Improved optimization of perturbation theory: Applications to the oscillator energy levels and Bose-Einstein condensate critical temperature

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(Received 21 January 2004; published 24 May 2004)

Improving perturbation theory via a variational optimization has generally produced in higher orders an embarrassingly large set of solutions, most of them unphysical (complex). We introduce an extension of the optimized perturbation method which leads to a drastic reduction of the number of acceptable solutions. The properties of this method are studied and it is then applied to the calculation of relevant quantities in different ϕ^4 models, such as the anharmonic oscillator energy levels and the critical Bose-Einstein condensation temperature shift ΔT_c recently investigated by various authors. Our present estimates of ΔT_c, incorporating the most recently available six and seven loop perturbative information, are in excellent agreement with all the available lattice numerical simulations. This represents a very substantial improvement over previous treatments.

DOI: 10.1103/PhysRevA.69.053624 PACS number(s): 03.75.Hh, 05.30.Jp, 11.10.Wx, 12.38.Cy

The variationally improved or optimized perturbation based on the linear δ expansion (LDE) [1–3] is a well-used modification of the usual perturbation theory, based on a reorganization of the interacting Lagrangian such that it depends on an arbitrary mass parameter, to be fixed by some optimization prescription. In $D=1$ theories, such as the quantum mechanical anharmonic oscillator [4], the LDE turns out to be equivalent [5] to the "order-dependent mapping" (ODM) resummation method [3]. At the same time, the principle of minimal sensitivity (PMS) [2] optimization, which takes extrema with respect to the mass parameter, is equivalent at large orders to a rescaling of the adjustable oscillator mass with perturbative order, which can essentially suppress the factorial large order behavior of ordinary perturbative coefficients. This appropriate rescaling of the adjustable mass gives a convergent series [5,6], e.g., for the oscillator energy levels [4] and related quantities. Any physical quantity whose ordinary perturbative sequence is available can then be evaluated to an order δ^k using simply modified Feynman rules as implied by the following formal substitution valid for a scalar field theory:

$$\omega \to \omega(1-\delta)^{1/2}; \quad g \to g\delta, \tag{1}$$

where ω and g are the mass and coupling, respectively. Note that for the $D=1$ quantum mechanical anharmonic oscillator described by a $g\phi^4$ theory no renormalization is needed [4], while for $D>1$ models, the parameters g and ω in Eq. (1) are to be considered implicitly bare parameters, and the procedure can be made in this case fully consistent [7,8] with the renormalization program of ordinary perturbation theory. In particular, appropriate renormalization takes into account properly any (mass or field) anomalous dimensions when the latter are relevant.

Now, a definite drawback of the optimization prescription is that it involves minimization of a polynomial equation of order k in the relevant mass parameter ω at perturbative order δ^k, such that more and more solutions, most of them being complex, are to be considered when increasing the order. This nonuniqueness of the optimized solution requires extra choice criteria, and thus may seriously obscure the interpretation and the convergence towards the correct result in many nontrivial cases where the exact nonperturbative result is totally unknown. Moreover, the fact that most solutions are complex is embarrassing, as one has to invoke still an extra criterion to select a (supposedly correct) real result. For some of the simplest models where the method applies, like the oscillator energy levels typically, fortunately all of the complex optimization solutions have actually small imaginary parts and are rapidly decreasing as the perturbative order increases (see, e.g., Bellet *et al.* in Refs. [6]), so that the convergence properties are not very obscured by this inconvenience of the PMS. But in less trivial situations, the imaginary parts of the PMS solutions may be large (see, e.g., Refs. [9–11]) and thus their physical intepretation unclear.

In this paper, we propose a simple generalization of the PMS criterion as performed on the LDE series, which turns out to lead to a drastic reduction of physically acceptable real optimization solutions at each successive perturbative order in all the physical cases we have applied it to. First, we treat the oscillator ground state energy, both in the large-N case [for the vector $O(N)$-symmetric ϕ^4 model] and the ordinary oscillator (scalar, $N=1$) ϕ^4 model. We then apply it to a less trivial and more interesting problem associated with the breakdown of perturbation theory near a critical point, namely the evaluation of the critical transition temperature for a dilute, weakly interacting homogeneous Bose gas. This has been the source of controversy for many years and recently several independent groups have provided comparable evaluations of the critical temperature. The relevant field theoretic framework is a ϕ^4 $D=3$ model (after dimensional reduction) with an $O(2)$ symmetry (see, e.g., Refs. [12,16] for reviews). Here, for completeness we also consider the large-N limit of the $O(N)$ symmetric model case where the exact next-to-leading $1/N$ result is known analytically. In all these cases, the method seems to give excellent approxima-

tions in comparison with the standard PMS ones.

I. BASIC METHOD AND THE OSCILLATOR ENERGY LEVELS

Let us start with the basic perturbative series of the oscillator ground state energy level as described by a $g\phi^4$ $D=1$ model with mass ω:

$$E_0^{(n)} = \frac{\omega}{2} + \omega \sum_{q=1}^{n} (-1)^{q+1} c_q \left(\frac{g}{\omega^3} \right)^q, \tag{2}$$

with $c_1=3/4$, $c_2=21/8$, etc. [4]. As discussed in the Introduction, the LDE procedure is implemented by the substitutions

$$\omega \to \omega(1-\delta)^{1/2}, \quad g \to g\delta, \tag{3}$$

into the perturbative series Eq. (2), then reexpanding the latter to order k in the new expansion parameter δ. Next, δ is set to the value $\delta=1$, such as to recover the original (massless) theory, while at any finite order k there remains a dependence on ω in the LDE result $E^{(k)}(\omega, \delta=1)$. The standard, mostly used optimization criterion is the principle of minimal sensitivity (PMS), requiring at each successive perturbative order k: $\partial E^{(k)}(\omega, \delta=1)/\partial\omega=0$ for optimal ω values. The modification (generalization) that we propose here is first to introduce extra arbitrary parameters, starting at order two with one more parameter:

$$\omega \to \omega(1-\delta)^{1/2}[1+(1-a)\delta]^{1/2}, \tag{4}$$

such that the modified Lagrangian still interpolates between the free field (massive) theory for $\delta=0$ and the original (massless) theory for $\delta=1$. Second, now the PMS criterion is generalized by requiring both $\partial E^{(2)}/\partial\omega=0$, and $\partial^2 E^{(2)}/\partial\omega^2=0$, which gives a system of two equations to be solved simultaneously for a and ω: For $a=1$, $E^{(2)}(\omega)$ has no real minimum, but an inflection point with an almost horizontal tangent, and a small value of $a-1$ makes this tangent horizontal, removing the embarrassment of complex extrema. At higher orders, the generalization is easily done with additional parameters and additional vanishing of higher derivatives of $E^{(k)}(\omega)$. For example, at third order, we introduce two parameters (a and b) as

$$\omega \to \omega(1-\delta)^{1/2}[1+(1-a)\delta+b\delta^2]^{1/2}, \tag{5}$$

together with the extra requirement on the third derivative: $\partial^3 E^{(3)}/\partial\omega^3=0$.[1]

The results of the method for the (scalar case) oscillator energy level are shown in Table I, up to order nine, where higher order generalizations of Eq. (5) are still reasonably tractable upon using efficient polynomial equation solver (we used mainly *Mathematica* [17]).[2] The real solution is unique (at least up to the highest perturbative orders that we analyzed, as given in Table I) and we also indicate for later discussion the corresponding values obtained for the extra interpolation parameter a as defined in Eqs. (4) and (5). [Note that actually the (unique) real optimization solution gives values for the other parameters $b,\ldots$ in, e.g., Eq. (5) at higher orders, which are almost always small with respect to a. More on this later.]

TABLE I. Improved-PMS results for the oscillator ground state energy level, at different orders k, with the corresponding values of the optimal mass parameter $\tilde{\omega}$ and the main interpolation parameter a. $E_{exact}=0.420\,804\,97\ldots(4g)^{1/3}$.

k	$E_{IPMS}/(4g)^{1/3}$	$\tilde{\omega}$	a
1	0.429 268	1.82	
2	0.418 483	2.04	1.05
3	0.422 341	2.18	1.08
4	0.419 138	2.28	1.09
5	0.423 496	2.37	1.11
6	0.415 23	2.45	1.12
7	0.436 015	2.51	1.15
8	0.380 812	2.59	1.15
9	0.647 259	2.60	1.25

We see that after approaching the exact answer to within 4×10^{-3} at fourth order, the approximation becomes worse in a way strongly reminiscent of the behavior of an asymptotic expansion with alternating signs evaluated at a finite value of its parameter. This is clearly related to the fact that the original series in Eq. (2) has factorially growing coefficients at large orders, and thus our alternative method apparently loses the property of compensating this divergent behavior by an appropriate rescaling [5,6] of the optimized mass parameter, unlike the standard LDE-PMS. More precisely, one can see from the third column of Table I that the values of the optimized mass parameter $\tilde{\omega}$, after increasing regularly with the perturbative order for the first few orders, appear rather to have a slower increase for the highest perturbative orders we could consider.

Nevertheless, we can try to better exploit these results by standard resummation methods applied on the obtained perturbative sequence, typically considering simple Padé approximants. Accordingly, let us define the following perturbative series:

$$E^{(p)} = E_1 + \sum_{q=1}^{p} (E_{q+1} - E_q) x^q, \tag{6}$$

where E_q denotes the results for the ground-state energy at orders q in Table I, so that Eq. (6) simply gives $E^{(p)}=E_p$ for $x=1$. Now a simple (3,3) Padé approximant of Eq. (6) using

[1] For completeness, note that we generalize Eq. (5) at arbitrary higher orders by simply adding to the expression within the right-handed bracket of Eq. (5), a term $b_k\delta^k$ at each order k, where b_k are new parameters.

[2] It is clearly becoming more cumbersome and CPU consuming to apply such a method at very large perturbative orders due to the increase in the number of parameters at successive orders.

TABLE II. Improved-PMS results for the oscillator ground state energy level at large N, for different orders k, with the corresponding values of the optimal mass parameter $\tilde{\omega}$ and interpolation parameter a. $E_{exact}=0.429\,267\,840\,9\ldots(4g/3)^{1/3}$.

k	$E_{IPMS}/(4g/3)^{1/3}$	$\tilde{\omega}$	a
1	0.429 268	1.26	
2	0.429 589	1.37	0.96
3	0.429 400	1.43	0.92
4	0.429 326	1.47	0.89
5	0.429 296	1.50	0.87
6	0.429 282	1.52	0.85
7	0.429 275	1.54	0.84
8	0.429 272	1.55	0.83
9	0.429 270	1.56	0.82

the first seven orders of Table I gives 0.420 841, and a (4,4) Padé using all the numbers of Table I gives 0.420 838, still an improvement, within 8×10^{-5} of the exact answer, in spite of the superficially disastrous results of the eighth and ninth orders.

Next, the same analysis is performed for the large-N approximation of the vector $O(N)$ symmetric oscillator, which consists of considering only the "cactus" Feynman graphs. As is well-known, the resulting perturbative series for the ground state energy of a form similar to Eq. (2) can be obtained simply in this case to arbitrary orders by solving the large-N gap equation exactly, and expanding in a perturbative series in g the corresponding exact large-N expression of the ground state energy. When written with the same normalization as in Eq. (2) the first few coefficients of this series are $c_1=1/4$, $c_2=1/4$, $c_3=1/2,\ldots$. However, an important difference with the scalar case oscillator is that the resulting perturbative series has no factorially growing coefficients at large orders and a finite convergence radius. As one can see, the method performs very well in this case, approaching rapidly very close to the exact solution when the order increases. Again, there is only one real positive solution at each order. One can also notice from the last column of Table II the more regular behavior of the extra interpolation parameter a, as compared to Table I, which now appears to tend towards a constant value. (The other parameters b, etc., corresponding to the analysis in Table II are numerically quite negligible with respect to a, so that we do not give their explicit values.)

In order to better understand why the method seems not so efficient when applied to a factorially divergent series, like Eq. (2) for the standard (scalar case) oscillator, let us consider now the even simpler functional integral of Euclidian ϕ^4 theory in zero dimension:

$$I(g,m)\equiv\frac{\sqrt{2}}{\Gamma(1/2)}\int_0^\infty dx\,\exp\{-m^2x^2/2-gx^4/4\}. \tag{7}$$

As is well-known, this can be expanded in a perturbative series in g with alternating signs, factorially growing coefficients at large orders:

$$I^{pert}(g,m)=\frac{1}{\Gamma(1/2)m}\sum_{p=0}^{\infty}(-1)^p\frac{\Gamma(2p+1/2)}{p!}\left(\frac{g}{m^4}\right)^p. \tag{8}$$

Now the exact result of the integral for $m=0$ is

$$I(g,0)=2^{-3/2}\frac{\sqrt{2}\Gamma(1/4)}{\Gamma(1/2)}g^{-1/4}=1.022\,765\,672\,1\ldots g^{-1/4}, \tag{9}$$

which we can compare with the results of the application of the method: 0.954 325, 1.086 577 1, 0.942 138 24, 1.244 391 803, 0.703 884 6, respectively, for orders one to five. Then, there is no real solution at order six. Nevertheless, a Padé approximant $P_{[1,1]}$ (thus constructed from the first three orders) still gives 1.017 54. Thus, the behavior is quite similar to the one of the (scalar) oscillator where there are also no physically acceptable solutions at higher orders. (However, the quality of the results from Padé approximants of higher orders appears to deteriorate more rapidly than in the previous oscillator case: For example, a $P_{[2,2]}$ thus constructed from the first five orders gives 1.056 77). This anyway confirms that the generalized method is not appropriate to turn a factorially divergent large order behavior into a more convergent one, while it definitely improves the LDE convergence when starting from a less divergent original series.

II. CRITICAL THEORY AND BEC CRITICAL TEMPERATURE

We now turn to the more recent and challenging problem of the BEC critical temperature evaluation. We first recall that, to $O(a^2n^{2/3})$, its functional form was found to be [13]

$$T_c=T_0\{1+c_1an^{1/3}+[c_2'\ln(an^{1/3})+c_2'']a^2n^{2/3}\}, \tag{10}$$

where T_0 is the ideal gas condensation temperature, a is the s-wave scattering length, n is the density and c_1, c_2', c_2'' are numerical coefficients. Some early analytical predictions included the self-consistent resummation schemes [14] ($c_1\simeq2.90$), the $1/N$ expansion at leading order [15] ($c_1\simeq2.33$) and at next to leading order [16] ($c_1\simeq1.71$), and also the LDE at second order [18] ($c_1\simeq3.06$). The numerical methods include essentially lattice simulations (LS). The most recent LS results are reported by the authors of Ref. [19] ($c_1=1.29\pm0.05$) and of Ref. [20] ($c_1=1.32\pm0.02$). Very recent analytical studies predict $c_1=1.27\pm0.11$ [21]. The problem is that these coefficients (except c_2') are sensitive to the infrared physics at the critical point and, so, no perturbative approach can be used to compute them. At the critical point, one can describe a weakly interacting dilute homogeneous Bose gas by an effective action analogous to an $O(2)$ scalar field model in three dimensions given by

$$S_\phi=\int d^3x\left(\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}\omega\phi^2+\frac{g}{4!}\phi^4\right), \tag{11}$$

where ϕ is a two-component real scalar field. The parameters ω and g are related to the original parameters of the nonrel-

TABLE III. Improved-PMS (IPMS) versus standard results for the large-N BEC ΔT_c at different orders k. $c_{1,exact}=2.328\,47\ldots$.

k	$c_{1,IPMS}$	a	$c_{1,bestPMS}$
2	2.163		2.163
3	1.85	1.44	$1.88\pm0.17I$
4	1.93	1.69	1.96
5	2.00	1.78	$1.91\pm0.03I$
6	2.04	1.83	1.94
7	2.08	1.89	$1.93\pm0.0015I$
8	2.10	1.88	1.935
9	2.12	1.89	1.95
10	2.14	1.91	$1.95\pm0.03I$

TABLE IV. Improved-PMS (IPMS) versus standard results for the large-N BEC ΔT_c at different orders k. $c_{1,exact}=2.328\,47\ldots$.

k	$c_{1,IPMS}$	a	$c_{1,bestPMS}$
2	2.852		2.852
3	2.367 71	1.49	$2.444\pm0.276I$
4	2.344 51	1.53	$2.244\pm0.20I$
5	2.336 86	1.56	$2.397\pm0.079I$
6	2.328 47	2.0	$2.333\pm0.08I$
7	2.328 47	2.0	$2.298\pm0.06I$
8	2.328 47	2.0	$2.342\pm0.04I$
9	2.328 47	2.0	$2.324\pm0.036I$

ativistic action by $\omega=-2m\mu$ and $g=48\pi amT$ [15,16] where μ represents the chemical potential, m the atomic mass and T the temperature. The leading order coefficient of the critical temperature shift can be expressed as [14]

$$c_1 = -(256/N)\pi^3[\zeta(3/2)]^{-4/3}\kappa, \tag{12}$$

where $g\kappa=\Delta\langle\phi^2\rangle\equiv\langle\phi^2\rangle_g-\langle\phi^2\rangle_0$. The subscripts g and 0 mean that the field fluctuations are to be evaluated in the presence and in the absence of interactions, respectively.

The implementation of LDE within this model is reviewed in previous applications [9–11,18]. Let us first consider the LDE in the large-N limit extending the relevant $O(2)$ model to an $O(N)$-symmetric one. The interest of such an approximation is that the exact c_1 result is known by direct evaluation [15]. The original perturbation series in this case can be written as

$$\langle\phi^2\rangle_g^{(k)} = -\frac{N\omega}{4\pi} + \frac{Ng}{3}\sum_{i=1}^{k-1} C_i\left(-\frac{gN}{6\omega}\right)^i, \tag{13}$$

where the perturbative coefficients are given by

$$C_i = \frac{3}{16\pi^3}\left(\frac{1}{8\pi}\right)^i\int_0^\infty dz\frac{z^2}{(z^2+1)(z^2+4)}[A(z)]^i, \tag{14}$$

with

$$A(z) = \frac{2}{z}\arctan\frac{z}{2}, \tag{15}$$

and $z=k/\omega$, and can be calculated analytically to arbitrary precision using, e.g., *Mathematica* [17] numerical integration. By applying the generalized LDE procedure as described above in Eqs. (4) and (5), etc., up to order δ^9, one obtains the results shown in Table III ($c_{1,IPMS}$), together with those obtained by the standard PMS optimization procedure, the latter for the best converging family of solutions $c_{1,bestPMS}$ (see Ref. [10] for details on the latter).

As illustrated and discussed in detail in Ref. [10], the convergence of the standard LDE-PMS method when applied to the exact large-N series Eq. (13) is actually very slow (leaving apart the accidentally good result at first order), so that only at very large perturbative orders $\sim$50 do the optimized values approach reasonably close to the exact large-N value, $c_1\simeq 2.328\,47\ldots$. Thus the advantage of the improved PMS approach is not obvious in this case, since it is evidently less algebraically tractable than the standard method at very large perturbative orders. Nevertheless, one can see from the improved-PMS results in Table III that the method performs much better, giving always a unique real solution and seemingly converging much faster towards the expected result. In fact, as motivated in Ref. [10], in order to study the eventual LDE convergence properties of the large-N expression of c_1, it is more instructive to consider an approximated form of the corresponding large-N perturbative series which takes into account only the relevant infrared limit of the auxiliary field propagator (see Ref. [10] for more details). This has the advantage of giving a simple geometric series with exact coefficients:

$$\langle\phi^2\rangle_{IR}^{(k)} = -\frac{N\omega}{4\pi} + \frac{gN}{3}\sum_{i=1}^{k-1} G_i\left(-\frac{gN}{6\omega}\right)^i, \tag{16}$$

where $G_i\equiv[(64\pi^2)(8\pi)^i]^{-1}$, such that the straightforward resummation of (16) is well defined for the relevant limit $\omega\rightarrow 0$, which accordingly can be reached smoothly in contrast with the genuine large-N series Eq. (13). The alternative method results applied on the original series (16) are shown in Table IV, together with those obtained by the standard PMS optimization procedure, the latter for the best converging family of solutions.

In that case, the convergence of the standard PMS method is faster in comparison of Table III, but the improved-PMS method performs even better: starting from order six and beyond, the exact result is always obtained as the unique physically acceptable solution. It is interesting to trace what happens in more detail. In fact, the solutions found at LDE orders $k\geqslant 6$ by applying the procedure is $a=2$, $b=c=d=\cdots=0$, which is easily seen by comparing with Eqs. (4) and (5) to correspond to a basic interpolation of the form

$$\omega\rightarrow\omega(1-2\delta+\delta^2+0)^{1/2} = \omega(1-\delta). \tag{17}$$

Then, applying the substitution (17) on any geometric series instead of the standard LDE substitution (1) canonical for a scalar theory, one can easily see from simple algebraic properties that the improved-PMS solution always reproduces the

exact solution, and this is at any arbitrary LDE order $k \geqslant 2$. This case exhibits a spectacular improvement over the standard LDE results, which converges only rather slowly and with nonzero (albeit small) imaginary parts, a source of much frustration. *A posteriori*, there is nothing particularly remarkable in this result which is essentially an algebraic accident of the noncanonical substitution Eq. (17) followed by the LDE when performed on a simple geometric series. However, what is perhaps more interesting is that our improved-PMS procedure is in that way guessing a more appropriate value of the rescaling power within the LDE substitution ansatz Eq. (3): Alternatively we could have parametrized the basic LDE substitution according to

$$\omega \to \omega(1-\delta)^{\gamma}, \tag{18}$$

with an arbitrary power γ to begin with, and then look for the best value of γ such that the LDE series converges faster towards the exact result, which is clearly the case for $\gamma=1$ in this large-N case. Now, it is worth noting that considering an arbitrary power coefficient γ according to Eq. (18) when applied to the BEC series turns out in practice to be essentially equivalent to modifying the simplest LDE substitution formula Eq. (1) by introducing the relevant critical exponent:

$$\omega \to \omega(1-\delta)^{1/\omega'}, \tag{19}$$

where $\omega'=2\Omega/(2-\eta)$, η is the anomalous dimension of the critical propagator $\sim 1/p^{2-\eta}$ and $\Omega \equiv \beta'(g_c)$, g_c being the critical coupling (see, e.g., Ref. [12]). This renormalization group inspired modification Eq. (19) of the standard LDE Eq. (3) is indeed the approach followed, e.g., in Refs. [21,22], where numerical values of ω' as obtained by different methods (including the variational perturbation theory [23]) are used in ansatz (19) prior to an (otherwise standard) PMS optimization. Now in our case, a major difference is that the relevant exponent $\omega' \simeq 2/a$ is simply guessed by the generalized optimization procedure, at the same time as obtaining as optimized solution the relevant physical quantity c_1. More precisely, the correct exact large-N value $\omega'=1$, $\Omega=1$ (see, e.g., Ref. [12]) is clearly guessed by our improved-PMS procedure, with $\omega'=2/a$, at least for the infrared approximated large-N case as illustrated in Table IV. Note also that the values of the parameter a at successive perturbative orders are also clearly approaching quite closely, though more slowly, the correct critical value when considering the genuine large-N series, as illustrated in Table III (noting, however, that in this case the other parameters b, etc. are small with respect to a but not strictly zero).

We will consider now mainly the physically relevant $N=2$ case, as well as the case $N=1$ and $N=4$ for comparison with other available lattice simulation and analytical results. We will see that many of the previous results for the large-N series generalize, at least qualitatively, to this less trivial case. For $N=2$, the quantity $\langle \phi^2 \rangle_g^{(k)}$ has been first evaluated, up to order δ^4, in Ref. [9]. Recently, higher order terms have been evaluated by Kastening [21] so that, to order δ^6, the perturbative series can be written as

TABLE V. Improved-PMS (IPMS) versus standard (PMS) results for c_1 at $N=2$ in the BEC case at different orders k. Also shown are the Padé approximants (PA) resummation results applied to the improved-PMS cases. The lattice results are $c_1=1.29\pm0.05$, 1.32 ± 0.02.

k	$c_{1,IPMS}$	a	$PA(c_{1,IPMS})$	$c_{1,PMS}$
2	3.06			3.06
3	0.98	1.95		$2.45\pm1.66I$
4	1.426	2.81	$P_{[1,1]}=1.347$	$1.53\pm2.32I$
5	1.247	2.75	$P_{[1,2]}=1.283$; $P_{[2,1]}=1.298$	$0.76\pm2.53I$
6	1.300	2.83	$P_{[2,2]}=1.286$	$2.40\pm1.69I$

$$\langle \phi^2 \rangle_g^{(6)} = -\frac{N\omega}{4\pi} + g\sum_{i=1}^{5} K_i \left(-\frac{g}{\omega}\right)^i, \tag{20}$$

where the coefficients are given by $K_1=3.221\,74\times10^{-5}$, $K_2=1.517\,92\times10^{-6}$, $K_3=9.665\,12\times10^{-8}$, $K_4\simeq 7.513\,66\times10^{-9}$, and $K_5\simeq 6.7493\times10^{-10}$.[3] The results of our alternative procedure are shown in Table V, together with the results from applying simple Padé approximants similar to the ones discussed above for the oscillator, i.e., defining a perturbative series similar to Eq. (6) but where the E_i are now replaced by the values of c_1 in Table V at successive orders k. One can notice that at orders $k=2,3$ and 4 the IPMS results quickly oscillate around the MC results. From order δ^4 onwards the oscillation is reduced drastically and the IPMS procedure generates stable results which agree remarkably well with the lattice results. Furthermore, we also indicated in Table V the corresponding values obtained at each order for the parameter a, whereas as already indicated, in the generalized LDE substitutions Eq. (5), etc., at higher orders the remaining parameters b, etc. are numerically smaller. Accordingly, to first order approximation,

$$\omega \to \omega[1-a\delta-(1-a)\delta^2+b\delta^2(1-\delta)+\cdots]^{1/2} \simeq \omega(1-\delta)^{a/2}, \tag{21}$$

so that in a rough approximation one has $\omega' \simeq 2/a$. Taking the latter approximate relation at face value would give, e.g., for the successive orders considered in Table V: $\omega' \sim 1.03$, 0.71, 0.73, 0.71, which do not compare badly with the reported values of $\omega'=0.8\pm0.04$ [12,23] obtained numerically by other methods. Note, however, that the present method does not pretend at this stage to accurately predict in that way the relevant critical exponent ω', Ω, etc. Indeed, in this $N=2$ case the other parameters b, etc. are small with respect to a but not strictly negligible. Nevertheless, though the es-

[3]Throughout this paper we use the perturbative loop coefficients results of Ref. [21] to evaluate all these coefficients, since in particular the latter are obtained as much as possible from exact integrals. There are therefore some very small differences in the lowest orders numerical results with respect to some previous analysis [9,10], since the latter used numerical integration, not always very precise. Note also a trivial difference of normalization with respect to Ref. [21] in our defining series, Eq. (20).

sential motivation of our approach introducing more interpolation parameters in Eq. (5) is to get rid of the unwanted complex optimization solutions, it is interestingly connected with the parametrization of corrections to scaling, the leading correction being correctly described according to Eq. (19) **(**see, e.g., Refs. [21,22] for further motivation of Eq. (19)**)**. However, to be complete one should note that, in contrast with the previous oscillator and large-N BEC series, the real IPMS solutions in Table V are not always unique: actually, at orders $k=4$ and $k=6$ there appears a second real (but negative) solution. But these extra solutions can be immediately eliminated as they correspond to largely unreasonable values of the interpolation parameters: for instance, for $k=4$ the extra real solution is $a\simeq -13.9$, $b\simeq 55.5$, and $\tilde{\omega}\simeq -0.02$, which gives $c_1\simeq -55.4$, while we can expect consistent values of these parameters to be reasonably close to their corresponding large-N values, i.e., typically a should be not too far from its large-N value $a=2$, as also supported from the results of the above mentioned leading corrections to scaling analysis.

At the same time, the results from the standard LDE-PMS optimization method shown in Table V also oscillate somehow around the MC result, as anticipated in Ref. [11], but the convergence is not at all obvious: after approaching rather closely the lattice result at order δ^4, the results of the next two higher orders depart sensibly from the lattice one. In the absence of any indication on the higher orders, we can hardly speculate but, by comparison with the results of Ref. [10] for the large-N case, as summarized in Tables III and IV, it appears possible that the standard LDE ultimately converges towards the same result, but in a much slower way than our improved method given in the left column of Table V. Moreover, the latter standard PMS results will look much less attractive if one recalls that they have been selected among several possible complex results with large imaginary parts [10], requiring an extra selection criteria,[4] in contrast to the improved version.

One can also note the remarkable results of the Padé approximants based on the improved-PMS results at different orders, even for the lowest order one $P_{[1,1]}$, though it only uses the second and third order IPMS results. This is certainly not coincidental and should be mainly attributed to the oscillatory property of the IPMS results. Indeed, the very same Padé approximants using instead (real parts of) the standard PMS results of the rightmost column of Table V are far away from any reasonable result: this would give, e.g., $P_{[1,1]}=4.26$ and $P_{[2,2]}=3.66$.

Next, we consider the completely similar calculation of the coefficient c_1 as defined in Eq. (12) but for the cases $N=1$ and $N=4$, respectively, for which lattice calculations of c_1 have been recently performed in Ref. [26]. The original perturbative series reads like Eq. (20) with now the relevant coefficients K_i given by $K_1=1.208\,097\times 10^{-5}$, $K_2=5.122\,985\,3\times 10^{-7}$, $K_3=2.964\,522\,5\times 10^{-8}$, $K_4\simeq 2.108\,495\,6\times 10^{-9}$, and $K_5\simeq 1.742\,0267\times 10^{-10}$, and $K_1=9.664\,779\,4\times 10^{-5}$, $K_2=5.464\,517\,7\times 10^{-6}$, $K_3=4.095\,845\,76\times 10^{-7}$, $K_4\simeq 3.702\,611\,98\times 10^{-8}$, and $K_5\simeq 3.833\,335\,8\times 10^{-9}$, respectively, for $N=1$ and $N=4$.

[4]More precisely, we selected [10] among the different LDE-PMS real or complex solutions the ones having the smallest $\mathrm{Re}[\tilde{\omega}]>0$ value of the optimized mass, motivated by the fact that the exact solution would correspond to $\omega\to 0$.

TABLE VI. Same as Table V (improved versus standard) PMS results but for $N=1$ at different orders k. The lattice result is 1.09 ± 0.09.

k	$c_{1,IPMS}$	a	PA($c_{1,IPMS}$)	$c_{1,PMS}$
2	2.65			2.65
3	0.817	1.95		$2.12\pm 1.47I$
4	1.237	2.88	$P_{[1,1]}=1.159$	$1.31\pm 2.06I$
5	1.047	2.77	$P_{[1,2]}=1.086$; $P_{[2,1]}=1.106$	$0.62\pm 2.29I$
6	1.114	2.90	$P_{[2,2]}=1.095$	$2.11\pm 1.54I$

Our results are given in Tables VI and VII, respectively, for $N=1$ and $N=4$. As one can see, the results from our alternative method are again showing very good convergence properties and approaching very closely the lattice results. The Padé approximants are also in excellent agreement with the latter. In addition, the corresponding values obtained for the parameter a, as related in first approximation to the critical exponent ω' in Eq. (18), appear qualitatively very consistent to known results for ω' for $N=1$ and $N=4$ [12,23]. On the other hand, the results from the standard LDE-PMS method show a behavior very similar to the case $N=2$: after approaching not too far from the lattice result at fourth order (but with very large imaginary parts), the results from higher orders are not satisfactory, with trends very similar to the case $N=2$.

We finally apply our alternative method to the other nonperturbative relevant coefficient c_2'' in the defining Eq. (10). This coefficient, which is not generally considered by most authors working on the BEC ΔT_c problem, was first evaluated with lattice simulations. To our knowledge, its sole analytical evaluation made use of the standard LDE-PMS [9,10] up to order δ^4 (five loops). This $O(a^2 n^{2/3})$ coefficient appearing in the ΔT_c expansion, Eq. (10), can be written as [13]

$$c_2''=-\frac{2}{3}[\zeta(3/2)]^{-5/3}b_2''+\frac{7}{9}[\zeta(3/2)]^{-8/3}(192\pi^3\kappa)^2+\frac{64\pi}{9}\zeta(1/2)\times[\zeta(3/2)]^{-5/3}\ln\zeta(3/2), \tag{22}$$

where b_2'' is

TABLE VII. Same as Table V (improved versus standard) PMS results but for $N=4$ at different orders k. The lattice result is 1.59 ± 0.10.

k	$c_{1,IPMS}$	a	PA($c_{1,IPMS}$)	$c_{1,PMS}$
2	3.75			3.75
3	1.222	1.94		$2.99\pm 1.99I$
4	1.665	2.68	$P_{[1,1]}=1.589$	$1.90\pm 2.74I$
5	1.524	2.66	$P_{[1,2]}=1.550$; $P_{[2,1]}=1.558$	$0.99\pm 3.01I$
6	1.556	2.74	$P_{[2,2]}=1.549$	$2.90\pm 1.98I$

TABLE VIII. Improved-PMS results for c_2'' in the $N=2$ BEC case at different orders k. The lattice result is $c_2''=75.7\pm0.4$.

k	$c''_{2,IPMS}$	Padé ($c''_{2,IPMS}$)	$c''_{2,bestPMS}$
2	101.2		101.2
3	69.83		$94.2\pm31.2I$
4	77.90	$P_{[1,1]}=76.25$	$75.0\pm41.1I$
5	73.46	$P_{[1,2]}=79.31$; $P_{[2,1]}=75.04$	$60.4\pm41.2I$

$$b_2''=32\pi\left\{\left[\frac{1}{2}\ln(128\pi^3)+\frac{1}{2}-72\pi^2\mathcal{R}-96\pi^2\kappa\right]\times\zeta(1/2)+\frac{\sqrt{\pi}}{2}-K_2-\frac{\ln 2}{2\sqrt{\pi}}[\zeta(1/2)]^2\right\},\tag{23}$$

with $K_2=-0.135\,083\,353\,73$ and $g^2\mathcal{R}=r_c$ where $r_c=-\Sigma(0)$ (Hugenholtz-Pines theorem). As before, the quantity κ is easily obtained from Eq. (20) whereas $\mathcal{R}$ can be obtained directly from the perturbative evaluation of $\Sigma(0)$. We refer the interested reader to Ref. [9] for the details and subtleties associated with this type of evaluation. Thanks to the recent availability of improved six-loop results [24] one obtains, at $N=2$,

$$\delta r_c^{(5)}=-\Sigma_{\rm ren}^{(5)}(0)=\frac{g\omega}{6\pi}+g^2A_2\left[\ln\left(\frac{M}{\omega}\right)-0.597\,75\right]+\omega^2\sum_{i=3}^{5}\left(\frac{-g}{\omega}\right)^i A_i,\tag{24}$$

where M is an arbitrary mass scale introduced by dimensional regularization and $A_2=1.407\,24\times10^{-3}$, $A_3=8.508\,88\times10^{-5}$, $A_4=3.572\,59\times10^{-6}$ and $A_5=2.253\,32\times10^{-7}$. As discussed in Ref. [9] it is interesting to note that the optimized $\tilde{\omega}$ defined by the standard PMS, or as well by the IPMS introduced in the present work, are scale independent to any order in δ so that $\tilde{\omega}$ is only g dependent. Note also that, contrary to $\Delta\langle\phi^2\rangle$, r_c is a divergent quantity. In this case, the optimization procedure must be implemented after renormalization as advocated in Ref. [8]. Finally, let us point out that the potential technical difficulty associated with the evaluation of r_c stems from the fact that, while $\Delta\langle\phi^2\rangle$ depends on the (finite) difference $\Sigma(p)-\Sigma(0)$, the former depends on the (divergent) $\Sigma(0)$ only. Here, we had access [25] to values of $\Sigma(0)$ up to order δ^5 (six loops) only. Unfortunately, at order δ^6 only the results for the joint $\Sigma(p)-\Sigma(0)$ contribution are known [24].

The results of the alternative method are shown in Table VIII below for $M=g/3$ which was the value chosen in the lattice evaluations [13]. As one can see, the results of our alternative method show again a remarkable agreement with the numerical lattice result up to the fifth order (six loop). Concerning the results of the standard PMS method, as shown in the right column of Table VIII, their behavior is again very similar to the ones for the c_1 coefficient above in Tables V–VII: after approaching very closely the lattice result at fourth (five-loop) order, the six-loop result starts to decrease a bit further.

III. CONCLUSION

We have introduced a conceptually rather simple generalization of the optimized (or variationally improved) perturbation method, which considerably reduces the number of irrelevant optimization solutions at each perturbative order. We have applied this variant of the PMS method to the calculation of certain nonperturbative quantities in the simple quantum mechanical oscillator as well as the more challenging BEC critical temperature determination.

Just like the usual LDE/PMS procedures, it is a recipe without a formal general justification. For quantities described by a perturbative series which is originally factorially divergent, such as the series relevant for the oscillator, we find that this procedure, though it helps in selecting only a few (if not always uniquely) among the many possible complex solutions, does not exhibit obvious convergence improvement behavior, in contrast with the standard method whose rigorous convergence is established in simple cases. Our experience from several toy series as well as the physical systems considered in this paper seems to point to the following behaviors which remain somewhat mysterious to us: Compared to the usual LDE/PMS procedures, it seems more superior conceptually when applied to series with finite radius of convergence or at least a less divergent series, like the ones relevant to the BEC critical temperature. When one deals with an asymptotic expansion (with zero radius of convergence but alternating signs) typical of those appearing in perturbative calculations for physical systems, it seems that the recipe breaks down at high order, but that at low order it gives excellent numerical results, provided that the asymptotic expansion is not too different in its first few orders from a series with finite radius of convergence (examples are provided by finite-N versus large-N theories).

Moreover, quite interestingly, the method appears in nontrivial cases to estimate numerically (by optimization) essentially correct values of the leading corrections to scaling behavior, since the introduction of more interpolation parameters in the LDE ansatz is, to a first approximation, equivalent to introducing the relevant critical exponent. Finally, our numerical results obtained for the BEC critical temperature when using the latest available perturbative information are in excellent agreement with the numerical lattice simulations results for the all available finite N cases (and also for the large-N case as compared to the exact analytic result).

ACKNOWLEDGMENTS

M.B.P. is partially supported by CNPq-Brazil. We thank Boris Kastening for useful communication relevant to the six loop evaluation of r_c.

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