|j-i|} - x^{2|j-i|+1} - x^{2|j-i|-1}, \tag{35a}$$

$$\langle E_i^2\rangle \to 2-2x, \tag{35b}$$

in the equations from the previous section. This is justified in Appendix C.

Figure 5 shows the fidelity of the DFS alone and the three-bit code concatenated with the DFS. We see that the bare DFS yields perfect protection for $x=1$ but poor protection for partial correlations. Concatenating the three-bit code with the DFS combines the properties of both codes; i.e., it yields perfect fidelity for $x=1$, but also gives some protection for a range of partial correlations. Compared with the bare three-bit code for $x=0$ the fidelity falls off twice as quickly in p, reflecting the fact there are twice as many physical qubits interacting with the uncorrelated noise fields. In gen-

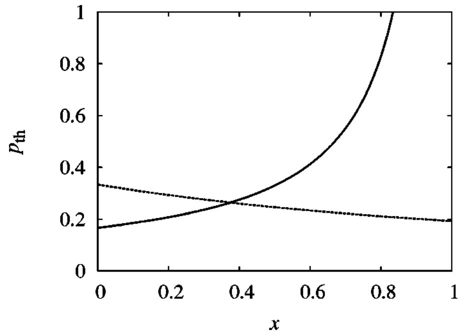

FIG. 6. The threshold curve for the three-bit code concatenated with DFS's (solid line). The threshold for the three-bit code without DFS's (dashed line).

eral concatenating with a k-bit DFS yields a worst-case reduction in the threshold by a factor of k for independent errors. Again, see Appendix C for the proof.

Concatenating the three-bit code many times and then adding the DFS yields a threshold curve just as before, although with a different dependence on x. The simplest approximation is to make the replacements from Eqs. (35) and keeping only correlations within a single (six-bit) block. This yields a threshold curve

$$p_{\rm th} = [3(12 - 24x + 16x^2 - 16x^3 + 24x^4 - 16x^5 + 6x^6 - 8x^7 + 12x^8 - 8x^9 + 2x^{10})]^{-1/2}, \tag{36}$$

which is shown in Fig. 6 along with the threshold for the three-bit code without DFS's for comparison. This illustrates that there is a crossover point $x_c \approx 0.4$ in the correlation strength below which traditional error correction provides the best protection and above which concatenating with a DFS is favorable.

V. CONCLUSION

We have evaluated the performance of quantum error correction in the presence of correlated noise, considering both traditional error correcting codes and decoherence free subspaces. We find that the presence of correlations can enhance the failure probability of traditional error correcting codes by several orders of magnitude for large codes and perfect correlations. This leads to a correlation-dependent threshold in the single-qubit error probability for concatenated codes, with the threshold going to zero for perfectly correlated noise. For intermediate correlations we find that the threshold may be reduced by an order of magnitude for qubit spacings on the order of 0.22 correlation lengths. Finally, concatenating a decoherence free subspace with a traditional code yields perfect protection in the limit of perfect correlation and a maximum reduction in the threshold for independent noise by a factor equal to the number of qubits which define the DFS.

While we have not considered a fault-tolerant scheme for implementing error correction, we believe that a similar model can be applied in a straightforward manner to yield a lower bound on the effect of correlations when the scale-up in the number of fundamental operations is taken into account.

APPENDIX A: QUANTIZED FIELD MODES

We consider a single logical qubit encoded with the three-bit repetition code correcting a single-bit-flip error. The environment is a bath of quantized field modes. The interaction Hamiltonian is

$$H_I = \hbar \sum_{i=1}^{3} \sigma_{ix} B_i, \tag{A1}$$

where the environment operator is

$$B_i = \sum_j g_j (b_j e^{-ik_j x_i} + b_j^\dagger e^{ik_j x_i}). \tag{A2}$$

Here b_j and $b_j^\dagger$ are annihilation and creation operators for the jth field mode. The time evolution operator is

$$U(t) = \exp\left(-\frac{i}{\hbar} H_I t\right), \tag{A3}$$

and the fidelity following error correction is

$$F_{\rm EC}^2 = \langle \psi(0) | {\rm Tr}_F \left\{ \sum_{n=0}^{3} \sigma_{jx} P_j U(t) | \psi(0)\rangle \chi_F \langle \psi(0) U^\dagger(t) P_j \sigma_{jx} \right\} \times | \psi(0)\rangle, \tag{A4}$$

where χ_F is the initial density operator representing the state of the environment and $|\psi(0)\rangle$ is the initial state of the qubits. Keeping all terms to fourth order in time this is

$$F_{\rm EC}^2 = 1 - \gamma \frac{t^4}{4} \sum_{m>n=1}^{3} [\langle B_n B_m B_n B_m \rangle + \langle B_n B_m B_m B_n \rangle + \langle B_m B_n B_n B_m \rangle + \langle B_m B_n B_m B_n \rangle]. \tag{A5}$$

Assuming the environment is initially in the vacuum state this becomes

$$F_{\rm EC}^2 = 1 - \gamma \frac{t^4}{4} \sum_{m>n=1}^{3} [4\langle B_n^2 \rangle \langle B_m^2 \rangle + 6\langle B_n B_m \rangle \langle B_m B_n \rangle + \langle B_n B_m \rangle^2 + \langle B_m B_n \rangle^2], \tag{A6}$$

and under the assumption that each qubit interacts equally with left-and right-going waves this reduces to

$$\begin{aligned} F_{\rm EC}^2 &= 1 - 2\gamma t^4 (\langle B_1 B_3 \rangle \langle B_3 B_1 \rangle + \langle B_1 B_2 \rangle \langle B_2 B_1 \rangle + \langle B_2 B_3 \rangle \times \langle B_3 B_2 \rangle) - \gamma t^4 (\langle B_1^2 \rangle \langle B_3^2 \rangle + \langle B_1^2 \rangle \langle B_2^2 \rangle + \langle B_2^2 \rangle \langle B_3^2 \rangle) \\ &= 1 - \gamma g^4 t^4 [3 + 2(e^{-2|x_1 - x_2|/l_0} + e^{-2|x_1 - x_3|/l_0} + e^{-2|x_2 - x_3|/l_0})], \end{aligned} \tag{A7}$$

where we have assumed $\langle B_i B_j \rangle = g^2 \exp(-|x_i - x_j|/l_0)$. The fidelity has the same form as for classical noise fields.

APPENDIX B: FIDELITY FOR THE ⟦7,1,3⟧ CODE

We calculate the fidelity for the ⟦7,1,3⟧ Hamming code in a manner analogous to the three-bit repetition code. The time evolution operator is

$$U(t) = \prod_{i=1}^{7} [\cos \Omega_i t + i \sin \Omega_i t(\cos \phi_i \sigma_{ix} + \sin \phi_i \sigma_{iy})], \tag{B1}$$

where ϕ_i is the phase of the ith noise field. We have used Mathematica to explicitly calculate the fidelity following error correction as

$$F^2 = |\langle\psi|U(t)|\psi\rangle|^2 + \sum_{i=1}^{7} |\langle\psi|\sigma_{ix}U(t)|\psi\rangle|^2 + \sum_{i=1}^{7} |\langle\psi|\sigma_{iy}U(t)|\psi\rangle|^2. \tag{B2}$$

The result to lowest order in p is

$$F^2 = 1 - \frac{1}{2}\sum_{i\neq j} \Omega_i^2\Omega_j^2 - 6 \sum_{\{i,j,k,l\}} \Omega_i\Omega_j\Omega_k\Omega_l(\cos \phi_i \cos \phi_j \cos \phi_k \cos \phi_l + \sin \phi_i \sin \phi_j \sin \phi_k \sin \phi_l), \tag{B3}$$

where $\{i,j,k,l\}$ in the second sum consists of the set

$$\{\{2,4,5,7\},\{3,4,5,6\},\{2,3,6,7\},\{1,2,3,4\},\{1,2,5,6\},\{1,3,5,7\},\{1,4,6,7\}\}. \tag{B4}$$

The first sum is just what we got by counting all the two-bit errors. The second sum is a new set of terms arising from the partial coherence of the errors and the structure of the ⟦7,1,3⟧ code. These terms do not contribute in the case of independent errors. Averaging over many realizations of the noise fields and assuming Gaussian correlations, the fidelity is

$$F^2 = 1 - p^2(21 + 24x^2 + 31x^4 + 14x^6 + 30x^8 + 4x^{10} + 2x^{12}). \tag{B5}$$

Comparing with Eq. (22) we see that the contribution of the new terms is the same size as the contribution from counting two-bit errors.

APPENDIX C: FIDELITY USING A DFS

The effect of concatenating the three-bit repetition code with a DFS can be accounted for by making the substitution $E_i \to E_{i1} - E_{i2}$ in the expressions for the fidelity. This form comes from the particular DFS used, but clearly if the fields are perfectly correlated, there will be no decoherence. Making this substitution we have the correlations

$$\begin{aligned}\langle E_iE_j\rangle &= \langle(E_{i1} - E_{i2})(E_{j1} - E_{j2})\rangle \\ &= \langle E_{i1}E_{j1}\rangle + \langle E_{i2}E_{j2}\rangle - \langle E_{i1}E_{j2}\rangle - \langle E_{i2}E_{j1}\rangle \\ &= 2x^{2|j-i|} - x^{2|j-i|+1} - x^{2j-i-1}\end{aligned} \tag{C1}$$

and

$$\langle E_i^2\rangle = \langle(E_{i1} - E_{i2})^2\rangle = \langle E_{i1}^2\rangle + \langle E_{i2}^2\rangle - 2\langle E_{i1}E_{i2}\rangle = 2 - 2x. \tag{C2}$$

For a different DFS, for example, the four-bit DFS of Zanardi and Rasetti [9], a different substitution will be necessary. However, in the limit of independent errors we can find the fidelity for a concatenation with any k-bit DFS. The appropriate substitution will replace E_i with a sum over k fields. Because the noise is independent, only correlations of the form $\langle E_{ij}^2\rangle$ will contribute. Then the appropriate correlations following substitution are

$$\langle E_iE_j\rangle \to 0 \tag{C3}$$

and

$$\langle E_i^2\rangle \to k\langle E_i^2\rangle. \tag{C4}$$

This yields a worst-case reduction in the threshold by a factor of k for independent noise.

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