PHYSICAL REVIEW A **69**, 053808 (2004)

Mechanocaloric and thermomechanical effects in Bose-Einstein-condensed systems

G. C. Marques
Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05389-970 São Paulo, SP, Brazil

V. S. Bagnato and S. R. Muniz
Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-970 São Carlos, SP, Brazil

D. Spehler
Université Louis Pasteur, I.U.T., Allée d'Athènes, 67300 Schiltigheim, France

(Received 23 September 2003; published 10 May 2004)

In this paper we extend previous hydrodynamic equations, governing the motion of Bose-Einstein-condensed fluids, to include temperature effects. This allows us to analyze some differences between a normal fluid and a Bose-Einstein-condensed one. We show that, in close analogy with superfluid ^{4}He, a Bose-Einstein-condensed fluid exhibits the mechanocaloric and thermomechanical effects. In our approach we can explain both effects without using the hypothesis that the Bose-Einstein-condensed fluid has zero entropy. Such ideas could be investigated in existing experiments.

DOI: 10.1103/PhysRevA.69.053808 PACS number(s): 03.75.Kk, 03.75.Hh, 03.75.Nt

I. INTRODUCTION

The achievement of Bose-Einstein condensation (BEC) in dilute atomic gases [1] represents the establishment of several new exciting possibilities. The investigation of degenerate quantum gases provides, for instance, a good testing ground for innumerable many-body theories developed along the past few decades. Those theories were mainly developed within the context of explaining the striking properties of ^{4}He below the λ point. Among other features, this fluid behaves as having zero viscosity (a superfluid) and also apparently as having zero entropy [2].

The zero-entropy hypothesis seemed to be the only explanation for two other (although related) remarkable properties of superfluid ^{4}He. The first notable property is called the thermomechanical effect (or the "fountain" effect), where a temperature gradient across the fluid produces motion of matter. The force responsible for imparting the motion to the fluid is a "thermoforce" (a term used first by London [3]). On the other hand, the motion of the superfluid also produces temperature gradients. This second effect is the so-called mechanocaloric effect. Both effects were very important to understand the behavior of superfluids, giving valuable hints about the nature of the processes involved.

In earlier times the first proposal of London [3] and Tisza [4], considering the superfluidity as a macroscopic manifestation of BEC, was somehow overruled by the good agreement of the hydrodynamic theory of Landau [5] to the first experiments measuring the second sound velocity [6]. Just later, after much of the development of the many-body theory, especially by Bogoliubov [7], it became clear that BEC was in fact behind the appearance of superfluidity. Even then, the hydrodynamic formalism has been quite useful for calculating and understanding the ^{4}He superfluid. Recently, after the experimental achievement of quantum degenerate gases, the formalisms developed initially to understand superfluidity of helium have been extended to explore the dilute atomic gases as well [8]. A full historical account of these developments, although exciting, is much beyond the scope of this present paper. The interested reader, however, could look, for instance, at Ref. [9], and the references therein.

In this paper we deal with some aspects of the hydrodynamics of a Bose-Einstein-condensed phase. In particular we show that both effects (the mechanocaloric and the thermomechanical effects) are always expected to happen in any Bose-Einstein condensate.

The way we deal with these two phenomena is derived from the same equation, showing the clear relation between them. Furthermore, we do not have to use the zero-entropy property for the superfluid or the BEC system to explain both effects.

The presentation of this paper is arranged as follows. In Sec. II we introduce the hydrodynamic quantization approach to the Bose-Einstein condensation of a system of charged spin-0 bosons under the action of external fields. A chemical potential is introduced later in order to discuss the equilibrium conditions for this system. In Sec. III we present the equations of motion and define averages over the ensemble.

In Sec. IV we write the hydrodynamic equations of Bose-Einstein condensates under the action of external fields. These equations do not take into account temperature effects.

The relevant equilibrium equation, a generalized Bernoulli equation, is introduced in Sec. V. From this zero temperature equilibrium condition we then propose a natural extension that allows us also to take into account the temperature effects.

The force equation and explicit expression for the thermoforce are presented in Sec. VI. Finally, the conclusions in Sec. VII close the paper.

II. HYDRODYNAMIC QUANTIZATION

In earlier papers [10,11] we have presented the hydrodynamic quantization approach and some aspects of the fluidity of a Bose-Einstein-condensed system. In this section we shall analyze these equations for the quite general and important case of a condensate in the presence of electric and magnetic fields and when the number of particles varies. In order to do so we introduce a chemical potential. Furthermore, the introduction of a chemical potential is also relevant in order to study the equilibrium conditions for condensation under the action of external fields [12].

We shall discuss here the field theoretic approach aimed at the description of a system of scalar Bose particles. Within the field theoretic approach, charged spin-0 bosons are associated with a complex scalar field $\psi(x)$. In what follows, x is a shorthand notation for a time and space point $[x=(t,\vec{x})]$.

The quantization of the theory is carried out by imposing equal-time commutation relations between the field and its canonically conjugated variable ψ^*, followed by a typical quantization scheme implemented with the introduction of the Fock space. This method is especially useful when we deal with systems whose number of particles is well defined, for example, in the description of scattering processes. One can use, however, an alternative procedure: the so-called hydrodynamic quantization of the field [13]. In this scheme we use a new set of variables $\rho(x)$ and $\varphi(x)$, which are more convenient in the description of properties related to Bose-Einstein condensation. These new variables are defined as

$$\rho(x) = \psi^*(x)\ \psi(x),$$

$$\psi(x) = \sqrt{\rho(x)}\ e^{i\varphi(x)}, \tag{2.1}$$

denominated density and phase variables, respectively. They are canonically conjugated to each other, as we shall see below. For this reason we impose the following commutation relations among the density operator and the phase operator:

$$[\hat{\rho}(x), \hat{\varphi}(x')] = i\,\delta(x-x'). \tag{2.2}$$

The quantization method, based on commutation relations among density and phase operators, is called a hydrodynamic quantization, and its application in superfluidity, where Bose-Einstein condensation is known to occur, was proposed by Landau [12]. The commutation relation in Eq. (2.2), in conjunction with an explicit representation for these operators, requires a departure from more usual quantization approaches.

An explicit representation of the algebra of operators (2.2) can be realized by making use of the so-called density representation. In this representation the density operator is a classical c number. That is,

$$\hat{\rho}(x) = \rho(x), \tag{2.3}$$

and the phase operator is represented by the operator

$$\hat{\varphi}(x) = -i\frac{\delta}{\delta\rho(x)}. \tag{2.4}$$

Within the density representation the states vectors are represented as functionals of the density

$$\psi = \psi[\rho]. \tag{2.5}$$

In Ref. [10] we give examples of how to construct wave functionals associated with the vacuum and Fock states.

The state functionals, as well as other relevant physical quantities, are written in terms of the density and phase operators (2.1), or in terms of integrals over space-time densities. We shall give below several examples that will be used through this paper. We start with the Lagrangian and the Hamiltonian.

In terms of the field ψ, the classical Lagrangian density $\mathcal{L}$, for the nonrelativistic scalar particles, is written as

$$\mathcal{L} = \frac{i}{2}\{\psi^*(x)[\partial_t\ \psi(x)] - [\partial_t\ \psi^*(x)]\psi(x)\} - \frac{\vec{\nabla}\psi^*(x)\cdot\vec{\nabla}\psi(x)}{2m} - \mathcal{H}^I[\psi^*(x)\ \psi(x)], \tag{2.6}$$

where $\mathcal{H}^I$ is the interaction Hamiltonian density. We define the Hamiltonian density $\mathcal{H}$ as composed of two terms:

$$\mathcal{H}(x) \equiv \frac{\vec{\nabla}\psi^*(x)\cdot\vec{\nabla}\psi(x)}{2m} + \mathcal{H}^I[\psi^*(x)\ \psi(x)] \equiv K + \mathcal{H}^I. \tag{2.7}$$

The first term in Eq. (2.7), containing derivatives of the field, is the kinetic energy term (K) and the second is the interaction Hamiltonian $\mathcal{H}^I$, which contains no field derivative terms.

Placing Eq. (2.1) into Eq. (2.6) one can see that the classical Lagrangian density can be written under the general form

$$\mathcal{L}(x) = -\rho(x)\frac{\partial\varphi(x)}{\partial t} - \mathcal{H}[x,\rho,\varphi], \tag{2.8}$$

thus showing that phase and density are canonically conjugated variables.

The quantum action is defined as

$$S = \int \mathcal{L}(x)\ dx. \tag{2.9}$$

If the system is under the action of external magnetic fields $(\vec{B}=\vec{\nabla}\times\vec{A})$ the kinetic term can be written, in terms of the variables $\rho(x)$ and $\varphi(x)$, as

$$K(\rho,\varphi,\vec{A}) = \rho(x)\frac{[\vec{\nabla}\varphi(x) - e\vec{A}(x)]^2}{2m} + \frac{[\vec{\nabla}\sqrt{\rho(x)}\ ]^2}{2m}, \tag{2.10}$$

where e is the electric charge of the spin-0 bosons.

Whereas, in the presence of an external potential $U(x)$, the interaction Hamiltonian takes the general form

$$\mathcal{H}^I(\rho) = \rho(x)\ U(x) + \rho(x)\ \varepsilon(\rho,x) \equiv \rho(x)\ U(x) + \mathcal{H}^{\text{int}}, \tag{2.11}$$

where $\varepsilon(\rho,x)$ in Eq. (2.11) is the per-particle interaction internal energy density. If we assume binary interactions among the particles, $\varepsilon(\rho,x)$ is given by

$$\varepsilon(\rho,x) = \frac{1}{2}\int d\vec{x}'\ V(\vec{x}-\vec{x}')\ \rho(\vec{x}'). \tag{2.12}$$

Let us consider the expression, in terms of density and phase, of other physically relevant quantities in the study of quantum fluids. We start with the expression for the momentum. The classical definition of momentum is the integral over the space of the momentum operator density ($\vec{\mathcal{P}}(x)$) of the field:

$$\vec{p} = \int d\vec{x}\vec{\mathcal{P}}(\vec{x}) = \frac{1}{2}\int dx\ \varphi^*(x)\vec{\nabla}\varphi(x) = \int d^3x\ \rho(x)\vec{\nabla}\varphi(x). \tag{2.13}$$

From Eq. (2.13) it follows that the per-particle momentum $\vec{P}(\vec{x})$ is given by

$$\vec{P}(\vec{x}) = \vec{\nabla}\varphi(x) = \vec{\nabla}\left(-i\frac{\delta}{\delta\rho(x)}\right) \equiv m\vec{V}(\vec{x}). \tag{2.14}$$

In quantum theory there is, however, another relevant momentum. This quantum momentum $[\vec{P}_q(x)]$ is defined as $\frac{1}{2}$ of the gradient of the log of the density. That is,

$$\vec{P}_q(\vec{x}) \equiv \vec{\nabla}[\ln\sqrt{\rho(x)}\ ] = m\ \vec{V}_q(x). \tag{2.15}$$

The quantum origin of $\vec{P}_q(\vec{x})$ defined in Eq. (2.15) is explained in Ref. [14].

The analysis of a hard sphere gas provides a good example for the understanding of both types of momenta [11]. The classical momentum $\vec{P}(\vec{x})$ is associated with the quantized vortices, whereas the quantum momentum $\vec{P}_q$ is more relevant in understanding the dependence of the fluid density upon the distance of the vortex center [11].

In terms of the velocities defined in Eqs. (2.13) and (2.14) one can write, by using Eqs. (2.10) and (2.11), the Hamiltonian density $\mathcal{H}$ as

$$\mathcal{H}[\rho,\varphi] = \rho(x)\frac{m}{2}[\vec{V}^2(x) + \vec{V}_q^2(x)] + \rho(x)[U(x) + e(\rho,x)]. \tag{2.16}$$

If the bosonic system is under the action of an external magnetic field the above expression remains valid for $\vec{V}(x)$ defined by

$$\vec{V}(x) = \frac{\vec{P}(x) - e\vec{A}(x)}{m}. \tag{2.17}$$

The interaction of the particles with external fields is carried out, in field theory, as usual. For the coupling with external magnetic fields ($\vec{B}=\vec{\nabla}\times\vec{A}$) we use the minimum substitution $\vec{\nabla}\rightarrow\vec{A}-ie\vec{A}$. The potential $U(x)$ takes into account the interaction with external electric fields as well as external gravitational fields.

All the above definitions are relevant in the understanding of fluidity aspects of Bose-Einstein-condensed systems.

In the following section we shall analyze the dynamical equations.

III. EQUATIONS OF MOTION AND ENSEMBLE AVERAGES

The time evolution of a physical quantity represented by the operator $O(x)$ is given by the Heisenberg equation of motion,

$$\frac{\partial O(x)}{\partial t} = i[H,O(x)]. \tag{3.1}$$

The Hamiltonian operator in Eq. (3.1) is the integral of the Hamiltonian density

$$H = \int dx\ \mathcal{H}[\varphi,\rho]. \tag{3.2}$$

Within the density representation the equations of motion of the density and phase operators are

$$\frac{\partial\varphi(x)}{\partial t} = -\frac{\delta H}{\delta\rho(x)}, \tag{3.3}$$

$$\frac{\partial\rho(x)}{\partial t} = \frac{\delta H}{\delta\varphi(x)}. \tag{3.4}$$

These equations follow also from Eq. (2.8) by taking ρ and φ as independent dynamical variables.

By taking the gradient of Eq. (3.3) we define a new equation that we name the force equation:

$$\frac{\partial\vec{P}(x)}{\partial t} = \vec{F}(x). \tag{3.5}$$

The force equation gives the rate of change of the per-particle momentum and is the relevant equation in the understanding of superfluidity and the thermoforce. As we shall see, the thermoforce is a new type of force that arises in a BEC system.

The local force operator $\vec{F}(x)$ is, formally, written as

$$\vec{F}(x) = -\vec{\nabla}\left(\frac{\delta H}{\delta\rho(x)}\right). \tag{3.6}$$

One of the interesting features of the hydrodynamic quantization is that the quantum equation of motion resembles that of a classical fluid (from this fact derives the "hydrodynamic" name). Using Hamiltonian (2.16), the time evolution equations are

$$-\frac{\partial\varphi(x)}{\partial t} = \frac{\vec{P}^2(x)}{2m} + U(x) + h(x) + \frac{\vec{P}_q^2}{2m} + \frac{1}{2m\rho}\vec{\nabla}\cdot(\rho\ \vec{P}_q), \tag{3.7}$$

$$\frac{\partial\rho\ (x)}{\partial t} = -\vec{\nabla}\cdot\frac{\rho(x)\vec{P}(x)}{m}, \tag{3.8}$$

$$\frac{\partial \vec{P}}{\partial t} = -\vec{\nabla}\left(\frac{\vec{P}^2}{2m}\right) + \vec{F}_{\text{ext}}(x) - \vec{\nabla} h(x) - \vec{\nabla}\left(\frac{\vec{P}_q^2}{2m} + \frac{1}{2m\rho}\vec{\nabla}\cdot(\rho\,\vec{P}_q)\right), \tag{3.9}$$

where $h(x)$ in Eqs. (3.7) and (3.9) is the per-particle enthalpy

$$h(x) \equiv \frac{\delta\,\mathcal{H}^{\text{int}}}{\delta\rho(x)} = e(\rho,x) + \frac{P(x)}{\rho(x)}. \tag{3.10}$$

The pressure $P(x)$ in Eq. (3.10) is, as can be inferred from Eqs. (3.10) and (2.11), given by

$$\frac{P(x)}{\rho(x)} = \int dx'\;\rho(x')\frac{\delta e(\rho,x')}{\delta\rho(x)} e(\rho(x'),x'). \tag{3.11}$$

The term $\vec{F}_{\text{ext}}(x)$ in Eq. (3.9) is the force exerted on each of the charged particles of the system as a result of the external potential $U(x)$,

$$\vec{F}_{\text{ext}}(x) = -\vec{\nabla} U(x). \tag{3.12}$$

Equations (3.7)–(3.9) are valid as time evolution equations of operators. In order to deal with classical c numbers one can consider expectation values and averages over the ensemble.

Any of the quantum versions of the equations of motion (3.7)–(3.9) are obviously valid when one considers expectation values. Defining the trace of an operator O as

$$\text{Tr}O(x) = \sum_{\psi}\langle\psi|O(x)|\psi\rangle\frac{1}{\langle\psi|\psi\rangle}, \tag{3.13}$$

we can write the following equations:

$$\frac{\partial}{\partial t}[\text{Tr}\;\varphi(x)] = -\,\text{Tr}\left(\frac{\delta\mathcal{H}}{\delta\rho(x)}\right), \tag{3.14}$$

$$\frac{\partial}{\partial t}[\text{Tr}\;\rho(x)] = \text{Tr}\left(\frac{\delta\mathcal{H}}{\delta\varphi(x)}\right). \tag{3.15}$$

Ehrenfest theorems can be defined as averages over the ensemble of quantum equations. In order to define averages over the ensemble, we define the quantum-mechanical partition function of a Bose system.

For the quantum action S, given by Eq. (2.9), we define the partition function as

$$Z = \text{Tr}\;e^{iS}. \tag{3.16}$$

This is just an extension of the usual finite temperature (T) definition of the partition function

$$Z = \text{Tr}\;(e^{-H/kT}). \tag{3.17}$$

The average over the ensemble of a physical quantity $O(x)$ is usually defined by

$$\langle O(x)\rangle = \frac{1}{Z}\text{Tr}\{e^{iS}\;O(x)\}. \tag{3.18}$$

Taking averages over the ensemble, one writes

$$\frac{\partial\langle\varphi(x)\rangle}{\partial t} = -\left\langle\frac{\delta\mathcal{H}}{\delta\rho(x)}\right\rangle, \tag{3.19}$$

$$\frac{\partial\langle\rho(x)\rangle}{\partial t} = +\left\langle\frac{\delta\mathcal{H}}{\delta\varphi(x)}\right\rangle. \tag{3.20}$$

Averages over the ensemble are relevant in the context of statistical mechanics. In the next section we shall consider matrix elements of the equations of motion under the form (3.14) and (3.15).

IV. HYDRODYNAMIC EQUATIONS FOR BOSE-EINSTEIN-CONDENSED STATES

One expects some basic distinctions, at the level of states, between a condensed system and a normal one. We have proposed, in Ref. [10], that the basic distinction can be traced back to special properties of the wave functional associated with BEC states. We have proposed that the wave functional of BEC systems is endowed with two distinctive properties [10,11].

A. Property (1) of the condensate wave functional

The wave function of the system associated with $\psi_c[\rho]$ [11,15] is a product of wave functions

$$\psi_c(x_1\cdots x_N) = \prod_{i=1}^{N}\psi_c(x_i). \tag{4.1}$$

The wave function $\psi_c(x)$ in Eq. (3.1) is, by definition, the condensate wave function which will be written as

$$\psi_c(x) = \sqrt{\rho_c(x)}\;e^{i\varphi_c(x)}. \tag{4.2}$$

The variables $\rho_c(x)$ and $\varphi_c(x)$ will be identified as the density and phase of the condensate.

B. Property (2) of the condensate wave functional

The condensate wave functional describes coherent states for which the following factorization property holds true:

$$\frac{\psi^*[\rho]\hat{\psi}(x)\;\psi[\rho]}{\psi^*[\rho]\;\psi[\rho]} = \psi_c(x) = \sqrt{\rho_c(x)}\;e^{i\varphi_c(x)} \tag{4.3}$$

and

$$\frac{\psi^*[\rho]\cdot\hat{\psi}(x_1)\cdots\hat{\psi}(x_N)\;\psi[\rho]}{\psi^*[\rho]\;\psi[\rho]} = \psi_c(x_1)\cdots\psi_c(x_N). \tag{4.4}$$

Property (1) is an obvious requirement in order that the wave functional be associated with a condensed state, whereas property (2) allows us to identify the condensate wave function with the order parameter of the phase transition [16,17].

The factorization property (3.4) is the analog of the "off-diagonal long-range order" introduced by Penrose and Onsager [18] in the context of superfluidity of ^{4}He.

In Refs. [10,11] we have shown that the wave functional

$$\psi_c[\rho] = \exp\left(\int \rho(x) \ln \psi_c(x) dx \right) \tag{4.5}$$

exhibits the properties above mentioned properties (1) and (2), thus providing an explicit example of such wave functionals.

The relevance of the factorization property in the understanding of superfluidity has been already emphasized by Anderson in Ref. [16]. By taking averages over states satisfying Eq. (4.4), we conclude that the wave function of the condensate, defined in Eq. (4.3), satisfies the equations

$$\frac{\partial \varphi_c(x)}{\partial t} = -\left(\frac{\delta \mathcal{H}}{\delta \rho(x)} \right) \Bigg|_{\substack{\rho=\rho_c(x) \\ \varphi=\varphi_c(x)}}, \tag{4.6}$$

$$\frac{\partial \rho_c(x)}{\partial t} = \left(\frac{\delta \mathcal{H}}{\delta \varphi(x)} \right) \Bigg|_{\substack{\rho=\rho_c(x) \\ \varphi=\varphi_c(x)}}. \tag{4.7}$$

Since a c-number field $\psi(x)$ is the wave function in the $\vec{x}$ representation of a state $|\psi\rangle$, Eqs. (4.6) and (4.7) specify the equation for the condensate wave function. They are c-number equations.

By taking the functional derivatives of $\mathcal{H}$ given by Eq. (2.2), now expressed as a functional of ρ and φ, we obtain the following set of equations:

$$\frac{\partial \rho_c(x)}{\partial t} + \vec{\nabla} \cdot \vec{J}_c(x) = 0, \tag{4.8}$$

$$-\frac{\partial \varphi_c(x)}{\partial t} = \frac{m}{2} \vec{V}^2(x) + \frac{m}{2} \vec{V}_q^2(x) - \frac{\vec{\nabla} \cdot \vec{J}_q(x)}{2\rho_c(x)} + U(x) + h(x), \tag{4.9}$$

where we have used for $\vec{V}, \vec{V}_q$, and $h(x)$ the definitions (2.7), (2.15), and (3.10). The density currents $\vec{J}_c$ and $\vec{J}_q$ are given by

$$\vec{J}_q(x) = \rho_c(x)\ \vec{V}_q(x), \tag{4.10}$$

$$\vec{J}_c(x) \equiv \rho_c(x)\ \vec{V}_c(x). \tag{4.11}$$

Taking the gradient of Eq. (3.9) we get the force equation

$$\begin{aligned} &\frac{\partial \vec{P}_c}{\partial t} + m(\vec{V}_c \cdot \vec{\nabla})\vec{V}_c + m(\vec{V}_q \cdot \vec{\nabla})\vec{V}_q \\ &\quad = e\ \vec{V}_c \wedge \vec{B} - \vec{\nabla}\left\{ U + h + \frac{1}{2\rho} \vec{\nabla} \cdot \rho\ \vec{V}_q \right\}. \end{aligned} \tag{4.12}$$

In the presence of an external magnetic field the motion is necessarily rotational since the rotational of the condensate velocity is related to the magnetic field by

$$\vec{\nabla} \times \vec{V}_c = -\frac{e}{m} \vec{\nabla} \times \vec{A} = -\frac{e}{m} \vec{B}. \tag{4.13}$$

In the absence of a magnetic field the motion of the condensate is irrotational.

Equation (3.8) is a continuity equation which we have shown to be true for the condensed component of the system.

As we shall see in the next section, Eq. (3.9) corresponds to a generalized Bernoulli equation. We refer to it as the equilibrium equation. It gives the condition of equilibrium for a Bose-Einstein-condensed fluid under the action of external fields.

Equation (3.13) is the dynamic equation satisfied by the fluid in motion under external fields. We shall see that it is an extension of one of Anderson's equations in Ref. [16].

This allows us to conclude that the dynamical equations for the density and phase of the wave function of the condensate, are Eqs. (4.8), (4.9), and (4.12). As a result of these equations we can conclude, on quite general grounds, that the condensed fluid satisfies the continuity equation and that in the absence of magnetic fields the motion of the fluid is irrotational.

V. EQUILIBRIUM EQUATION

We show in this section that, for stationary states, Eq. (4.9) gives the equilibrium condition when the system is under the action of external fields and that this equilibrium condition is a generalized Bernoulli equation.

As usual, we assume that the equilibrium condition for a system under the action of external fields is the chemical potential (μ) to be constant:

$$\mu = \mu_0. \tag{5.1}$$

Remembering that the field ψ is the wave function of a state ψ in the $\vec{r}$ representation, the field associated with a stationary state is

$$\psi(\vec{x},t) = e^{i[\mu_0 t + \varphi_c(\vec{x})]} \sqrt{\rho_c(\vec{x})}, \tag{5.2}$$

where μ_0 is a constant.

For a stationary state the equilibrium equation, for the phase given by Eq. (5.2), is

$$\mu_0 = h(x) + U(x) + \frac{m\ \vec{V}_c^2(x)}{2} + \frac{m}{2} \vec{V}_q^2(x) - \frac{\vec{\nabla} \cdot J_q(x)}{2\rho_c(x)}. \tag{5.3}$$

Taking now into account the explicit expression for $h(x)$, we write

$$\begin{aligned} \mu_0 = \varepsilon(x, \rho_c(x)) + \frac{P(x)}{\rho_c(x)} + \frac{m}{2} \vec{V}_c^2(x) + U(x) + \frac{m}{2} \vec{V}_q^2(x) \\ - \frac{\vec{\nabla} \cdot \vec{J}_q(x)}{2\rho_c(x)}. \end{aligned} \tag{5.4}$$

In order to show that μ_0 is the chemical potential we multiply Eq. (5.4) by $\rho_c(x)$ and integrate over $\vec{x}$. The result can be written as

$$\mu_0\ N_c = E_c + \int P(\vec{x})\ d^3x, \tag{5.5}$$

where N_c in Eq. (5.5) is the number of particles in the condensate

$$N_c = \int d^3x\ \rho_c(\vec{x}), \tag{5.6}$$

and E_c is the condensate energy defined as the integral of the density Hamiltonian:

$$E_c = \int d^3x\ \mathcal{H}[\rho_c, \varphi_c]. \tag{5.7}$$

Expression (5.5) shows that μ_0 is the chemical potential since $\mu_0 N_c$ is the total enthalpy of the system [12].

For a uniform condensate, the Bogoliubov condensate [20] is

$$\rho_c(x) = \rho_{0c}. \tag{5.8}$$

The quantum velocity vanishes so that for a uniform condensate, Eq. (4.3) is Bernoulli's equation:

$$\mu_0 = \varepsilon(x, \rho_{0c}(x)) + m\frac{P(x)}{\rho_{0c}} + \frac{m}{2}\vec{V}_c^2(x) + U(x). \tag{5.9}$$

Since the first four terms of Eq. (4.4) correspond to the usual Bernoulli equation, one can predict that deviations from the usual fluid is expected as a result of the quantum velocity term. This prediction deserves to be tested experimentally.

We notice that for a non-self-interacting system,

$$h(x) = 0. \tag{5.10}$$

Such non-self-interacting fluid, the fluid described by Eqs. (4.2) and (4.3), is a Mahdelung fluid [19]. Therefore, neglecting self-interaction of particles, the BEC is a Mahdelung fluid.

One can extend some of the previous equations to finite temperatures. In order to do so we just recall that the equilibrium condition for a fluid under external fields takes the form [12]

$$\mu = g(P,T) + \mu', \tag{5.11}$$

where $g(P,T)$ is the per-particle Gibbs energy in the absence of the external fields and μ' takes into account the external fields.

Remembering also that at zero temperatures the per-particle Gibbs free energy is equal to the per-particle enthalpy

$$g(P,0) = \varepsilon(x, \rho_{0c}) + m\frac{P(x)}{\rho_{0c}(x)} = h(x), \tag{5.12}$$

we can write Eq. (4.6) under the following form:

$$\mu_0 = g(P,0) + \frac{m}{2}\vec{V}_c^2(x) + U(x). \tag{5.13}$$

The natural extension of Eq. (5.5) to finite temperatures, in view of Eq. (5.13), is

$$\begin{aligned}\mu_0 &= g(P,T) + \frac{m\ \vec{V}_c^2(x)}{2} + U(x)\\ &= \varepsilon(x,\rho_0) + m\frac{P(x)}{\rho_0} - Ts(T) + m\frac{\vec{V}_c^2}{2} + U(x).\end{aligned} \tag{5.14}$$

Notice that for nonstationary states the generalization of Eq. (4.11) to finite temperatures should be, from Eqs. (5.2) and (5.14),

$$\frac{\partial\,\varphi_c}{\partial\,t} = g(P,T) + m\frac{\vec{V}_c^2}{2} + U(x). \tag{5.15}$$

The interesting aspect of Eq. (5.14) is that one can imagine an *entropy filter*. That is, one can devise experiments by means of which a low entropy component is separated from a large entropy component of the BEC fluid. This would be similar to what happens in superfluid ^{4}He.

In fact, one can think of two experimental setups in order to check the validity of the predictions made here. These experiments are analogous to the ones done with superfluid ^{4}He. Let us consider two containers (A and B) of a Bose-Einstein-condensed fluid in which we keep the temperature initially constant on each side and keep the density constant. At this point we consider no external fields. Suppose that these containers are now physically connected by a tube, in which we can have a flow of the condensate. Since

$$dg = \frac{1}{\rho}dP - s\ dT \tag{5.16}$$

it follows from Eq. (5.14), that

$$\frac{1}{\rho}dP - s\ dT = -d\left(\frac{m\ \vec{V}_c^2}{2}\right), \tag{5.17}$$

so that the flow through the superleak occurs until $\vec{V}_c = 0$. Under this condition, it follows from Eq. (5.17) that

$$dP = s(T)\ dT. \tag{5.18}$$

The temperature difference giving rise to a pressure difference is the thermomechanical effect. Equation (5.18) is London's relation [3].

If the containers discussed above are kept at constant pressure, then from Eq. (5.16) it follows that

$$d\left(\frac{m\ \vec{V}_c^2(x)}{2}\right) = s\ dT. \tag{5.19}$$

As a consequence of Eq. (5.19) one can see that if there is mass flow in one of the containers, it will be colder than the other one. That is, the mechanocaloric effect is expected to happen in any BEC system.

We can see, from Eq. (5.19), that in any BEC system a gradient temperature leads to motion of the fluid and vice versa. It follows, also from Eq. (5.19) that, along a stream line, the difference between the kinetic energy of the fluid at two points B and A, in each container, is

$$\frac{m\ \vec{V}_c^2(B)}{2} - \frac{m\ \vec{V}_c^2(A)}{2} = s(T)(T_B - T_A). \tag{5.20}$$

From Eq. (5.20) it follows that if the fluid flows from A to B $[V(B) > V(A)]$, the container B will be warmer. On the other hand, if one of the containers is now considered as a thick tube the pressure dP will cause a fountain of Bose-Einstein condensate. We predict, in this way, the fountain effect for any BEC system.

In superfluid ^{4}He these two phenomena are explained by assuming that the superfluid component is a zero-entropy fluid. We have seen here that this is not necessary.

VI. FORCE EQUATION

In this section we will show that the phenomenon of superfluidity for a BEC system is intimately connected to the two effects present in ^{4}He, the thermomechanic and mechanocaloric effects. Usually, they are considered as independent properties of the superfluid helium. We shall give, also in this section, formal expressions for the thermoforce.

First, we recall that

$$dg = \frac{1}{\rho} dP - s(T)\ dT. \tag{6.1}$$

Now we take the gradient of Eq. (5.14), and the dynamic equation for a nearly uniform fluid becomes

$$\frac{\partial \vec{P}_c}{\partial t} + m(\vec{V}_c \cdot \vec{\nabla})\vec{V}_c = \vec{F}_{\text{ext}} - \frac{m}{\rho}\vec{\nabla} P + e\ \vec{V}_c \times \vec{B} + s(T)\vec{\nabla} T. \tag{6.2}$$

All terms in Eq. (6.2) have a simple interpretation. The term in the left-hand side of Eq. (6.2) is the total derivative of $\vec{P}_c$, that is,

$$\frac{D\vec{P}_c}{Dt} \equiv \left(\frac{\partial}{\partial t} + \vec{V}_c \cdot \vec{\nabla}\right)\vec{P}_c. \tag{6.3}$$

The first term in the right-hand side of Eq. (6.2) is the external force

$$\vec{F}_{\text{ext}} = -\vec{\nabla} U. \tag{6.4}$$

The second term in the right-hand side of Eq. (6.2) is the usual force that induces motion in the fluid as a result of pressure gradients,

$$\vec{F}_1 = -\frac{m}{\rho}\vec{\nabla} P. \tag{6.5}$$

The third term in the right-hand side of Eq. (6.2) is the Lorentz force due to the external magnetic field,

$$\vec{F}_{\text{Lorentz}} = e\ \vec{V}_c \times \vec{B}. \tag{6.6}$$

The last term in Eq. (6.2) is our main contribution to the BEC motion. We propose that this term is the thermoforce:

$$\vec{F}_{\text{ther}} = s(T)\vec{\nabla} T. \tag{6.7}$$

The presence of the thermoforce in Eq. (6.2) means that there will be motion of the fluid as a result of temperatures gradients. The motion is towards the greatest temperatures.

Putting all this together, we write

$$\frac{D\vec{P}_c}{Dt} \equiv \vec{F}_{\text{ext}} + \vec{F}_{\text{ther}} + \frac{1}{\rho}\vec{\nabla} P + e\ \vec{V}_c \times \vec{B}. \tag{6.8}$$

Except for the thermoforce, Eq. (6.8) is just the Euler equation for an ideal fluid. Under similar assumptions made here, Anderson derived the usual Euler equation for ^{4}He [16].

The conclusion, therefore, is that the same phenomenon (Bose-Einstein condensation) is responsible for all the striking properties of superfluid ^{4}He: the superfluidity and both the related thermomechanical and mechanocaloric effects. In fact, in principle, we expect that any BEC system should present those phenomena. At least one of them has already been experimentally confirmed [21].

VII. CONCLUSIONS

We have shown in this paper that the thermomechanical and mechanocaloric effects are always expected to occur in a Bose-Einstein condensate. Besides the understanding of these effects, we have shown that they can be predicted by using the same equation that leads to the Euler equation (the equation that we have named force equation).

We have proposed a generalized Bernoulli equation that would lead, for nonuniform density fluids, to a departure from the usual Bernoulli equation. The generalized Bernoulli equation is valid for stationary BEC states. For time-independent states the chemical potential is zero; in this case the Bose-Einstein-condensed state is characterized as a zero Gibbs energy fluid.

Another important property of a superfluid is the fact that it is irrotational [22]. As a consequence of the presented calculation, we have deduced that this is not generally the case. It only happens in the absence of external magnetic fields.

We have derived an expression for the thermoforce. It is interesting to note that the same expression was also found by London [3], using a different approach. Experimental evidence for the specific form (6.7) is, to our knowledge, missing. However, it predicts that as the temperature is lowered the thermoforce tends to zero [since $\lim_{T\to 0} s(T) \to 0$]. This fact is quite well known in superfluid ^{4}He. The thermoforce becomes negligible for temperatures below 0.6 K [3].

Another important contribution of this present paper is that in order to explain some aspects of the BEC phenomenon, there is no need for the zero-entropy fluid hypothesis.

Finally, we would like to mention that some of our specific quantitative predictions contained in the presented equations can be tested using present experimental conditions.

ACKNOWLEDGMENTS

This work was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and by Programa de Apoio a Núcleos de Excelência (PRONEX).

---

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269**, 198 (1995); C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *ibid.* **75**, 3969 (1995).

[2] See, for example, R. P. Feynmann, in *Statistical Mechanics, A Set of Lectures* (Benjamin Cummings Publishing, New York, 1972).

[3] F. London, in *Superfluids, Vol. II—Macroscopic Theory of Superfluid Helium* (Dover, New York, 1954); See also F. London, Phys. Rev. **54**, 947 (1938).

[4] L. Tisza, Nature (London) **141**, 913 (1938).

[5] L. D. Landau, J. Phys. (Moscow) **5**, 71 (1941).

[6] J. P. Pellam and R. B. Scott, Phys. Rev. **76**, 869 (1949).

[7] N. Bogoliubov, J. Phys. (USSR) **5**, 23 (1947).

[8] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, 2002).

[9] A. Griffin, in *Proceedings of the International School of Physics "Enrico Fermi", Course CXL*, edited by M. Inguscio, S. Stringari, and C. E. Wieman, (IOP, Bristol, 1999).

[10] G. C. Marques and V. S. Bagnato, Phys. Rev. A **61**, 053607 (2000).

[11] G. C. Marques, V. Bagnato, and D. Spehler, Phys. Rev. A **63**, 043607 (2001).

[12] L. Landau, *Statistical Physics* (Addison-Wesley, Reading, MA, 1980).

[13] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[14] G. C. Marques, Phys. Rev. A **8**, 1577 (1994).

[15] E. Recami and G. Salesi, Phys. Rev. A **57**, 98 (1998).

[16] P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966).

[17] P. W. Anderson, *Basic Notions of Condensed Matter Physics* (Addison-Wesley, Reading, MA, 1983).

[18] O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).

[19] E. Mahdelung, Z. Phys. **40**, 332 (1926).

[20] G. C. Marques, Phys. Rev. B **44**, 12485 (1991).

[21] O. M. Marago, S. A. Hopkins, J. Arlt, E Hodby, G. Hechenblaikner, and C. J. Foot, Phys. Rev. Lett. **84**, 2056 (2000).

[22] I. M Khalatnikov, *Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965); R. J. Donnely, *Quantized Vortices in Liquid He II* (Cambridge University Press, Cambridge, 1991).