

Gaussian entanglement of formation

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We introduce a Gaussian version of the entanglement of formation adapted to bipartite Gaussian states by considering decompositions into pure Gaussian states only. We show that this quantity is an entanglement monotone under Gaussian operations and provide a simplified computation for states of arbitrary many modes. For the case of one mode per site the remaining variational problem can be solved analytically. If the considered state is in addition symmetric with respect to interchanging the two modes, we prove additivity of the considered entanglement measure. Moreover, in this case and considering only a single copy, our entanglement measure coincides with the true entanglement of formation.

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I. INTRODUCTION

One of the main novelties of quantum information theory is to consider entanglement no longer merely as an apparent paradoxical feature of correlated quantum systems, but rather as a resource for quantum information processing purposes. This new point of view naturally raises the question regarding the quantification of this resource. How much entanglement is contained in a given state? For pure bipartite states there is, under reasonable assumptions, a simple and unique answer to this question, namely, the von Neumann entropy of the reduced state [1–3]. For mixed states there are several entanglement measures [4], which can be distinguished due to their operational meaning and mathematical properties. Such a measure should be nonincreasing under mixing as well as under local operations and classical communication (LOCC), and it should return the right value for pure states. The largest measure fulfilling these requirements is the entanglement of formation E_F [5]. Operationally, it quantifies the minimal amount of entanglement, which is needed in order to prepare the state by mixing pure entangled states. It is therefore defined as an infimum

$$E_F(\rho) = \inf \left\{ \sum_k p_k E(\Psi_k) \mid \rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \right\} \quad (1)$$

over all (possibly continuous) convex decompositions of the state into pure states with respective entanglement $E(\Psi) = S(\text{tr}_B[|\Psi\rangle\langle\Psi|])$, where $S(X) = -\text{tr}[X \log X]$ is the von Neumann entropy. By its definition calculating E_F is a highly nontrivial optimization problem, which becomes numerically intractable very rapidly if we increase the dimensions of the Hilbert spaces. Remarkably, there exist analytical expressions for two-qubit systems [6] as well as for highly symmetric states [7,8].

Recently, E_F was calculated for continuous variable states, namely, for symmetric Gaussian states of two modes [9]. In general, Gaussian states are distinguished among other continuous variable states due to several reasons. Experimentally, they are relatively easy to create and arise naturally as states of the light field of a laser [10] or in atomic

ensembles interacting with light [11]. For this and other reasons they play a more and more important role in quantum information theory [12].

Theoretically, despite the underlying infinite dimensional Hilbert space, they are completely characterized by finitely many quantities—the first and second moments of cononical operators. Moreover, they stand out due to several extremal properties.¹ In fact, the calculation of E_F for symmetric two-mode Gaussian states depends crucially on the fact that for given “EPR correlations” [9] two-mode squeezed Gaussian states are the cheapest regarding entanglement. This implies that in this particular case there is a decomposition in terms of pure Gaussian states, which is optimal for E_F in Eq. (1).

On the one hand, this raises the question, whether this is generally true for all Gaussian states. On the other hand, one may, motivated by the operational interpretation of E_F , restrict Eq. (1) to decompositions into Gaussian states from the very beginning. After all, Gaussian states arise naturally, whereas the experimental difficulties of preparing an arbitrary pure continuous variable state are by no means simply characterized by the amount of its entanglement. For these reasons we will in the following investigate the Gaussian entanglement of formation E_G to quantify the entanglement of bipartite Gaussian states by taking the infimum in Eq. (1) only over decompositions into pure Gaussian states.

This article is organized as follows. In Sec. II we recall basic notions concerning Gaussian states. Section III defines the Gaussian entanglement of formation and provides a major simplification concerning its evaluation for bipartite Gaussian states of arbitrary many modes. In Sec. IV we prove that E_G is indeed a (Gaussian) entanglement monotone, in the sense that it is nonincreasing under Gaussian local operations and classical communication (GLOCC). The case of general two-mode Gaussian states is solved analytically in Sec. V. The special case of symmetric Gaussian states, for which it was proven in Ref. [9] that $E_G = E_F$ is

¹For example, Gaussian states have maximal entropy among all states with given first and second moments of the canonical observables.

discussed in detail in Sec. VI, where we give an alternative calculation of E_G , which is in turn utilized in Sec. VII in order to prove additivity of E_G for this particular case. Finally, Sec. VIII applies the measure to some examples which arise when a two-mode squeezed state is sent through optical fibers. The Appendix proves a lemma about decompositions of classical Gaussian probability distributions.

II. GAUSSIAN STATES

Consider a bosonic system of n modes, where each mode is characterized by a pair Q_k, P_k of canonical (position and momentum) operators. If we set $R=(Q_1, P_1, \dots, Q_n, P_n)$ the canonical commutation relations are governed by the symplectic matrix

$$\sigma \equiv \bigoplus_{k=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2)$$

via $[R_k, R_l] = i\sigma_{kl}$. A state is called a Gaussian state if it is completely characterized by the first and second moments of the canonical operators R_k in the sense that the corresponding Wigner function is a Gaussian. Utilizing Weyl displacement operators $W_\xi \equiv e^{i\xi^T \sigma R}$, the first moments $d_k \equiv \text{tr}[\rho R_k]$ can be changed arbitrarily by local unitaries. Hence, all the information about the entanglement of the state is contained in the covariance matrix (CM)

$$\gamma_{kl} \equiv \text{tr}[\rho\{R_k - d_k \mathbf{1}, R_l - d_l \mathbf{1}\}_+], \quad (3)$$

where $\{\dots, \dots\}_+$ denotes the anticommutator. By definition the matrix γ is real and symmetric, and due to Heisenberg's uncertainty relation it has to satisfy $\gamma \geq i\sigma$. For pure Gaussian states we have $\det(\gamma) = 1$ or, equivalently, $(\sigma\gamma)^2 = -\mathbf{1}$.

In the following we denote the density operator corresponding to the Gaussian state with covariance matrix γ and displacement vector d by $\rho_{(\gamma, d)}$. If the latter is a bipartite state, its tensor product structure corresponds to a partition of the n modes into two subsets.

An important decomposition of $\rho_{(\gamma, 0)}$ into pure Gaussian states is given by

$$\rho_{(\gamma, d)} \propto \int d^{2n} \xi \rho_{(\gamma_p, d-\xi)} e^{(-1/4)\xi^T (\gamma - \gamma_p)^{-1} \xi}, \quad (4)$$

where $\gamma_p \leq \gamma$ is the covariance matrix of a pure Gaussian state. Since displacements of the form $\rho_{(\gamma, 0)} \mapsto \rho_{(\gamma, d)}$ are local operations, Eq. (4) tells us that starting with $\rho_{(\gamma', 0)}$ we can obtain every Gaussian state with CM $\gamma \geq \gamma'$ by means of LOCC operations.

III. GAUSSIAN ENTANGLEMENT OF FORMATION

We define the Gaussian entanglement of formation E_G for a bipartite Gaussian state $\rho_{(\gamma, d)}$ by

$$E_G(\rho_{(\gamma, d)}) \equiv \inf_{\mu} \left\{ \int \mu(d\gamma_p, dD) E(\rho_{(\gamma_p, D)}) \right\}, \quad (5)$$

$$\rho_{(\gamma, d)} = \int \mu(d\gamma_p, dD) \rho_{(\gamma_p, D)}, \quad (6)$$

where the infimum is taken over all probability measures μ characterizing convex decompositions of $\rho_{(\gamma, d)}$ into pure Gaussian states $\rho_{(\gamma_p, D)}$, and $E(\rho_{(\gamma_p, D)})$ is the von Neumann entropy of the reduced state. For pure $n \times n$ mode Gaussian states this quantity can be readily expressed in terms of the symplectic eigenvalues of the reduced CM. Denote these eigenvalues by $a_k, k=1, \dots, n$. We have $a_k \geq 1$ and define $r_k \geq 0$ by $a_k = \cosh r_k$. Then

$$E(\rho_{(\gamma, d)}) = \sum_k H(r_k), \quad (7)$$

where

$$H(r) = \cosh^2(r) \log_2(\cosh^2 r) - \sinh^2(r) \log_2(\sinh^2 r). \quad (8)$$

To obtain this expression note that any pure $n \times n$ Gaussian state is locally equivalent (via unitary GLOCC) to the tensor product of n two-mode squeezed states with squeezing parameters r_k [13]. For each tensor factor the entanglement is given by the above formula. Since the symplectic spectrum of the reduced CM is invariant under local unitary GLOCC the r_k can be computed directly from γ_p as described above.

The integrals in Eqs. (5) and (6) are taken over the space \mathbb{R}^n of displacements and over the set of admissible pure state covariance matrices. The following proposition tells us that it is sufficient to only consider the measures μ , which vanish for all but one covariance matrix.

Proposition 1. The Gaussian entanglement of formation for the bipartite Gaussian state $\rho_{(\gamma, d)}$ is given by

$$E_G(\rho_{(\gamma, d)}) = \inf_{\gamma_p} \{ E(\rho_{(\gamma_p, 0)}) \mid \gamma_p \leq \gamma \}, \quad (9)$$

where the infimum is taken over pure Gaussian states with CM γ_p .

Proof. The proof can be divided into three steps.

(i) The problem can be reformulated in terms of classical Gaussian distributions, by considering Wigner functions instead of density operators. This is formally achieved by taking the trace of the decomposition in Eq. (6) with the space displaced parity operator $\mathcal{P}_\xi = W_\xi \mathcal{P} W_\xi^*$ [14]. Then

$$\text{tr}[\mathcal{P}_\xi \rho_{(\gamma, d)}] = |\gamma|^{-1/2} \exp[-(\sigma\xi + d)^T \gamma^{-1} (\sigma\xi + d)] \quad (10)$$

is up to a normalization factor equal to the Wigner function of $\rho_{(\gamma, d)}$, which in turn completely determines the state.

(ii) All the states $\rho_{(\gamma_p, d)}$ contributing to Eqs. (5) and (6) must have smaller covariance matrices $\gamma_p \leq \gamma$, i.e.,

$$\mu[\{(\gamma_p, D) \mid \exists_x: \langle x, (\gamma - \gamma_p)x \rangle < 0\}] = 0. \quad (11)$$

The idea of the proof is that the tails of a Gaussian with CM that is too large would give rise to an increasing and in the end overflowing contribution if we only move far enough away from the center. This is mathematically formalized in Lemma 3 stated and proven in the Appendix. To apply Lemma 3 via Eq. (10) to Eq. (5) we need in addition, that the inverse is operator monotone on positive matrices $\gamma > 0$.

(iii) Assume that $\tilde{\mu}$ is a measure corresponding to an

optimal decomposition of $\rho_{(\gamma,d)}$ giving rise to the infimum in Eq. (5). Then

$$E_G(\rho_{(\gamma,d)}) = \int_{\gamma_p \leq \gamma} \tilde{\mu}(d\gamma_p, dD) E(\rho_{(\gamma_p, D)}) \quad (12)$$

$$\geq \inf_{\gamma_p} \{E(\rho_{(\gamma_p, 0)}) | \gamma_p \leq \gamma\}. \quad (13)$$

However, by using a Gaussian decomposition of the form in Eq. (4) we know that equality in Eq. (13) can be achieved for a measure $\tilde{\mu}$ which is Gaussian in D and a delta function with respect to γ_p . ■

Proposition 1 considerably simplifies the calculation of E_G , since the optimization is reduced from the set of all possible decompositions to the set of pure states satisfying the matrix inequality $\gamma_p \leq \gamma$. Before we proceed to calculate E_G analytically for the two-mode case, we will show that E_G is indeed a proper entanglement monotone. To confine the argument to CM's (rather than density matrices) we make use of Proposition 1.

IV. MONOTONICITY UNDER GAUSSIAN OPERATIONS

For E_G to serve as a good Gaussian entanglement measure it should not increase under GLOCC. More precisely, if a GLOCC operation maps an initial Gaussian state ρ onto a state ρ_i with probability p_i , then E_G should not increase on average, i.e.,

$$E_G(\rho) \geq \sum_i p_i E_G(\rho_i). \quad (14)$$

That this is the case is quickly seen using the characterization of Gaussian operations given in Refs. [15,16]. In fact, in the Gaussian case monotonicity holds even for probabilistic operations. That is, $E_G(\rho) \geq E_G(\rho_i)$ is satisfied for every single term in Eq. (14).

It was shown in Refs. [15,16] that the change of the CM γ of a Gaussian state under (probabilistic) Gaussian operations takes the form of a Schur complement

$$\gamma \mapsto G(\gamma) = \tilde{\Gamma}_1 - \tilde{\Gamma}_{12} \frac{1}{\tilde{\Gamma}_2 + \gamma} \tilde{\Gamma}_{12}^T \equiv S(\tilde{\Gamma} + 0 \oplus \gamma). \quad (15)$$

Here

$$\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_2 \end{pmatrix}$$

is the $4n \times n$ CM of the state characterizing the operation G and $\tilde{\Gamma} \geq 0$ is the CM of the partially transposed state.²

For positive matrices $A \geq B$ implies $S(A) \geq S(B)$ (see Ref. [17]). Consequently, if $\gamma \geq \gamma_p$, then the transposed CM fulfills $G(\gamma) \geq G(\gamma_p)$.

Every GLOCC can be decomposed into a *pure* GLOCC

G_p mapping pure states onto pure states, and the addition of classical Gaussian noise. This decomposition can easily be shown using the above mentioned ordering of the Schur complements for ordered matrices. The decomposition then reads $G(\gamma_p) = G_p(\gamma_p) + P$, where the noise is characterized by some positive matrix $P \geq 0$, which is usually state-dependent. Therefore we have $G(\gamma) \geq G(\gamma_p) \geq G_p(\gamma_p)$ and since the latter CM corresponds to a pure state, which can be obtained from γ_p by a local Gaussian operation, its entanglement is certainly smaller than that of γ_p [18]. It follows that E_G cannot increase under GLOCC.

V. THE GENERAL TWO-MODE CASE

Now that we have assured that E_G is a good measure of entanglement in a Gaussian setting, we set to compute it for the case of two Gaussian modes in an arbitrary mixed state. The CM γ of any two-mode Gaussian state can be brought to the normal form [19,20]

$$\gamma = \begin{pmatrix} n_a & k_q \\ k_q & n_b \end{pmatrix} \oplus \begin{pmatrix} n_a & k_p \\ k_p & n_b \end{pmatrix} \equiv C_q \oplus C_p, \quad (16)$$

with $k_q \geq |k_p|$ by local unitary Gaussian operations. The block structure corresponds to a direct sum of position and momentum space, i.e., we have reordered $R = (Q_1, Q_2, P_1, P_2)$.

Since the normal form in Eq. (16) is unique the parameters (n_a, n_b, k_q, k_p) provide a complete set of local invariants.

The first step towards calculating E_G for these states is to show that there is always a pure state γ_p , which is optimal for Eq. (9) and has the same block structure as γ in Eq. (16). To this end we will first provide a general parametrization for pure state CM's, which accounts for the direct sum with respect to configuration and momentum space.

Lemma 1. A real symmetric matrix γ_p is the covariance matrix of a pure Gaussian state of n modes iff there exist real symmetric $n \times n$ matrices X and Y with $X > 0$ such that

$$\gamma_p = \begin{pmatrix} X & XY \\ YX & YXY + X^{-1} \end{pmatrix}, \quad (17)$$

where the block structure corresponds to a direct sum of configuration and momentum space.

Proof. A covariance matrix γ_p corresponds to a pure Gaussian state if $(\gamma_p \sigma)^2 = -\mathbf{1}$. If we write

$$\gamma_p = \begin{pmatrix} X & C \\ C^T & D \end{pmatrix},$$

with $X, D > 0$, then this is equivalent to

$$XD = 1 + C^2, \quad (18)$$

$$DC = (DC)^T, \quad (19)$$

$$CX = (CX)^T. \quad (20)$$

Equation (20) implies that $Y := -X^{-1}C = (X^{-1}C)^T = Y^T$ is indeed symmetric. Moreover, Eq. (18) leads to $D = X^{-1}(1 + C^2)$

²Partial transposition of an $n \times n$ Gaussian density matrix with CM γ yielding a Gaussian operator with CM $\tilde{\gamma} \equiv \Lambda \gamma \Lambda$, where $\Lambda = 1_{2n} \oplus \text{diag}(1, -1, \dots, 1, -1)$.

$=X^{-1}+YXY$. Hence, every covariance matrix of a pure Gaussian state is of the form in Eq. (17).

Conversely, every such matrix γ_p with $X>0$ is positive definite and has symplectic eigenvalues equal to one since the spectrum of $-(\gamma_p\sigma)^2$ is the symplectic spectrum squared of γ_p . Thus every matrix γ_p is an admissible covariance matrix corresponding to a pure Gaussian state. ■

The covariance matrix in the normal form of Eq. (16) only contains terms which are quadratic in the momenta but has no linear contributions. This implies that the state remains invariant under momentum reversal $P\mapsto -P$ and since this can be interpreted as complex conjugation, the respective density operator is real (in position representation).

Equation (17) gives the covariance matrix of a pure Gaussian state with respective wave function

$$\Psi(x) = |\pi X|^{-1/4} \exp\left[-\frac{1}{2}x^T(X^{-1} - iY)x\right], \quad (21)$$

which in turn becomes real if $Y=0$. The following lemma shows that for two-mode states we can, in fact, restrict to these real pure states in the calculation of $E_G(\gamma)$.

Lemma 2. Let $\gamma=C_q\oplus C_p$ be the covariance matrix of a two-mode Gaussian state. Then there exists a pure state with covariance matrix of the same block structure which minimizes E_G for γ .

Proof. We will show that for every $\gamma_p(X,Y)\leq\gamma$ of the form in Eq. (17) the covariance matrix $\gamma_p(X,0)$ leads to an improvement for E_G . First note that the block structure of γ implies that $\gamma\geq\gamma_p(X,\pm Y)$ and thus

$$\gamma\geq\frac{1}{2}[\gamma_p(X,Y)+\gamma_p(X,-Y)] \quad (22)$$

$$=\gamma_p(X,0)+0\oplus(YXY) \quad (23)$$

$$\geq\gamma_p(X,0). \quad (24)$$

Therefore $\gamma_p(X,0)$ is an admissible covariance matrix for the E_G optimization problem.

In order to show that $\gamma_p(X,0)$ is less entangled than $\gamma_p(X,Y)$ we make explicit use of the assumption that we deal with two-mode states. In this case the entanglement is a monotonous function of the determinant of the reduced covariance matrix. The difference of the respective determinants can be calculated straightforward and it is given by

$$\det[\gamma_p^{(A)}(X,Y)]-\det[\gamma_p^{(A)}(X,0)] \quad (25)$$

$$=Y_{12}^2\det[X]\geq 0, \quad (26)$$

which completes the proof. ■

According to Lemma 2 the remaining task for calculating E_G is to find the CM $\gamma_p=X\oplus X^{-1}$ which has minimal entanglement under the constraint that

$$C_p^{-1}\leq X\leq C_q. \quad (27)$$

This inequality has a simple graphical depiction stemming from the fact that the set of positive semidefinite matrices X

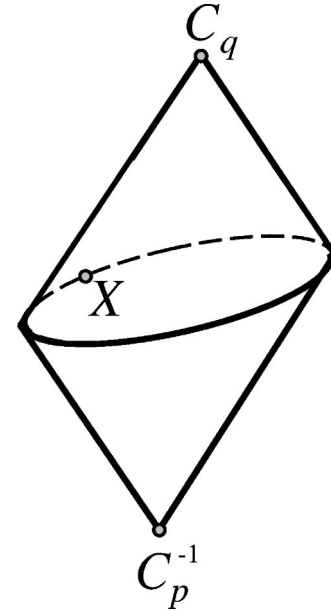


FIG. 1. For any two-mode Gaussian state with CM $\gamma=C_q\oplus C_p$ the Gaussian entanglement of formation is given by the entanglement of the least entangled pure state with CM $\gamma_p=X\oplus X^{-1}$ which is such that $C_p^{-1}\leq X\leq C_q$. Moreover, the optimal X can be shown to lie on the rim of the intersection of the forward and backward cones of C_p^{-1} and C_q , respectively.

satisfying an inequality as, e.g., $C_q-X\geq 0$, form a cone, which is equivalent to the (backward) light cone of C_q in Minkowski space: if we expand a Hermitian 2×2 matrix in terms of Pauli matrices (and the identity), the expansion coefficients play the role of the space-time coordinates and the Minkowski norm is simply given by the determinant of the matrix. Hence, by Eq. (27) X has to lie in the backward cone of C_q and in the forward cone of C_p^{-1} (see Fig. 1).

Instead of minimizing the entropy of the reduced state under this constraint, we may as well minimize the determinant of one of the local covariance matrices

$$m(X)\equiv 1+\frac{X_{12}^2}{\det X}=1+\frac{(X^{-1})_{12}^2}{\det(X^{-1})} \quad (28)$$

$$=X_{11}(X^{-1})_{11}, \quad (29)$$

since, as already stated, this is a monotonically increasing function of entanglement. Thus we have to find

$$\min_x\{m(X)|C_p^{-1}\leq X\leq C_q\} \quad (30)$$

over the real, symmetric 2×2 matrices X .

In fact, for the optimal X both inequalities have to be saturated, i.e.,

$$\det(C_q-X)=\det(X-C_p^{-1})=0. \quad (31)$$

In order to see this assume we are given a matrix X with $C_p^{-1}\leq X\leq C_q$. Then we can decrease the value of $m(X)$ with a matrix $\hat{X}:=X+\epsilon\mathbf{1}$ by increasing $\epsilon>0$ until $C_q-\hat{X}$ is of rank

1. However, by Eq. (28) the same argument holds for $X^{-1} < C_p$.

To depict it geometrically again, the optimal X has to lie on the rim given by the intersection of the backward and forward cones of C_q and C_q^{-1} , respectively. Hence, we have reduced the number of free parameters in the calculation of $E_G(\gamma)$ to one angle, which parametrizes the ellipse of this intersection.

For every explicitly given CM γ minimizing $m(X)$ on this ellipse is now straightforward. Writing down the resulting value for E_G in terms of the general parameters (n_a, n_b, k_q, k_p) of Eq. (16) leads, however, to quite cumbersome formulas involving the roots of a fourth order polynomial. Since not much insight is coming out of these expressions we refrain from writing them down explicitly and continue with discussing the special case $n_a = n_b$ for which we obtain a simple formula for E_G .

Nevertheless, for an arbitrary but explicitly given γ the remaining variation under the constraint in Eq. (31) is a simple exercise which can be solved analytically with the help of any computer algebra program. For some examples, see Sec. VIII.

VI. SYMMETRIC STATES

Symmetric two-mode Gaussian states with CM of the form in Eq. (16) with $n_a = n_b \equiv n$ arise naturally when the two beams of a two-mode squeezed state are sent through identical lossy fibers [21] (see also Sec. VIII). The entanglement of formation E_F of these states was calculated in Ref. [9] and it was proven that a decomposition into Gaussian states gives rise to the optimal value. Together with the obvious fact that E_G is an upper bound for E_F this implies that $E_G = E_F$ in this case. Since the calculation of E_F is however quite technical and in order to make the present article more self-contained, we provide in the following a simpler way to obtain E_G .

In principle we could utilize the general results of the previous section, which simplify greatly for the symmetric case. However, we give an alternative proof and reduce the result to the fact that the optimal γ_p in Eq. (9) has the same logarithmic negativity [22] as γ . A similar argument is used in Sec. VII to prove additivity of E_G .

Proposition 2. (E_G for symmetric states). For symmetric 1×1 Gaussian states, i.e., states whose CM γ is characterized by local invariants (n, n, k_q, k_p) , the Gaussian entanglement of formation is given by

$$E_G(\gamma) = H(r_0), \tag{32}$$

where the minimum two-mode squeezing required is given by

$$r_0 = \frac{1}{2} \ln[(n - k_q)(n + k_p)] \tag{33}$$

and $H(r)$ is defined in Eq. (8).

Proof. First, instead of γ we consider the locally equivalent

CM γ' which is obtained from γ by squeezing³ both Q_A and Q_B by $\lambda = [(n + k_p)/(n - k_q)]^{1/4}$. Clearly the CM γ' has the same E_G as γ . It is straightforward to check that the pure two-mode squeezed state with two-mode squeezing parameter r_0 and corresponding CM $\gamma(r_0)$ is indeed smaller than γ' .

That there can be no pure state γ_p with less entanglement satisfying $\gamma_p \leq \gamma'$ follows from the monotonic dependence of pure state entanglement on the two-mode squeezing parameter: any pure two-mode Gaussian state is locally equivalent to a two-mode squeezed state $\gamma(r)$ and its entanglement is given by $H(r)$. An important entanglement-related characteristic of these CM's are the *symplectic eigenvalues* of the partially transposed CM [22], in particular those smaller than one. They are invariant under local unitary Gaussian operations and for the two-mode squeezed state given by $e^{\pm r}$. For the symmetric CM γ they are $\sqrt{(n \mp k_q)(n \pm k_p)}$. Thus the smallest symplectic eigenvalues of γ and $\gamma(r_0)$ coincide.

For positive matrices $A \geq B$ implies $a_k \geq b_k$, where $a_k(b_k)$ denote the ordered symplectic eigenvalues of $A(B)$ [18]. Since the ordering $A \geq B$ is preserved under the partial transposition, all pure states with less entanglement than $\gamma(r_0)$ cannot possibly satisfy $\gamma \geq \gamma_p$, hence r_0 is optimal. ■

Thus for symmetric states γ the optimal pure state γ_p is characterized by the fact that the smallest symplectic eigenvalues $s_1(\tilde{\gamma})$ and $s_1(\tilde{\gamma}_p)$ of the two partially transposed CM's are identical. According to Ref. [22] this implies that the logarithmic negativity, $E_N(\gamma) = -\frac{1}{2} \ln[s_1(\tilde{\gamma})]$ of both states is the same, i.e., in the optimal decomposition pure Gaussian states are mixed such that “no negativity is lost” in the mixing process. For nonsymmetric states this is no longer possible and $s_1(\tilde{\gamma})$ is strictly larger than $s_1(\tilde{\gamma}_p)$, i.e., more entanglement is needed to form γ than required by its negativity.

VII. ADDITIVITY

One longstanding question about the entanglement of formation is if it is additive, that is, whether $E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)$ or whether one may get an “entanglement discount” when generating several states at a time. Here we show that for symmetric Gaussian states the Gaussian entanglement of formation E_G is additive. Since for these states E_G was shown [9] to equal E_F this may hint at additivity of even the latter quantity for Gaussian states.

Proposition 3 (E_G is additive for symmetric states). Let $\gamma_l, l = 1, \dots, N$ describe symmetric Gaussian states with local invariants $(n_l, n_l, k_{q,l}, k_{p,l})$ and let $\Gamma = \bigoplus_{l=1}^N \gamma_l$ describe the tensor product of these states, then

$$E_G(\Gamma) = \sum_l E_G(\gamma_l). \tag{34}$$

Proof. Let the logarithmic negativity of the k th state be r_k ,

³“Squeezing the γ ” describes the local unitary operation that (in the Heisenberg picture) multiplies (divides) the operators $Q_{A,B}(P_{A,B})$ by λ .

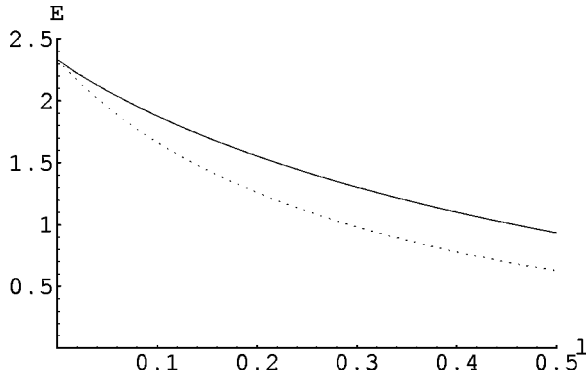


FIG. 2. Gaussian entanglement of formation (in units of ebits) for a TMSS with $r=1$ after transmission through a lossy optical fiber at temperature $\tau=0$. The plot shows $E_G(\rho)$ versus transmission length l/l_A for the symmetric and asymmetric setting (dotted and solid line, respectively).

and assume $r_{k+1} \leq r_k$. To show, additivity, we use again that $A \geq B > 0$ implies $a_k \geq b_k$ for the ordered symplectic eigenvalues of $A(B)$.

Let $\Gamma_p \leq \Gamma$ be pure $N \times N$ -mode CM. Consider the partially transposed CM $\tilde{\Gamma}$. We have

$$\tilde{\Gamma} \geq \tilde{\Gamma}_p,$$

which implies $s_k \geq s'_k$, where $\{s_k, k=1, \dots, N\}$ denote the (descendingly ordered) symplectic eigenvalues of $\tilde{\Gamma}$, and s'_k the same for $\tilde{\Gamma}_p$.

All pure bipartite Gaussian states are locally equivalent to a tensor product of two-mode squeezed states [13] with (ordered) two-mode squeezing parameters t_k . For the following we only need to look at the N smallest symplectic eigenvalues. For these we have

$$s'_k = e^{-t_k} \quad \text{for } \Gamma_p$$

and

$$s_k = e^{-r_k} \quad \text{for } \Gamma.$$

Hence $\Gamma \geq \Gamma_p$ implies $t_k \geq r_k$, i.e., the optimal joint decomposition is the tensor product of the optimal decompositions for the individual copies. Thus E_G is additive. ■

For the nonsymmetric case the optimal individual decomposition does no longer allow $t_k = r_k$, i.e., more entanglement than required by the logarithmic negativity must be expended to produce ρ . Therefore, the previous argument does not hold and the question of additivity of E_G remains open for general 1×1 Gaussian states.

VIII. EXAMPLES

In this section we will apply the Gaussian entanglement of formation to a simple practical example. Consider a two-mode squeezed state (TMSS) with CM

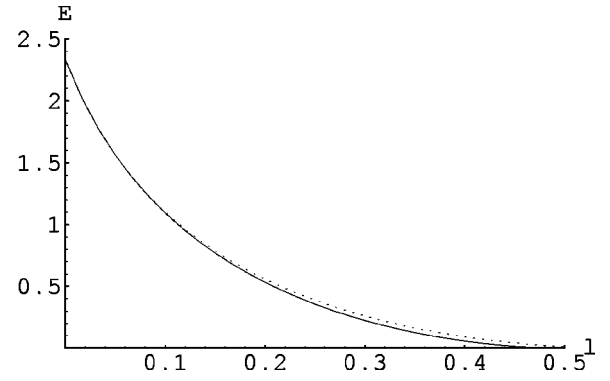


FIG. 3. Gaussian entanglement of formation (in units of ebits) for a TMSS with $r=1$ after transmission through a lossy optical fiber at temperature $\tau=1$. The plot shows $E_G(\rho)$ versus transmission length l/l_A for the symmetric and asymmetric setting (dotted and solid line, respectively).

$$\gamma = \begin{pmatrix} c & 0 & -s & 0 \\ 0 & c & 0 & s \\ -s & 0 & c & 0 \\ 0 & s & 0 & c \end{pmatrix}, \quad (35)$$

$$c = \cosh 2r, \quad s = \sinh 2r,$$

which is to be distributed between two parties by means of a lossy optical fiber. There are two external settings for the transmission of the state which may lead to different values for the distributed entanglement: The source could be placed either at one party's site ("asymmetric setting") or halfway between both parties ("symmetric setting"). In the former case, one mode is transmitted through the whole length of the fiber while the other one is retained unaffected, in the latter both modes are transmitted through half the length of the fiber each. It turns out that depending on thermal noise and transmission length, one or the other setting yields more entanglement for the distributed state from a given squeezing of the initial state.

According, to Ref. [23] the Gaussian state after transmission through the fiber has a CM

$$\gamma' = \begin{pmatrix} c'_1 & 0 & -s' & 0 \\ 0 & c'_1 & 0 & s' \\ -s' & 0 & c'_2 & 0 \\ 0 & s' & 0 & c'_2 \end{pmatrix}, \quad (36)$$

with $c'_i = cT_i^2 + (2N_{\text{th}} + 1)(1 - T_i^2)$ for $i=1, 2$ and $s' = sT_1T_2$. Here T_1, T_2 are the transmission coefficients of the fiber for the respective modes and the fiber is in a thermal state with mean number of photons N_{th} , which is in turn related to a "temperature" τ by $N_{\text{th}} = [\exp(1/\tau) - 1]^{-1}$. Note that in quantum optical settings using optical frequencies we have $\tau=0$.

Depending on the setting, the transmission coefficients take on the values $T_1 = e^{-l/l_A}, T_2 = 1$ (asymmetric setting) or $T_1 = T_2 = e^{-l/(2l_A)}$ (symmetric setting) where l/l_A denotes the total length l of the fiber in units of the absorption length l_A .

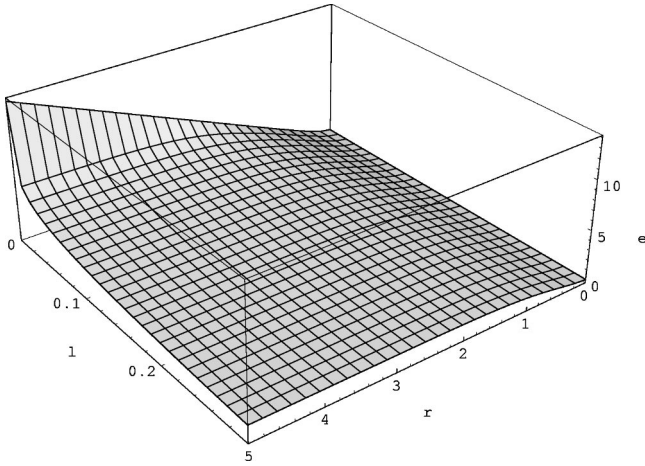


FIG. 4. Gaussian entanglement of formation (in units of ebits) for a TMSS with initial squeezing r after symmetric transmission through a lossy optical fiber at zero temperature. l/l_A is the transmission length in units of the absorption length.

As an example, we compare the two settings for a TMSS with $r=1$. While for temperature $\tau=0$ (Fig. 2) the asymmetric setting always yields a higher entanglement for the final state, at finite temperature $\tau=1$ (Fig. 3) the symmetric setting is to be preferred for longer ranges.

Figure 4 shows E_G as a function of the initial squeezing r and the transmission length l for the symmetric setting at zero temperature. As already indicated in Refs. [24,25] increasing the squeezing over a certain threshold has only a negligible effect on the transmitted entanglement already after a small fraction of the absorption length.

IX. CONCLUSION

We introduced a Gaussian version of the entanglement of formation by taking into account only decompositions into pure Gaussian states. On Gaussian states this is a proper entanglement measure in the sense it is nonincreasing under GLOCC operations. Moreover, it is an upper bound for the full entanglement of formation which is tight and additive at least for symmetric two-mode Gaussian states. However, it remains an open question whether this is true for all Gaussian states.

We have shown how to analytically calculate E_G for all two-mode Gaussian states and given a simple formula for the symmetric case. For multimode bipartite states, the (numerical) computation of E_G becomes rapidly more difficult: Although Proposition 1 still allows us to replace the minimization over all possible decomposition into pure Gaussian states by the minimization over the (for the $n \times n$ case) $2n(2n+1)$ parameter set of pure $2n$ -mode covariance matrices the further simplifications used in the 1×1 case are no longer available and the minimization is no longer easy. To compute E_G for interesting multimode states such as two copies on nonsymmetric 1×1 states or bound entangled states [26] new methods need to be developed.

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APPENDIX: PROOF OF LEMMA 3

The following lemma about decompositions of classical multivariate Gaussian distributions is used in the proof of Proposition 1.

Lemma 3. Let

$$G(A, a, x) = |A|^{1/2} (2\pi)^{-n/2} e^{-(1/2)(x-a)^T A (x-a)}, \quad x, a \in \mathbb{R}^n \quad (\text{A1})$$

be a Gaussian probability distribution with symmetric $A > 0$ and consider an arbitrary convex decomposition of $G(A, 0, x)$ into other Gaussian distributions of this form:

$$G(A, 0, x) = \int \mu(dB, db) G(B, b, x). \quad (\text{A2})$$

Then all the distributions contributing to this decomposition have to satisfy $B \geq A$ in the sense that

$$\mu[\{(B, b) | \exists x: x^T (A - B)x > 0\}] = 0. \quad (\text{A3})$$

Proof. Let us first define a set

$$S(x, \epsilon) = \{(B, b) | x^T A x - \epsilon \|x\|^2 \geq x^T B x\}. \quad (\text{A4})$$

Integrating Eq. (A2) only over $S(x, \epsilon)$ leads then to a lower bound on $G(A, 0, x)$:

$$G(A, 0, x) \geq \int_{S(x, \epsilon)} \mu(dB, db) G(B, b, x). \quad (\text{A5})$$

Inserting $G(A, 0, x)$ and $G(B, b, x)$ we can symmetrize inequality (A5) with respect to $x \mapsto -x$, which leads to

$$\begin{aligned} |A|^{1/2} e^{-(1/2)x^T A x} &\geq \int_{S(x, \epsilon)} \mu(dB, db) \\ &\times |B|^{1/2} \cosh(x^T B b) e^{-(1/2)(x^T B x + b^T B b)}. \end{aligned} \quad (\text{A6})$$

Utilizing $\cosh \geq 1$ and the defining property of the set $S(x, \epsilon)$ we have

$$|A|^{1/2} e^{(\epsilon/2)\|x\|^2} \geq \int_{S(x, \epsilon)} \mu(dB, db) |B|^{1/2} e^{(1/2)b^T B b}. \quad (\text{A7})$$

Note that the right-hand side of Eq. (A7) does no longer depend on the norm of x but merely on its angular components. Taking the limit $\|x\| \rightarrow \infty$ implies then that $S(x, \epsilon)$ is of measure zero for every $\epsilon > 0$. Moreover, every countable union of such sets is of measure zero. In particular,

$$\bigcup_{x \in \mathbb{Q}^n} \bigcup_{m \in \mathbb{N}} S\left(x, \frac{1}{m}\right) \quad (\text{A8})$$

$$= \left\{ (B, b) \left| \exists x \in \mathbb{Q}^n, \exists m \in \mathbb{N} : x^T A x \geq x^T B x + \frac{1}{m} \|x\|^2 \right. \right\} \quad (\text{A9}) \quad = \{(B, b) | \exists x \in \mathbb{R}^n : x^T (A - B)x > 0\}, \quad (\text{A11})$$

$$= \{(B, b) | \exists x \in \mathbb{Q}^n : x^T A x > x^T B x\} \quad (\text{A10}) \quad \text{where we have, of course, used that } \mathbb{Q}^n \text{ is dense in } \mathbb{R}^n. \quad \blacksquare$$

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