Two-qubit entanglement dynamics in a symmetry-broken environment

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We study the temporal evolution of entanglement pertaining to two qubits interacting with a thermal bath. In particular we consider the simplest nontrivial spin bath models where symmetry breaking occurs and treat them by mean field approximation. We analytically find decoherence free entangled states as well as entangled states with an exponential decay of the quantum correlation at finite temperature.

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I. INTRODUCTION

Since 1935 [1,2], entanglement has been recognized as one of the most puzzling features of quantum mechanics. However, it is nowadays a widespread opinion that it also represents a fundamental resource for many quantum information protocols. As such, entanglement deserves to be analyzed in all respects. A primary concern is its robustness against environmental effects, and a supplied literature exists aimed at preserving entanglement coherence [3–9]. More recently, attention has been devoted to the problem of *thermal entanglement* [10], i.e., quantifying entanglement arising in spin chains at thermal equilibrium with a bath. In this approach environment determines the temperature T to allow for a thermal distribution of system energy levels, while the detailed interaction between system and environment is not an essential part of the matter. The same is true also for those works that focus on entanglement decoherence [11,12] (also known as *disentanglement* [13,14]). In this context the study of entanglement time behavior is carried on with a master equation formalism and Markovian approximation [15] or, more generally, with arguments provided by spin-boson models.

In the present paper we are going to envisage an approach to the problem along the line introduced for the first time in Ref. [16] (a similar outline but supported by numerical means is also present in [17]). There, the authors considered a one spin system interacting with a fermionic environment endowed with a structure capable of symmetry breaking [18]. It was shown by analytical methods that coherence time increases as magnetic order enlarges or, in other terms, as temperature decreases. Here we extend this argument to a two-qubit system plunged in a fermionic environment described by *transverse Ising model* (TIM) and *Ising model* (IM) [18]. We shall examine the time evolution of concurrence of the bipartite system [19], and find environment-limited concurrences as well as unlimited ones according to environment ordering level.

The paper is organized as follows: in Sec. II we introduce the model by referring to [16] and we revise some results. In Sec. III we extend the model to a bipartite systems, and we present the results of paradigmatic cases in Sec. IV. Finally, Sec. V is for conclusions. Explicit calculations are reported in Appendixes A and B.

II. THE MODEL

We consider the general scenario of a system and a bath described by Hamiltonians H_s and H_B respectively, and interacting through the Hamiltonian H_{sB}. The total Hamiltonian is then $H=H_s+H_{sB}+H_B$, and the initial density matrix is assumed to be factorized, i.e., $\rho(0)=\rho_s\otimes\rho_B$. We are looking for the time evolution of the reduced system density matrix $\rho_s(t)$; in particular we are interested in its off diagonal elements, the so called "coherences." If H does not depend on time the total density matrix will evolve accordingly to

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}. \tag{1}$$

We can then obtain the reduced density matrix by tracing out the bath degrees of freedom in Eq. (1)

$$\rho_s(t) = \mathrm{tr}_B\rho(t). \tag{2}$$

We now follow the line sketched in Ref. [16] to introduce the model for a spin system interacting with a spin bath. First of all, we assume the bath density matrix having a thermal distribution, that is $\rho_B=(e^{-H_B/T})/Z$, with T the bath temperature multiplied by the Boltzmann constant, and $Z=\mathrm{tr}(e^{-H_B/T})$ the partition function. Furthermore, we ask the bath Hamiltonian to be a "symmetry breakable" one, that is endowed with phase transition in the degrees of freedom that provide the coupling with the system. The simplest Hamiltonian with these requirements is a long ranged *Ising model*-like one (IM). We add to it a transverse field to include a more general case in the analysis, dealing eventually with a *transverse Ising model* bath Hamiltonian (TIM). The differences between the two models are minimal as coherence and entanglement is concerning and, in any case, we will be able to find results for IM in the limit of no transverse field for TIM. These peculiar environment Hamiltonians will be studied through mean field approximation [18].

A. TIM environment

Let us consider $N+1$ spin-$\frac{1}{2}$, and let S_j^α be the α component ($\alpha=x,y,z$) of the jth spin ($j=0,1,\ldots,N$). The label $j=0$ refers to the system operators while $j=1,\ldots,N$ to the bath operators. Furthermore, $S_j^\pm=(S_j^x\pm iS_j^y)/2$ are the spin flip operators, and $|0\rangle$ and $|1\rangle$ are the lower and upper eigenstates of S^z. The following Hamiltonians define the energy of the system, of the TIM bath and of the interaction between them:

$$H_s = -\mu_0 S_0^z, \tag{3a}$$

$$H_{sB} = -\frac{J_0}{\sqrt{N}} S_0^z \sum_k S_k^z, \tag{3b}$$

$$H_B = -w\sum_k S_k^x - \frac{J}{N}\sum_{i,k} S_i^z S_k^z, \tag{3c}$$

where μ_0 is the coupling constant with an external magnetic field parallel to the $\hat{z}$ axis, J_0, J are exchange coupling constants and w is the strength of the transverse field; they are all non-negative constants. The indices of the sums run from 1 to N. Equation (3c) describes a material in which spins compete to align along the positive direction of $\hat{x}$ axis or along $\hat{z}$ axis following a ferromagnetic behavior; of course in the latter case the absolute direction of alignment is not important since the Hamiltonian is symmetric in z operators. We can notice that energy exchanges between system and bath are not included in the interaction Hamiltonian; this will generate a pure dephasing dynamics, in which energy will be conserved, and temporal evolution analytically solved.

The main difficulty with Eqs. (3a)–(3c) is represented by the nonlinear term in H_B. For this reason it is helpful to approximate it with a mean field bath Hamiltonian, as explained in [16]:

$$H_B^{mf} = -w\sum_k S_k^x - 2Jm\sum_k S_k^z + m^2 JN. \tag{4}$$

In the above equation m is the order parameter of the phase transition. Its absolute value ranges from 0 to $\frac{1}{2}$ as long as temperature ranges from the critical value $T_c = J/2$ to 0: the greater $|m|$ the larger the magnetic order of the bath along $\hat{z}$ axis. In the following we are going to consider only positive values for m since results are sign-independent. Everything remains true with the substitution $m \to -m$. This is a consequence of H_B z symmetry, that is not lost in H_B^{mf}. The order parameter m is implicitly defined by the following self-consistent equation for the quantity $\Theta = \pm\sqrt{w^2 + 4m^2J^2}$ (also Θ's sign, written here for the sake of precision, is irrelevant, for the same reasons of m's):

$$\frac{\Theta}{J} = \tanh\frac{\Theta}{2T}. \tag{5}$$

It is worth noting that from Eq. (5) we have $\Theta \to J$ for $T \to 0$; furthermore, from the definition of Θ, we can see it tends to $2mJ$ in the limit of no transverse field ($w \to 0$).

Together with Eq. (5) we must consider the following condition on the transverse field to obtain an ordered phase with TIM:

$$\frac{w}{J} < \tanh\left(\frac{w}{2T}\right). \tag{6}$$

This condition is not satisfied in the range of temperatures above T_c; for this reason the whole formalism we are using is valid only in the broken phase.

With the linearized mean field bath Hamiltonian it is possible to evaluate the coherence of the system (see Appendix A):

$$\begin{aligned} S_0^-(t) &= \mathrm{tr}_B\{e^{-iH^{mf}t}[S_0^-(0) \otimes \rho_B]e^{iH^{mf}t}\} \\ &= \frac{1}{Z}\mathrm{tr}_B[e^{-iH^{mf}t}(|0\rangle\langle 1| \otimes e^{-H_B^{mf}/T})e^{iH^{mf}t}] \\ &= S_0^-(0) r_{TIM}(t), \end{aligned} \tag{7}$$

where $H^{mf} = H_s + H_{sB} + H_B^{mf}$, and

$$r_{TIM}(t) = \left[\cos\left(\frac{tmJJ_0}{\Theta\sqrt{N}}\right) + i\frac{\Theta}{J}\sin\left(\frac{tmJJ_0}{\Theta\sqrt{N}}\right)\right]. \tag{8}$$

Equation (7) tells us that the time evolution of the off diagonal term of the system density matrix, responsible for the coherence of the system, is enclosed in the time behavior of the complex valued factor $r_{TIM}(t)$. In particular, in order to find system decoherence, we ask whether and when this factor's absolute value goes to zero. In the limit of large N we can approximate it as

$$|r_{TIM}(t)| \approx \exp\left[-\frac{J_0^2 m^2 t^2}{2}\left(\frac{J^2}{\Theta^2} - 1\right)\right]. \tag{9}$$

We can see from Eq. (9) that the system coherence decays exponentially with time. The coherence time is

$$\tau_{TIM} = \frac{|\Theta|}{J_0 m}\sqrt{\frac{2}{J^2 - \Theta^2}}, \tag{10}$$

and increases as temperature decreases; for $T=0$ it is $\tau = \infty$, and the system remains coherent. This is quite a counterintuitive effect since collective quantum properties of materials endowed with phase transition disappear as ordering increases (see for instance Ref. [20]). The factor t^2 in the exponent denotes the intrinsically reversible nature of the process, in contrast to irreversibility introduced by Markovian approach, and is closely related to the "Zeno effect" [21]. In particular the periodicity of $r_{TIM}(t)$ in Eq. (8) leads to the so-called "recoherences" on a Poincaré time scale. Decoherence takes place in the limit of an environment with infinite degrees of freedom; besides, the same limit is necessary to support the mean field theory approach we adopted. Thus in this context the limit $N \to \infty$ has a double function: to take into account the decoherence process and to give a meaning to the mean field approximation written above.

We briefly notice here that the factor $r_{TIM}(t)$ in Eq. (9) is exactly alike to $r_{as}(t)$ of Eq. (32) in [16]. But as far as that paper is concerned we must point out some inaccuracies: the final result (32) is correct, but the intermediate steps to find it are not. In particular the general formula (11) applies only if the 3×3 matrices Λ_k commute, and this is not true when you look at Eq. (29) of that article. For this reason the intermediate formula (30) is wrong and the oscillations showed in Fig. 1 are not present.

B. Limit of no transverse field: IM environment

In the limit of $w \to 0$ we obtain from Eqs. (3a)–(3c) the IM Hamiltonians which lead to

$$|r_{IM}(t)| \approx \exp\left[-\frac{J_0^2 t^2}{2}\left(\frac{1}{4}-m^2\right)\right]. \tag{11}$$

We can notice the same behavior as for TIM bath, but slightly more transparent: the coherence time is $\tau_{IM} = (2/J_0)\sqrt{2/(1-4m^2)}$ and its limits are $\tau_{IM}^{(T=T_c)} = 2\sqrt{2}/J_0$ and $\tau_{IM}^{(T=0)} = \infty$. We note that coherence explicit dependence on bath coupling constant J has disappeared in this case; only interaction coupling constant J_0 enters coherence expression when the bath is an IM one. Otherwise, the J coupling is indirectly present in Eq. (11) because it has a role in determining the order parameter m by means of Eq. (5).

III. THE EXTENSION

In this section we extend results obtained in the previous one by considering a two-qubit system, and studying the time evolution of their entanglement. We assume that the system qubits, labeled by 01 and 02, interact between them and with environment, that is symmetry-breakable and modeled by TIM Hamiltonians generalizing those of Eqs. (3a)–(3c):

$$H_s = -\xi_0 S_{01}^z S_{02}^z, \tag{12a}$$

$$H_{sB} = -\frac{J_0}{\sqrt{N}}(S_{01}^z + S_{02}^z)\sum_k S_k^z, \tag{12b}$$

$$H_B = -w\sum_k S_k^x - \frac{J}{N}\sum_{i,k} S_i^z S_k^z. \tag{12c}$$

In the above equations ξ_0 represents the coupling constant between the qubits. We have discarded both local interactions, like that between qubits and an external magnetic field, and local couplings with environment degrees of freedom, a situation resembling a "collective" system-environment pairing [7].

As a measure of entanglement between two qubits we adopt the so called "concurrence" [19], which ranges from 0 for separable states to 1 for maximally entangled states. The concurrence is given by

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{13}$$

where λ_1, λ_2, λ_3 and λ_4 are the square roots of the eigenvalues, in decreasing order, of the matrix $R = \rho_s\tilde{\rho}_s$. Here ρ_s is the density matrix of the 2 system qubits, and $\tilde{\rho}_s$ is the "time reversed" matrix given by

$$\tilde{\rho}_s = (\sigma_{01}^y \otimes \sigma_{02}^y)\rho_s^*(\sigma_{01}^y \otimes \sigma_{02}^y), \tag{14}$$

where σ's are the usual Pauli matrices. The symbol ρ_s^* means complex conjugation of the matrix ρ_s in the standard basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.

We assume that the qubits are initially decoupled from the environment, and the bath having a thermal density matrix $\rho_B = (e^{-H_B/T})/Z$. Therefore, we can write the whole state as

$$\rho = |\Psi\rangle\langle\Psi| \otimes \rho_B \tag{15}$$

with a generic system pure state:

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle, \quad |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1. \tag{16}$$

The steps to find time evolution of Eq. (15) are similar to those leading to Eq. (7) (see Appendix A), but now operators are represented by 4×4 matrices, being our system composed by two qubits. After mean field approximation (4) for the bath Hamiltonian and some elementary algebra we obtain the reduced density matrix as

$$\rho_s(t) = \mathrm{tr}_B(\rho(t)) = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta A^* e^{-(1/2)it\xi_0} & \alpha^*\gamma A^* e^{-(1/2)it\xi_0} & \alpha^*\delta B^* \\ \alpha\beta^* A e^{(1/2)it\xi_0} & |\beta|^2 & \beta^*\gamma & \beta^*\delta A^* e^{(1/2)it\xi_0} \\ \alpha\gamma^* A e^{(1/2)it\xi_0} & \beta\gamma^* & |\gamma|^2 & \gamma^*\delta A^* e^{(1/2)it\xi_0} \\ \alpha\delta^* B & \beta\delta^* A e^{-(1/2)it\xi_0} & \gamma\delta^* A e^{-1/2it\xi_0} & |\delta|^2 \end{pmatrix}, \tag{17}$$

where the coefficients

$$A = \left[\cos\left(\frac{tmJJ_0}{\Theta\sqrt{N}}\right) + i\frac{\Theta}{J}\sin\left(\frac{tmJJ_0}{\Theta\sqrt{N}}\right)\right]^N, \tag{18a}$$

$$B = \left[\cos\left(\frac{2tmJJ_0}{\Theta\sqrt{N}}\right) + i\frac{\Theta}{J}\sin\left(\frac{2tmJJ_0}{\Theta\sqrt{N}}\right)\right]^N, \tag{18b}$$

characterize the time dependence of the concurrence. From the above expression of $\rho_s(t)$ we can find the matrix $R(t)$ and its eigenvalues, and from them, as explained, the final concurrence of the system. The complete expression for $R(t)$ and for coefficients of Eqs. (18a) and (18b) is given in Appendix B. In the following we are going to consider some paradigmatic cases for the initial state (15).

IV. PARADIGMATIC CASES

A. Case 1

Let us set $\alpha=\delta=0$ in Eq. (16) for the initial state of the system. We obtain $|\Psi\rangle=\beta|01\rangle+\gamma|10\rangle$ and R matrix reduces to

$$R(t)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2|\beta|^2|\gamma|^2 & 2\beta^*|\beta|^2\gamma & 0 \\ 0 & 2\beta\gamma^*|\gamma|^2 & 2|\beta|^2|\gamma|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

whose square rooted eigenvalues are

$$\lambda_1 = 2|\beta||\gamma|, \tag{20a}$$

$$\lambda_2=\lambda_3=\lambda_4=0. \tag{20b}$$

This leads to the following concurrence:

$$C_{TIM}=2|\beta||\gamma|. \tag{21}$$

The entanglement results time independent, so the state does not perceive the presence of the environment. The reason is that $|\Psi\rangle$ is an eigenstate of the interaction Hamiltonian and so it represents a decoherence free entangled state [7]. Since w is not present in the concurrence written above we know that the expression for the concurrence would be exactly the same for an IM environment.

B. Case 2

Now we set $\beta=\gamma=0$ in Eq. (16) and obtain the state $|\Psi\rangle=\alpha|00\rangle+\delta|11\rangle$. The R matrix becomes

$$R(t)=\begin{pmatrix} |\alpha|^2|\delta|^2(1+|B|^2) & 0 & 0 & 2\alpha^*|\alpha|^2\delta B^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\alpha\delta^*|\delta|^2 B & 0 & 0 & |\alpha|^2|\delta|^2(1+|B|^2) \end{pmatrix}, \tag{22}$$

with square rooted eigenvalues in decreasing order:

$$\lambda_1 = |\alpha||\delta|(|B|+1), \tag{23a}$$

$$\lambda_2 = |\alpha||\delta|(||B|-1|), \tag{23b}$$

$$\lambda_3=\lambda_4=0. \tag{23c}$$

From Eqs. (18a) and (18b), for large N, we get

$$|B| \approx \exp\left[-2J_0^2 m^2 t^2\left(\frac{J^2}{\Theta^2}-1\right)\right]. \tag{24}$$

Then, by using concurrence definition and Eqs. (23a)–(23c), we arrive at

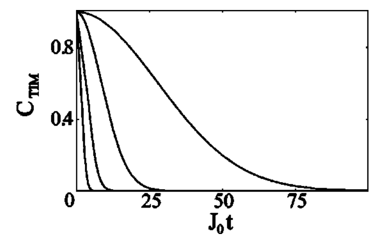

FIG. 1. Concurrence versus scaled time $J_0 t$. Curves from the left to the right are for $T/T_c=\{0.75, 0.50, 0.35, 0.25\}$. Values of the other parameters are $w=0.1$ and $J=2$.

$$C_{TIM}=2|\alpha||\delta||B|=2|\alpha||\delta|\exp\left[-2J_0^2 m^2 t^2\left(\frac{J^2}{\Theta^2}-1\right)\right]. \tag{25}$$

The time behavior of the concurrence just obtained is shown in Fig. 1 for different values of the ratio T/T_c. We notice that in this case the qubits perceive the presence of the thermal bath, which spoils entanglement between them; in fact the initial state is no longer an eigenstate of the interaction Hamiltonian. Only for zero temperature the order parameter reaches its saturation value and the concurrence remains constant. The behavior is very similar to that of one-qubit system coherence described by Eq. (7), but entanglement decoherence is exactly twice faster than one-qubit decoherence. This result agrees with what was found in [12]. Furthermore, together with the previous case, it falls within the general limitations represented by the *universal disentangling machine* [13].

In the limit $w\to 0$ we obtain the concurrence for an IM bath:

$$C_{IM}=2|\alpha||\delta|\exp\left[-2J_0^2 t^2\left(\frac{1}{4}-m^2\right)\right]. \tag{26}$$

Analogously to what was already noticed for the single-qubit coherence, in this limit the factor J disappears from the explicit concurrence expression. The only exchange coupling constant that enters in the decoherence time for the concurrence is J_0.

C. Case 3

If we set $\alpha=\beta=0$ we obtain a product state $|\Psi\rangle=\gamma|10\rangle+\delta|11\rangle=(\gamma|0\rangle+\delta|1\rangle)|1\rangle$, which trivially gives

$$R(t)=(\mathbf{0}) \Rightarrow C=0. \tag{27}$$

In this case TIM Hamiltonians are not able to induce entanglement between system qubits.

D. Case 4

If we set $\alpha=\beta=\gamma=\delta=\frac{1}{2}$ we obtain again a separable initial state, but different from the previous one: $|\Psi\rangle=\frac{1}{2}(|00\rangle+|01\rangle+|10\rangle+|11\rangle)=(1/\sqrt{2})(|0\rangle+|1\rangle)(1/\sqrt{2})(|0\rangle+|1\rangle)$. In this case the R matrix is not trivial:

$$R(t) = \frac{1}{16}\begin{pmatrix} 1+|B|^2-2|A|^2e^{-it\xi_0} & U_{\xi_0} & U_{\xi_0} & 2B^*-2(A^*)^2e^{-it\xi_0} \\ -V_{\xi_0} & 2-2|A|^2e^{it\xi_0} & 2-2|A|^2e^{it\xi_0} & -U_{\xi_0} \\ -V_{\xi_0} & 2-2|A|^2e^{it\xi_0} & 2-2|A|^2e^{it\xi_0} & -U_{\xi_0} \\ 2B-2A^2e^{-it\xi_0} & V_{\xi_0} & V_{\xi_0} & 1+|B|^2-2|A|^2e^{-it\xi_0} \end{pmatrix}, \tag{28}$$

where

$$U_{\xi_0} = [2A^*e^{-(1/2)it\xi_0} - (A^* + AB^*)e^{(1/2)it\xi_0}], \tag{29a}$$

$$V_{\xi_0} = [2Ae^{-(1/2)it\xi_0} - (A + A^*B)e^{(1/2)it\xi_0}]. \tag{29b}$$

Concurrence is valuable explicitly, but the expression is much too cumbersome and therefore is not reported here. We only show in Fig. 2 its behavior. The concurrence starts from its null value and increases because of the interaction between system qubits. If there was no disentanglement it would reach its maximum and decrease again giving rise to oscillations of equal amplitude. Nevertheless, the presence of environment alters this temporal behavior damping the oscillations. For suitable values of coupling constants it can even prevent qubits from entangling at all. The interesting question of the maximal entanglement generation under dephasing processes arises naturally in this case [11].

V. CONCLUSION

We have studied time behavior of entanglement between two qubits dipped in a large symmetry-breakable fermionic environment, below the critical temperature T_c. In the frame of mean field theory analytical results are provided for concurrence of the bipartite system, with temperature as a parameter of the problem. Hamiltonians involved in the discussion are those typical of *transverse Ising models* (TIM), capable of magnetic ordering under suitable conditions. To assign them a physical meaning we notice that, upon addition of a transverse field in H_s, our model resembles an array of Rydberg atoms interacting with a cavity mode of the radiation field [16]. Nevertheless such an assumption for H_s makes the problem unsolvable by analytical techniques, and requires numerical investigation that we plan to accomplish in the near future. Beside that an important improvement would be to overcome mean field approximations adopted in the text, by including the effect of fluctuations, or by applying the *spin wave* approach [22] to the bath.

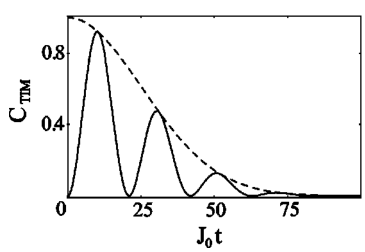

FIG. 2. Concurrence versus the scaled time J_0t. The plot shows the limiting role of decoherence (dotted line) that falls down exponentially, on entanglement (continuous line). The value of parameters are $w=0.1$, $\xi_0=0.3$, $J=2$, and $T/T_c=.25$.

What comes out from the paper is quite a counterintuitive conservation of entanglement in a bath with strong interactions: the bigger the coupling strength (or the lower the ratio T/T_c) the longer the time qubits remain entangled [Eq. (25) and Fig. 1]. In some cases entangled qubits do not perceive environment at all, and the system state is a *decoherence free* one [Eq. (21)]. Several connections with results from the field of entanglement decoherence are provided.

It could be interesting to compare the studied low-temperature scenario with the high-temperature one, above the critical value T_c. The major hindrance to this task is represented by the nonlinearity of bath Hamiltonians, like Eq. (12), which prevents us from finding the exact analytic dynamical solution. Nevertheless, it is possible to tackle the problem in the frame of the Ising model at infinite temperature. In this limit the bath density matrix turns out to be the identity operator, since each energy level has the same weight; furthermore Hamiltonians entering time evolution commute among themselves. This allows us to evaluate the concurrence as $C_{IM}^{T=\infty}=2|\alpha||\delta|\exp(-J_0^2t^2/2)$. A straightforward comparison can be settled with concurrence written in Eq. (26), deduced within the Ising model as well; when $T\to T_c$ $(m\to 0)$ the latter tends to the same value of $C_{IM}^{T=\infty}$. This shows that above critical temperature entanglement dynamics become independent of temperature itself, and of bath ordering properties (at least in the region of validity of assumptions made in the text). This prevents coherence time from decreasing further once the temperature goes beyond the critical value T_c.

In conclusion, we believe that the presented analysis can be useful for a more complete knowledge about entanglement dynamical properties.

APPENDIX A

1. Exponentiation of suitable matrices

Let us define a 2×2 traceless matrix $\mathcal{A}$ as

$$\mathcal{A} = (a\sigma_x + b\sigma_z) = \begin{pmatrix} b & a \\ a & -b \end{pmatrix}, \tag{A1}$$

with a, b real coefficients. The exponentiation of A gives

$$e^{\mathcal{A}} = (\cosh q)I + \left(\frac{\sinh q}{q}\right)\mathcal{A}; e^{i\mathcal{A}} = (\cos q)I + i\left(\frac{\sin q}{q}\right)\mathcal{A} \tag{A2}$$

with $q = \sqrt{(a^2+b^2)}$. Therefore

$$\mathrm{tr}(e^{\mathcal{A}}) = 2(\cosh q), \quad \mathrm{tr}(e^{i\mathcal{A}}) = 2(\cos q). \tag{A3}$$

Let us extend these arguments to three matrices $\mathcal{I}$, $\mathcal{R}$, and $\mathcal{I}'$ of the same form of $\mathcal{A}$:

$$\begin{aligned}\mathrm{tr}[e^{i\mathcal{I}}e^{\mathcal{R}}e^{i\mathcal{I}'}] &= \mathrm{tr}\left\{\left[(\cos x)I + i\left(\frac{\sin x}{x}\right)\mathcal{I}\right]\left[(\cosh y)I + \left(\frac{\sinh y}{y}\right)\mathcal{R}\right]\left[(\cos z)I + i\left(\frac{\sin z}{z}\right)\mathcal{I}'\right]\right\} \\ &= (\cosh y)\left[2(\cos x)(\cos z) + i(\cos z)\left(\frac{\sin x}{x}\right)\left(\frac{\tanh y}{y}\right)\mathrm{tr}(\mathcal{I}\mathcal{R}) + i(\cos x)\left(\frac{\tanh y}{y}\right)\left(\frac{\sin z}{z}\right)\mathrm{tr}(\mathcal{R}\mathcal{I}') - \left(\frac{\sin x}{x}\right)\left(\frac{\sin z}{z}\right)\mathrm{tr}(\mathcal{I}\mathcal{I}')\right],\end{aligned} \tag{A4}$$

where x, y, z are respectively related to the elements of $\mathcal{I}$, $\mathcal{R}$, $\mathcal{I}'$ as q was related to A.

2. Coherence expression for TIM

As an example of calculation we report the steps that lead to Eq. (9). All other calculations are easier than this one and can be performed following the same line.

The time evolution of the total density matrix is

$$\begin{aligned}\rho(t) = \frac{e^{-m^2\tilde{J}N/T}}{z}\left\{\exp\left\{it\sum_k\left[\left(\frac{J_0}{\sqrt{N}}S_0^z + 2mJ\right)S_k^z + wS_k^x\right]\right\}\rho_s\right. \\ \times\exp\left\{(1/T)\sum_k(wS_k^x + 2mJS_k^z)\right\} \\ \left.\times\exp\left\{-it\sum_k\left[\left(\frac{J_0}{\sqrt{N}}S_0^z + 2mJ\right)S_k^z + wS_k^x\right]\right\}\right\}.\end{aligned} \tag{A5}$$

First, the partition function results:

$$\begin{aligned}Z &= e^{-m^2\tilde{J}N/T}\mathrm{tr}\left\{\exp\left[(1/T)\sum_k(wS_k^x + 2mJS_k^z)\right]\right\} \\ &= e^{-m^2\tilde{J}N/T}\prod_k \mathrm{tr}[e^{(wS_k^x+2mJS_k^z)/T}].\end{aligned} \tag{A6}$$

By virtue of Eq. (A3) we find

$$Z = e^{-m^2\tilde{J}N/T} \quad 2^N\left\{\cosh\left[\frac{\Theta}{2T}\right]^N\right\}. \tag{A7}$$

Notice that the constant $e^{-m^2\tilde{J}N/T}$ in the partition function simplifies with that present in Eq. (A5).

Let us now study the time evolution of the operator $S_0^- = |0\rangle\langle 1|$ that represents the off diagonal part of the density matrix:

$$\begin{aligned}S_0^-(t) &= \left[2\cosh\left(\frac{\Theta}{2T}\right)\right]^{-N}\mathrm{tr}_B\left\{\prod_k e^{it[((J_0/\sqrt{N})S_0^z+2mJ)S_k^z+wS_k^x]}e^{(wS_k^x+2mJS_k^z)/T}|0\rangle\langle 1|\prod_k e^{-it[((J_0/\sqrt{N})S_0^z+2mJ)S_k^z+wS_k^x]}\right\} \\ &= S_0^-(0)\left[2\cosh\left(\frac{\Theta}{2T}\right)\right]^{-N}\prod_k tr_B\{e^{i\mathcal{I}}e^{\mathcal{R}}e^{i\mathcal{I}'}\},\end{aligned} \tag{A8}$$

where

$$\mathcal{I} = t\left[\left(\frac{J_0}{2\sqrt{N}} + 2mJ\right)S_k^z + wS_k^x\right], \tag{A9a}$$

$$\mathcal{R} = (wS_k^x + 2mJS_k^z)/T, \tag{A9b}$$

$$\mathcal{I}' = -t\left[\left(-\frac{J_0}{2\sqrt{N}} + 2mJ\right)S_k^z + wS_k^x\right]. \tag{A9c}$$

In order to use Eq. (A4) we evaluate the following quantities:

$$x = \frac{t}{2}\sqrt{\Theta^2 + 2\frac{mJJ_0}{\sqrt{N}} + O\left(\frac{1}{N}\right)}, \tag{A10a}$$

$$y = \frac{\Theta}{2T} \Rightarrow \left(\frac{\tanh y}{y}\right) = \frac{2T}{J}, \tag{A10b}$$

$$z = \frac{t}{2}\sqrt{\Theta^2 - 2\frac{mJJ_0}{\sqrt{N}} + O\left(\frac{1}{N}\right)} \tag{A10c}$$

and

$$\mathrm{tr}(\mathcal{I}\mathcal{R}) = \frac{t}{2T}\left(\frac{mJJ_0}{\sqrt{N}} + \Theta^2\right), \tag{A11a}$$

$$\mathrm{tr}(\mathcal{R}\mathcal{I}') = \frac{t}{2T}\left(\frac{mJJ_0}{\sqrt{N}} - \Theta^2\right), \tag{A11b}$$

$$\mathrm{tr}(\mathcal{I}\mathcal{I}') = -\frac{t^2}{2}\left(\Theta^2 - \frac{1}{4}\frac{J_0^2}{N}\right) = -\frac{t^2\Theta^2}{2} + O\left(\frac{1}{N}\right). \tag{A11c}$$

Then, substituting these into Eq. (A4) and performing the product we obtain

$$\prod_k \mathrm{tr}\{e^{i\mathcal{I}}e^{\mathcal{R}}e^{i\mathcal{I}'}\} = 2^N\left(\cosh\frac{\Theta}{2T}\right)^N\left[\cos\left(\frac{tmJJ_0}{\Theta\sqrt{N}}\right) + i\frac{\Theta}{J}\sin\left(\frac{tmJJ_0}{\Theta\sqrt{N}}\right)\right]^N. \tag{A12}$$

We can recognize in the second member of Eq. (A12) the constant $r_{TIM}(t)$ defined in Eq. (8); the absolute value of it, in the limit of large N, gives the result of Eq. (9). The other quantities of the article come out with similar calculations.

APPENDIX B

Complete *R* matrix for TIM

Let us begin with the time dependent density matrix expression for TIM Hamiltonians (12a)–(12c). After mean field approximation (4) we obtain

$$\begin{aligned}\rho(t) &= \frac{1}{Z}[e^{-it(H_s+H_{sB}+H_B^{mf})}\rho_s e^{-H_B^{mf}/T}e^{it(H_s+H_{sB}+H_B^{mf})}]\\ &= \frac{1}{Z}\exp\left\{it\left[\sum_k\left(\frac{J_0}{\sqrt{N}}(S_{01}^z+S_{02}^z)+2mJ\right)S_k^z+\sum_k wS_k^x\right]\right\}\rho_s'\exp\left\{(1/T)\sum_k(wS_k^x+2mJS_k^z)\right\}\\ &\quad\times\exp\left\{-it\left[\sum_k\left(\frac{J_0}{\sqrt{N}}(S_{01}^z+S_{02}^z)+2mJ\right)S_k^z+\sum_k wS_k^x\right]\right\},\end{aligned} \tag{B1}$$

where we have set $\rho_s' = e^{it\xi_0 S_{01}^z S_{02}^z}\rho_s e^{-it\xi_0 S_{01}^z S_{02}^z}$.

The constants present in Eqs. (18a) and (18b) are found by complex conjugation of the following quantities, evaluated in a similar manner as the one seen in Appendix A:

$$A^* = \frac{1}{Z}\prod_k \mathrm{tr}_B\{e^{it[2mJS_k^z+wS_k^x]}e^{(wS_k^x+2mJS_k^z)/T}e^{-it[(J_0/\sqrt{N}+2mJ)S_k^z+wS_k^x]}\}, \tag{B2a}$$

$$B^* = \frac{1}{Z}\prod_k \mathrm{tr}_B\{e^{it[(-J_0/\sqrt{N}+2mJ)S_k^z+wS_k^x]}e^{(wS_k^x+2mJS_k^z)/T}e^{-it[(J_0/\sqrt{N}+2mJ)S_k^z+wS_k^x]}\}, \tag{B2b}$$

$$D^* = \frac{1}{Z}\prod_k \mathrm{tr}_B\{e^{it[(-J_0/\sqrt{N}+2mJ)S_k^z+wS_k^x]}e^{(wS_k^x+2mJS_k^z)/T}e^{-it[2mJS_k^z+wS_k^x]}\}. \tag{B2c}$$

After calculations it is an easy task to verify that $A^* = D^*$, and for this reason the constant D does not appear in Eqs. (18a) and (18b).

The matrix $R(t)$ for TIM is

$$R(t) = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}, \tag{B3}$$

$$R_1 = \begin{pmatrix} |\alpha|^2|\delta|^2(1+|B|^2) - 2\alpha^*\beta\gamma\delta^*|A|^2e^{-it\xi_0} & 2\alpha^*\beta|\gamma|^2A^*e^{-(1/2)it\xi_0} - |\alpha|^2\gamma^*\delta(A^*+AB^*)e^{(1/2)it\xi_0} \\ \alpha\beta^*|\delta|^2(A+A^*B)e^{(1/2)it\xi_0} - 2|\beta|^2\gamma\delta^*Ae^{-(1/2)it\xi_0} & -2\alpha\beta^*\gamma^*\delta|A|^2e^{it\xi_0} + 2|\beta|^2|\gamma|^2 \end{pmatrix}, \tag{B4}$$

$$R_2 = \begin{pmatrix} 2\alpha^*|\beta|^2\gamma A^*e^{-(1/2)it\xi_0} - |\alpha|^2\beta^*\delta(A^*+AB^*)e^{(1/2)it\xi_0} & 2\alpha^*|\alpha|^2\delta B^* - 2(\alpha^*)^2\beta\gamma(A^*)^2e^{-it\xi_0} \\ -2\alpha(\beta^*)^2\delta|A|^2e^{it\xi_0} + 2\beta^*|\beta|^2\gamma & |\alpha|^2\beta^*\delta(A^*+AB^*)e^{(1/2)it\xi_0} - 2\alpha^*|\beta|^2\gamma A^*e^{-(1/2)it\xi_0} \end{pmatrix}, \tag{B5}$$

$$R_3 = \begin{pmatrix} \alpha\gamma^*|\delta|^2(A + A^*B^*)e^{(1/2)it\xi_0} - 2\beta|\gamma|^2\delta^* A e^{-(1/2)it\xi_0} & -2\alpha(\gamma^*)^2\delta|A|^2 e^{it\xi_0} + 2\beta\gamma^*|\gamma|^2 \\ 2\alpha\delta^*|\delta|^2 B - 2\beta\gamma(\delta^*)^2 A^2 e^{-it\xi_0} & 2\beta|\gamma|^2\delta^* A e^{-(1/2)it\xi_0} - \alpha\gamma^*|\delta|^2(A + A^*B)e^{(1/2)it\xi_0} \end{pmatrix}, \tag{B6}$$

$$R_4 = \begin{pmatrix} -2\alpha\beta^*\gamma^*\delta|A|^2 e^{it\xi_0} + 2|\beta|^2|\gamma|^2 & |\alpha|^2\gamma^*\delta(A^* + AB^*)e^{(1/2)it\xi_0} - 2\alpha^*\beta|\gamma|^2 A^* e^{-(1/2)it\xi_0} \\ 2|\beta|^2\gamma\delta A e^{-(1/2)it\xi_0} - \alpha\beta^*|\delta|^2(A + A^*B)e^{(1/2)it\xi_0} & |\alpha|^2|\delta|^2(1 + |B|^2) - 2\alpha^*\beta\gamma\delta^*|A|^2 e^{-it\xi_0} \end{pmatrix}. \tag{B7}$$

From it we have extracted all particular cases treated in the text.

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