Short-time decoherence for general system-environment interactions

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A short-time approximation is developed for system-environmental bath mode interactions involving a general non-Hermitian system operator Λ and its conjugate $\Lambda^\dagger$ in order to evaluate the onset of decoherence at low temperatures in quantum systems interacting with the environment. The developed approach is complementary to Markovian approximations and appropriate for the evaluation of quantum computing schemes. An example of a spin system coupled to a bosonic heat bath via $\Lambda \propto \sigma_-$ is worked out in detail.

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I. INTRODUCTION

A quantum system exposed to environmental modes is described by the reduced density matrix, and its evolution deviates from the ideal, usually pure-state, dynamics. For short times, appropriate for quantum computing gate functions and, generally, for controlled quantum dynamics, approximation schemes for the density matrix have been developed recently [1–3]. The present work derives a rather general short-time approximation which applies for models with system-bath interactions involving a general system operator. It thus extends the previously known approach [2,3] which was limited to couplings involving a single Hermitian system operator.

We consider an open quantum system with the Hamiltonian

$$H = H_S + H_B + H_I. \tag{1}$$

Here H_S describes the system proper. It is coupled to the environment (bath), described by H_B. The system and bath are coupled by the interaction H_I. The bath has been traditionally modeled [1–9] by a large number of uncoupled bosonic modes: namely, harmonic oscillators (with ground-state energy shifted to zero)

$$H_B = \sum_k \omega_k b_k^\dagger b_k. \tag{2}$$

Here b_k are the bosonic annihilation operators corresponding to the bath modes, and from now on we use the convention $\hbar = 1$.

In most of this work, we consider the general system H_S, and we assume that the interaction with the bath involves the system operator Λ that couples linearly [10–14] to the bath modes:

$$H_I = \Lambda \sum_k g_k b_k^\dagger + \Lambda^\dagger \sum_k g_k^* b_k, \tag{3}$$

with the interaction constants g_k.

Let $R(t)$ denote the overall density matrix. It is commonly assumed [1–14] that at time $t=0$ the system and bath are not entangled, and the bath modes are thermalized:

$$R(0) = \rho(0) \prod_k \theta_k, \tag{4}$$

where

$$\theta_k = \mathrm{Z}_k^{-1} e^{-\beta \omega_k b_k^\dagger b_k}, \tag{5}$$

with $\beta = 1/kT$ and

$$\mathrm{Z}_k \equiv (1 - e^{-\beta \omega_k})^{-1}. \tag{6}$$

We point out that while the quantum system S, described by the reduced density matrix $\rho(t)$, is small, typically two state (qubit) or several qubit, the bath has many degrees of freedom. The combined effect of the bath modes on the system can be large even if each of them is influenced little by the system. This has been the basis for arguments for the harmonic approximation for the bath modes [1–9] and the linearity of the interaction, as well as for the Markovian approximations [10–14] that assume that the bath modes are "reset" to the thermal state by the "rest of the universe" on time scales shorter than any dynamical time of the system interacting with the bath.

The frequencies of the oscillators of the bath are usually assumed to be distributed from zero to some cutoff value ω_c. The bath modes with the frequencies close to the energy gaps of the system, $\Delta E_{ij} = E_i - E_j$, contribute to the "resonant" thermalization and decoherence processes. Within the Markovian schemes, the diagonal elements of the reduced density matrix of the system,

$$\rho(t) = \mathrm{Tr}_B R(t), \tag{7}$$

approach the thermal values $\propto e^{-E_i/kT}$ for large times exponentially, on a time scale T_1. The off-diagonal elements vanish, which represents decoherence, on a time scale T_2, which, for resonant processes, is given by $T_2 \simeq 2T_1$. However, generally decoherence is expected to be faster than thermalization because, in addition to resonant processes, it can involve virtual processes that do not conserve energy. It has been argued that this additional "pure" decoherence is dominated by the bath modes with near-zero frequencies [10,14,15]. At

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low temperatures, this "pure decoherence" is expected [16] to make $T_2 \ll T_1$.

Since the resetting of these low-frequency modes to the thermal state occurs on time scales $1/kT=\beta$, the Markovian approach cannot be used at low temperatures [10,14,15]. Specifically, for quantum computing in solid-state semiconductor-heterostructure architectures [16–23], temperatures as low as a few 10 mK are needed. This brings the thermal time scale to $\beta \sim 10^{-9}$ sec, which is close to the single-qubit control times 10^{-11}–10^{-7} sec [16–23]. Alternatives to the Markovian approximation have been suggested [24–29].

In this work, we generalize the recently suggested scheme [2,3], applicable for Hermitian Λ only, to a wider class of interaction Hamiltonians. We treat the case when the system operator Λ entering the interaction [see Eq. (3)], is not Hermitian. In actual applications in quantum computing, calculations with only a single qubit or few qubits are necessary for evaluation of the local "noise," to use the criteria for quantum error correction [30–35]. For example, the system Hamiltonian is frequently taken proportional to the Pauli matrix σ_z. The interaction operator Λ can be proportional to σ_x, which is Hermitian. Such cases are covered by the short-time approximation developed earlier [2,3]. However, one can also consider models with $\Lambda \propto \sigma_-$. Similarly, models with non-Hermitian Λ are encountered in quantum optics [11]. In Sec. II, we develop our short-time approximation scheme. Results for a spin-boson-type model are given in Sec. III.

II. SHORT-TIME APPROXIMATION

In this section we obtain a general expression for the time evolution operator of the system (1)–(3) within the short-time approximation. The system operators H_S and Λ need not be specified at this stage; the derivation is quite general.

In order to define "short time," we consider dimensionless combinations involving the time variable t. There are several time scales in the problem. These include the inverse of the cutoff frequency of the bath modes, $1/\omega_c$, the thermal time $\beta=1/kT$, and the internal characteristic times of the system $1/\Delta E_{ij}$. Also, there are time scales associated with the system-bath interaction-generated thermalization and decoherence, $T_{1,2}$. The shortest time scale at low temperatures (when β is large) is typically $1/\omega_c$. The most straightforward expansion in t yields a series in powers of $\omega_c t$. The aim of developing more sophisticated short-time approximations [2,3] is to preserve unitarity and obtain expressions approximately valid up to intermediate times, of the order of the system and interaction-generated time scales. The latter property can only be argued for heuristically in most cases and checked by model calculations.

The overall density matrix, assuming a time-independent Hamiltonian over the quantum-computation gate function time intervals [16–23], evolves according to

$$R(t) = U(t)R(0)[U(t)]^\dagger, \tag{8}$$

where

$$U(t) = e^{-i(H_S+H_B+H_I)t} \tag{9}$$

is the evolution operator.

The general idea of our approach is the following. We break the exponential operator in Eq. (9) into products of simpler exponentials. This involves an approximation, but allows us to replace system operators by their eigenvalues, when spectral representations are used, and then calculate the trace of $R(t)$ over the bath modes, obtaining explicit expressions for the elements of the reduced density matrix of the system. For Hermitian coupling operators $\Lambda^\dagger=\Lambda$, our approach reduces to known results [2,3].

We split the exponential evolution operator into terms that do not have any noncommuting system operators in them. This requires an approximation. For short times, we start by using the factorization [36–38]

$$e^{-i(H_S+H_B+H_I)t+O(t^3)} = e^{-(i/2)H_S t}e^{-i(H_I+H_B)t}e^{-(i/2)H_S t}, \tag{10}$$

where we have neglected terms of the third and higher orders in t, in the exponent. The middle exponential in Eq. (10),

$$\Xi \equiv e^{-i(H_B+H_I)t} = e^{-i(H_B+\Lambda G^\dagger+\Lambda^\dagger G)t}, \tag{11}$$

where

$$G \equiv \sum_k g_k^* b_k, \tag{12}$$

still involves noncommuting terms as long as Λ is non-Hermitian. In terms of the Hermitian operators

$$L \equiv \frac{1}{2}(\Lambda + \Lambda^\dagger), \tag{13}$$

$$M \equiv \frac{i}{2}(\Lambda - \Lambda^\dagger), \tag{14}$$

we have

$$\Lambda G^\dagger + \Lambda^\dagger G = L(G + G^\dagger) + iM(G - G^\dagger). \tag{15}$$

We then carry out two additional short-time factorizations within the same quadratic-in-t (in the exponent) order of approximation:

$$\Xi = e^{(1/2)[M(G-G^\dagger)-iH_B]t}e^{(i/2)H_B t} \times e^{-i[H_B+L(G+G^\dagger)]t}e^{(i/2)H_B t}e^{(1/2)[M(G-G^\dagger)-iH_B]t}. \tag{16}$$

This factorization is chosen in such a way that Ξ remains unitary, and for $M=0$ or $L=0$ the expression is identical to that used for the Hermitian case [2,3]. The evolution operator then takes the form

$$U = e^{-(i/2)H_S t}\ \Xi\ e^{-(i/2)H_S t}, \tag{17}$$

with Ξ from Eq. (16), which is an approximation in terms of a product of several unitary operators.

It has been recognized [2,3] that an approximation of this sort is superior to the straightforward expansion in powers of t (or more exactly, $\omega_c t$). The evolution operator is factorized in terms of unitary operators. We used thrice the approximation

$$e^{-i(\mathcal{X}+\mathcal{Y})t+O(t^3)} = e^{-i\mathcal{Y}t/2}e^{-i\mathcal{X}t}e^{-i\mathcal{Y}t/2}. \tag{18}$$

The next-order approximation can be obtained, for instance, by using the relation

$$e^{-i(\mathcal{X}+\mathcal{Y})t+O(t^4)} = e^{-i\mathcal{W}t^3}e^{-i\mathcal{Y}t/2}e^{-i\mathcal{X}t}e^{-i\mathcal{Y}t/2}e^{-i\mathcal{W}t^3}, \tag{19}$$

where

$$\mathcal{W} \equiv \frac{1}{24}[\mathcal{Y},[\mathcal{Y},\mathcal{X}]] - \frac{1}{48}[\mathcal{X},[\mathcal{X},\mathcal{Y}]]. \tag{20}$$

Specifically, for the factorization (10), we have

$$[\mathcal{Y},[\mathcal{Y},\mathcal{X}]] = [H_B,[H_I,H_S]] + [H_I,[H_I,H_S]], \tag{21}$$

$$[\mathcal{X},[\mathcal{X},\mathcal{Y}]] = [H_S,[H_S,H_I]]. \tag{22}$$

These relations illustrate that the present approximation is not perturbative in powers of H_I and also that H_B, which commutes with H_S, drops out of some commutators that enter the higher-order correction terms. This suggests that a redefinition of the energies of the modes of H_B might have only a limited effect on the corrections and serves as one of the heuristic arguments for the validity of the approximation beyond the shortest time scale $1/\omega_c$.

Our goal is to approximate the reduced density matrix of the system. We consider its energy-basis matrix elements

$$\rho_{mn}(t) = \mathrm{Tr}_B\langle m|UR(0)U^\dagger|n\rangle, \tag{23}$$

where

$$H_S|n\rangle = E_n|n\rangle. \tag{24}$$

We next use the factorization (10) and (16) to systematically replace system operators by *c*-numbers by inserting decompositions of the unit operator in the bases defined by H_S, L, and M. First, we collect the expressions (4), (16), (17), and (24) and use two energy-basis decompositions of unity to get

$$\rho_{mn}(t) = \sum_{p\ q} e^{(i/2)(E_n+E_q-E_m-E_p)t}\rho_{pq}(0) \times \mathrm{Tr}_B\left[\langle m|\Xi|p\rangle\prod_k \theta_k\langle q|\Xi^\dagger|n\rangle\right]. \tag{25}$$

We now define the eigenstates of L and M:

$$L|\lambda\rangle = \lambda|\lambda\rangle, \tag{26}$$

$$M|\mu\rangle = \mu|\mu\rangle. \tag{27}$$

The operators Ξ and $\Xi^\dagger$ introduce exponentials in Eq. (25) that contain either L or M in the power. By appropriately inserting $\Sigma_\lambda|\lambda\rangle\langle\lambda|$ or $\Sigma_\mu|\mu\rangle\langle\mu|$ between these exponentials, we can convert all the remaining system operators to *c*-numbers. For convenience, let us define the operators

$$\pi_{\alpha\beta\gamma} = |\alpha\rangle\langle\alpha|\beta\rangle\langle\beta|\gamma\rangle\langle\gamma| \tag{28}$$

and

$$\mathcal{U}_{s_1,s_2,s_3} = e^{s_1 g_k^* b_k t + s_2 g_k b_k^\dagger t - i s_3 \omega_k b_k^\dagger b_k t}. \tag{29}$$

The resulting expression for the trace entering Eq. (25) is

$$\mathrm{Tr}_B\left\{\langle m|\Xi|p\rangle\prod_k \theta_k\langle q|\Xi^\dagger|n\rangle\right\} = \sum_{\mu_j\lambda_j}\langle m|\pi_{\mu_1\lambda_1\mu_2}|p\rangle\langle q|\pi_{\mu_3\lambda_2\mu_4}|n\rangle\prod_k \mathcal{T}_k, \tag{30}$$

where the indices λ and μ label the eigenstates of L and M, respectively, and

$$\begin{aligned}\mathcal{T}_k = \mathrm{Tr}_k\{&\mathcal{U}_{(1/2)\mu_1,-(1/2)\mu_1,1/2}\mathcal{U}_{0,0,-1/2}\mathcal{U}_{-i\lambda_1,-i\lambda_1,1}\\ &\times\mathcal{U}_{0,0,-1/2}\mathcal{U}_{(1/2)\mu_2,-(1/2)\mu_2,1/2}\theta_k\mathcal{U}_{-(1/2)\mu_3,(1/2)\mu_3,-1/2}\\ &\times\mathcal{U}_{0,0,1/2}\mathcal{U}_{i\lambda_2,i\lambda_2,-1}\mathcal{U}_{0,0,1/2}\mathcal{U}_{-(1/2)\mu_4,(1/2)\mu_4,-1/2}\}.\end{aligned} \tag{31}$$

In order to calculate the trace over the *k*th bath mode in Eq. (31), we rearrange the operators using the cyclic property, in such a way that the formula (A1), derived in Appendix A, can be used to simplify products of two or three operators $\mathcal{U}$ at a time. For example, we can transfer the operator $\mathcal{U}_{(1/2)\mu_1,-(1/2)\mu_1,(1/2)}$ to the right-hand side, getting the combination

$$\mathcal{U}_{-(1/2)\mu_4,(1/2)\mu_4,-1/2}\mathcal{U}_{(1/2)\mu_1,-(1/2)\mu_1,1/2}, \tag{32}$$

inside the trace, and use the identity (A1). We then transfer $\mathcal{U}_{0,0,-1/2}$ to the right-hand side and repeat the process, now for the three rightmost operators. After several steps we arrive at the following expression for the trace:

$$\mathcal{T}_k = \mathcal{R}_k\mathrm{Tr}_k\{\theta_k\mathcal{U}_{u,-u^*,0}\}, \tag{33}$$

where

$$u = \frac{i}{\omega_k t}(e^{-i\omega_k t/2}-1)\mu_-'' - \frac{2i}{\omega_k t}e^{-i\omega_k t/2}\sin\frac{\omega_k t}{2}\lambda_- + \frac{i}{\omega_k t}e^{-i\omega_k t/2}(e^{-i\omega_k t/2}-1)\mu_-' \tag{34}$$

and

$$\mathcal{R}_k = \exp\left\{-\frac{i|g_k|^2}{\omega_k^2}\left[\left(\sin\frac{\omega_k t}{2} - \frac{\omega_k t}{2}\right)(\mu_-'\mu_+' + \mu_-''\mu_+'') + (\sin\omega_k t - \omega_k t)\lambda_-\lambda_+ - 4\sin\frac{\omega_k t}{2}\sin^2\frac{\omega_k t}{4}\mu_-'\mu_+'' - 2\sin^2\frac{\omega_k t}{2}(\lambda_-\mu_+'' - \lambda_+\mu_-')\right]\right\}. \tag{35}$$

Here we introduced the variables

$$\mu_\pm' = \mu_1 \pm \mu_4, \tag{36}$$

$$\mu_\pm'' = \mu_2 \pm \mu_3, \tag{37}$$

and

$$\lambda_{\pm} = \lambda_1 \pm \lambda_2. \tag{38}$$

The trace in Eq. (33) can be evaluated, for instance, by using the coherent-states technique (see Appendix B):

$$\mathcal{T}_k = \mathcal{R}_k e^{(pq/2)|g_k|^2 t^2 \coth(\beta\omega_k/2)}. \tag{39}$$

The expression which follows from Eqs. (30), (34), (35), and (39) is

$$\prod_k \mathcal{T}_k = \exp[-\mathcal{P}(t)], \tag{40}$$

where

$$\begin{aligned}\mathcal{P} = {} & B^2(t)(\lambda_-^2 + \mu'_-\mu''_-) + B^2(t/2)(\mu''_- - \mu'_-)^2 - F(t)(\mu''_- - \mu'_-)\lambda_- \\ & - iC(t)\lambda_-\lambda_+ - iC(t/2)(\mu'_-\mu'_+ + \mu''_-\mu''_+) + iS(t)(\lambda_-\mu''_+ \\ & - \lambda_+\mu'_-) - iC_1(t)\mu'_-\mu''_+.\end{aligned} \tag{41}$$

The coefficients here are the spectral sums over the bath modes:

$$B^2(t) = 2\sum_k \frac{|g_k|^2}{\omega_k^2}\sin^2\frac{\omega_k t}{2}\coth\frac{\beta\omega_k}{2}, \tag{42}$$

$$C(t) = \sum_k \frac{|g_k|^2}{\omega_k^2}(\omega_k t - \sin\,\omega_k t). \tag{43}$$

These functions are well known [39,40]. The result also involves the new spectral functions

$$S(t) = -2\sum_k \frac{|g_k|^2}{\omega_k^2}\sin^2\frac{\omega_k t}{2}, \tag{44}$$

$$F(t) = 4\sum_k \frac{|g_k|^2}{\omega_k^2}\sin^2\frac{\omega_k t}{4}\sin\frac{\omega_k t}{2}\coth\frac{\beta\omega_k}{2}. \tag{45}$$

Furthermore, for the sake of convenience we defined

$$C_1(t) = 2C(t/2) - C(t). \tag{46}$$

By using Eqs. (25) and (40), we obtain our final result for the density matrix evolution:

$$\begin{aligned}\rho_{mn}(t) = {} & \sum_{p,q}\sum_{\mu_j\lambda_j} e^{(i/2)(E_n+E_q-E_m-E_p)t}\rho_{pq}(0)\langle m|\pi_{\mu_1\lambda_1\mu_2}|p\rangle \\ & \times\langle q|\pi_{\mu_3\lambda_2\mu_4}|n\rangle e^{-\mathcal{P}},\end{aligned} \tag{47}$$

where the first sum over p and q is over the energy eigenstates of the system; the second sum is over λ_1, λ_2, and $\mu_1, \ldots, \mu_4$, which label the eigenstates of the operators L and M, respectively; see Eqs. (26) and (27).

III. DISCUSSION AND APPLICATION

The result (47) looks formidable in the general case. However, in most applications evaluation of decoherence will require short-time expressions for the reduced density matrix of a single qubit. Few- and multiple-qubit systems will have to be treated by utilizing additive quantities [41–43], accounting for quantum error correction (requiring measurement), etc. For a two-state system—a qubit—the summation in Eq. (47) involves $2^8 = 64$ terms, each a product of several factors, the calculation of which is straightforward. Still, the required bookkeeping is cumbersome, and we utilized the symbolic language Mathematica to carry out the calculation for an illustrative example.

We consider the model [44] defined by

$$H = \mathcal{A}\sigma_z + \sum_k \omega_k b_k^\dagger b_k + \sum_k (g_k\sigma_- b_k^\dagger + g_k^*\sigma_+ b_k), \tag{48}$$

where $\mathcal{A} \geqslant 0$ is a constant, $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ and σ_z are the Pauli matrices, $b_k^\dagger$ and b_k are the bosonic creation and annihilation operators, and g_k are the coupling constants. Physically this model may describe, for example, a qubit interacting with a bath of phonons or a two-level molecule in an electromagnetic field. In the latter case, this is a variant of the multimode Jaynes-Cummings model [11,45]. Certain spectral properties of this model, the field-theoretic counterpart of which is known as the Lee field theory, are known analytically—e.g., [46]. However, the trace over the bosonic modes, to obtain the reduced density matrix for the spin, has not been obtained exactly.

For the model (48) we have $\Lambda = \sigma_-$ and $\Lambda^\dagger = \sigma_+$, so that $L = \sigma_x/2$ and $M = \sigma_y/2$. We have $|\lambda_{1,2}\rangle = (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$, with eigenvalues $\lambda_{1,2} = \pm 1/2$, and $|\mu_{1,2}\rangle = (|\uparrow\rangle \pm i|\downarrow\rangle)/\sqrt{2}$, with eigenvalues $\mu_{1,2} = \pm 1/2$. For the initial state, let us assume that the spin at $t = 0$ is in the excited state $|\uparrow\rangle\langle\uparrow|$, so that the initial density matrix has the form

$$\rho(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{49}$$

Calculation in Mathematica yields the following results for the density matrix elements: $\rho_{12}(t) = 0$ and

$$\begin{aligned}4\rho_{11}(t) = {} & 2 + e^{-2B^2(t)} + e^{-4B^2(t/2)}\cosh(2\,F) \\ & + 2e^{-2B^2(t/2)}\sinh(B_1)\cos(S) + 2e^{-B^2(t/2)}\cos(C_1)\sin(S) \\ & + ie^{-B^2(t)-B^2(t/2)}[e^{iC_1}\sinh(-iS + F) + e^{-iC_1}\sinh(-iS \\ & - F)],\end{aligned} \tag{50}$$

where C_1 was defined in Eq. (46) and

$$B_1(t) = 2B^2(t/2) - B^2(t). \tag{51}$$

Where not explicitly shown, the argument of all the spectral functions entering Eq. (50) is t.

In order to obtain irreversible behavior and evaluate a measure of decoherence, we consider the continuum limit of infinitely many bath modes. We introduce the density of the bosonic bath states $\mathcal{D}(\omega)$, incorporating a large-frequency cutoff ω_c, and replace the summations in Eqs. (42)–(45) by integrations over ω [9,10,39,47]. For instance, Eq. (42) takes the form

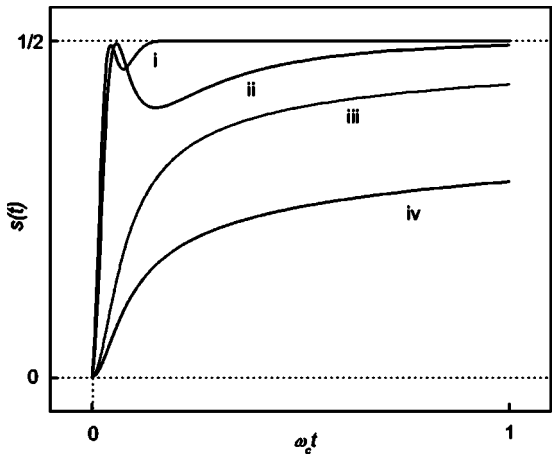

FIG. 1. Schematic behavior of $s(t)$ for different values of Ω, decreasing from i to iv.

$$B^2(t) = \int_0^\infty d\omega \frac{\mathcal{D}(\omega)|g(\omega)|^2}{\omega^2} \sin^2\frac{\omega t}{2} \coth\frac{\beta\omega}{2}. \qquad (52)$$

We will use the standard Ohmic-dissipation [9] expression, with an exponential cutoff, for an illustrative calculation:

$$\mathcal{D}(\omega)|g(\omega)|^2 = \Omega\omega e^{-\omega/\omega_c}, \qquad (53)$$

where Ω is a constant.

We point out that the results obtained for the density matrix elements depend on the dimensionless variable $\omega_c t$, as well as on the dimensionless parameters Ω and $\omega_c\beta$ ($=\hbar\omega_c/kT$, where we remind the reader that $\hbar$, set to 1, must be restored in the final results). Interestingly, the results do not depend explicitly on the energy gap parameter $\mathcal{A}$; see Eq. (48). This illustrates the point that short-time approximations do not capture the "resonant" relaxation processes, but rather only account for "virtual" decoherence processes dominated by the low-frequency bath modes. However, the short-time approximations of the type considered here are meaningful only for systems with a well-defined separation of the resonant versus virtual decoherence processes—i.e., for $\hbar/\mathcal{A} \gg 1/\omega_c$. For such systems, $\hbar/\mathcal{A} = 1/\mathcal{A}$ defines one of the "intermediate" time scales beyond which the approximation cannot be trusted.

As an example, we calculated a measure of deviation of a qubit from a pure state in terms of the "linear entropy" [41,43,48]:

$$s(t) = 1 - \mathrm{Tr}\,[\rho^2(t)]. \qquad (54)$$

Figure 1 schematically illustrates the behavior of $s(t)$ for different Ω values for the case $\omega_c^{-1} \ll \beta$. The values of $s(t)$ increase from zero, corresponding to a pure state, to $1/2$, corresponding to a completely mixed state, with superimposed oscillations. For Ohmic dissipation, three time regimes can be identified [40]. The shortest time scale is set by $t < O(1/\omega_c)$. The quantum-fluctuation-dominated regime corresponds to $O(1/\omega_c) < t < O(1/kT)$. The thermal-fluctuation-dominated regime is $t > O(1/kT)$. Our short-time approximation yields reasonable results in the first two regimes. For $t > O(1/kT)$ it cannot correctly reproduce the process of thermalization. Instead, it predicts an approach to the maximally mixed state.

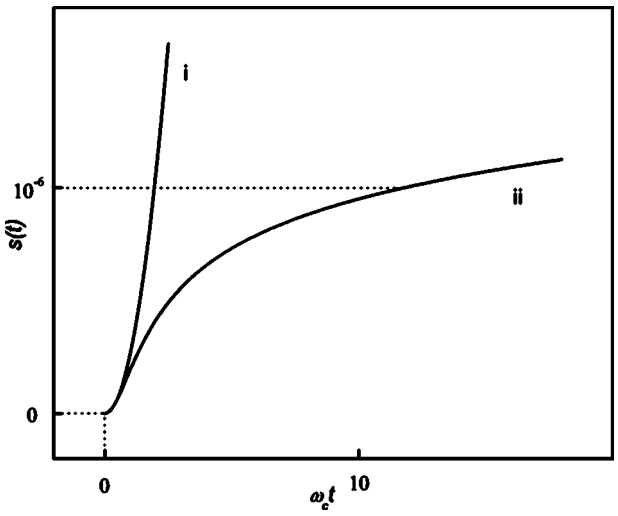

FIG. 2. The comparison between the $O(t^2)$ expansion, i, and the short-time approximation, ii.

Figure 2 corresponds to the parameter values typical for low temperatures and appropriate for quantum computing applications, $\omega_c\beta = 10^3$, with $\Omega = 1.5 \times 10^{-7}$ chosen to represent weak enough coupling to the bath to have the decoherence measure reach the threshold for fault tolerance, of order 10^{-6}, for "gate" times well exceeding $1/\omega_c$, here for $\omega_c t$ over 10. The leading-order quadratic expansion in powers of the time variable t is also shown. Its validity is limited to $t < O(1/\omega_c)$ and it cannot be used for evaluation of quantum-computing models.

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APPENDIX A

Our aim is to derive a relation of the form

$$\mathcal{U}_{v_1,w_1,x}\mathcal{U}_{v_3,w_3,0}\mathcal{U}_{v_2,w_2,-x} = \kappa\mathcal{U}_{p,q,0}, \qquad (A1)$$

where the operator $\mathcal{U}_{s_1,s_2,s_3}$ was defined in Eq. (29). Consider the quantity

$$\Delta = e^{x(b^\dagger+\alpha_1)(b+\alpha_2)} e^{\gamma_1 b^\dagger + \gamma_2 b} e^{-x(b^\dagger+\beta_1)(b+\beta_2)}, \qquad (A2)$$

where $b^\dagger$ and b are the bosonic creation and annihilation operators, and x, α_i, and β_i are c-numbers. Let us use the identity [11]

$$e^{\alpha b - \beta b^\dagger} f(b, b^\dagger) e^{-\alpha b + \beta b^\dagger} = f(b + \beta, b^\dagger + \alpha) \qquad (A3)$$

to represent the first and third exponentials in Δ in the form

$$e^{x(b^\dagger+\alpha_1)(b+\alpha_2)} = e^{\alpha_1 b - \alpha_2 b^\dagger} e^{x b^\dagger b} e^{-\alpha_1 b + \alpha_2 b^\dagger}, \qquad (A4)$$

$$e^{-x(b^\dagger+\beta_1)(b+\beta_2)} = e^{\beta_1 b - \beta_2 b^\dagger} e^{-xb^\dagger b} e^{-\beta_1 b + \beta_2 b^\dagger}. \quad \text{(A5)}$$

We then combine the second exponential in Eq. (A2) and the last and first exponentials in Eqs. (A4) and (A5), by utilizing the identity

$$e^{\alpha b} e^{\beta b^\dagger} = e^{(1/2)\alpha\beta} e^{\alpha b + \beta b^\dagger}, \quad \text{(A6)}$$

which follows from Eq. (A3). The resulting exponential operator, with exponent linear in b and $b^\dagger$, is sandwiched between $e^{xb^\dagger b}$ and $e^{-xb^\dagger b}$. Therefore, the following identity can be utilized [11]:

$$e^{xb^\dagger b} f(b, b^\dagger) e^{-xb^\dagger b} = f(be^{-x}, b^\dagger e^{x}). \quad \text{(A7)}$$

Once again using Eq. (A6), we arrive at the expression

$$\Delta = e^{\nu b + \mu b^\dagger + r}, \quad \text{(A8)}$$

where

$$\mu = (\alpha_2 - \beta_2)(e^x - 1) + \gamma_1 e^x, \quad \text{(A9)}$$

$$\nu = (\beta_1 - \alpha_1)(e^{-x} - 1) + \gamma_2 e^{-x}, \quad \text{(A10)}$$

and

$$r = -2(\alpha_1\beta_2 - \alpha_2\beta_1)\sinh^2\frac{x}{2} + (\alpha_1\alpha_2 - \beta_1\beta_2)\sinh x + \frac{1}{2}\gamma_1(\alpha_1 + \beta_1)(e^x - 1) + \frac{1}{2}\gamma_2(\alpha_2 + \beta_2)(e^{-x} - 1). \quad \text{(A11)}$$

Now Eq. (A1) follows, with

$$\kappa = \exp\left[\frac{2|g_k|^2}{x^2\omega_k^2}\sin^2\left(\frac{x\omega_k t}{2}\right)(v_1 w_2 - v_2 w_1)\right] \times \exp\left[\frac{i|g_k|^2}{x^2\omega_k^2}[\sin(x\omega_k t)(v_1 w_1 - v_2 w_2) + x\omega_k t(v_2 w_2 - v_1 w_1)]\right] \exp\left[\frac{i|g_k|^2 t}{2x\omega_k}[(e^{-ix\omega_k t} - 1)w_3(v_1 - v_2) + (e^{ix\omega_k t} - 1)v_3(w_1 - w_2)]\right], \quad \text{(A12)}$$

and

$$p = -\frac{i}{x\omega_k t}(e^{ix\omega_k t} - 1)(v_1 + v_2) + v_3 e^{ix\omega_k t}, \quad \text{(A13)}$$

$$q = \frac{i}{x\omega_k t}(e^{-ix\omega_k t} - 1)(w_1 + w_2) + w_3 e^{-ix\omega_k t}. \quad \text{(A14)}$$

APPENDIX B

Let us calculate the trace in Eq. (33) which has the form

$$\mathcal{T} \equiv \mathrm{Tr}\{e^{\delta b^\dagger b} e^{vb + wb^\dagger}\}, \quad \text{(B1)}$$

where we omitted the index k since all the calculations here are in the space of a single mode. We use the coherent-state technique [11]. The coherent states $|z\rangle$ by definition are eigenstates of the annihilation operator b:

$$b|z\rangle = z|z\rangle, \quad \text{(B2)}$$

with complex eigenvalues $z = x + iy$. These states are not orthogonal,

$$\langle z_1|z_2\rangle = \exp\left(z_1^* z_2 - \frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2\right), \quad \text{(B3)}$$

and they form an overcomplete set. The identity operator can be written as

$$\int d^2 z |z\rangle\langle z| = 1, \quad \text{(B4)}$$

where integration in the complex plane is defined via

$$d^2 z = \frac{1}{\pi} dx dy. \quad \text{(B5)}$$

We represent the trace (B1) by the coherent-state integral using the relation

$$\mathrm{Tr}\, A = \int d^2 z \langle z|A|z\rangle, \quad \text{(B6)}$$

where A is an arbitrary operator. We then use the normal ordering, $\mathcal{N}$, formula for bosonic operators, represented schematically (see [11] for details) by

$$e^{\delta b^\dagger b} = \mathcal{N} e^{b^\dagger (e^\delta - 1) b}. \quad \text{(B7)}$$

The second term in the trace in Eq. (B1) is split by using Eq. (A6). All instances of b and $b^\dagger$ can then be replaced by z and z^*, and the integral evaluated to yield the expression for the trace:

$$\mathcal{T} = \frac{e^{(wv/2)\coth(\delta/2)}}{1 - e^\delta}. \quad \text{(B8)}$$

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