

## Entanglement echoes in quantum computation

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We study the stability of entanglement in a quantum computer implementing an efficient quantum algorithm, which simulates a quantum chaotic dynamics. For this purpose, we perform a forward-backward evolution of an initial state in which two qubits are in a maximally entangled Bell state. If the dynamics is reversed after an evolution time  $t_r$ , there is an echo of the entanglement between these two qubits at time  $t_e = 2t_r$ . Perturbations attenuate the pairwise entanglement echo and generate entanglement between these two qubits and the other qubits of the quantum computer.

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The development of new techniques which could enhance the reliability of quantum computation is intrinsically connected with the study of its stability. Every physical implementation of a quantum computer will have to deal with errors, due to the coupling with the environment or to an imperfect control of the computer hardware. Therefore an accurate study of the stability of a quantum computer, while it is running quantum algorithms, is demanded [1].

Entanglement is arguably the most peculiar feature of quantum systems, with no analog in classical mechanics. Furthermore, it is an important physical resource, which is at the basis of many quantum information protocols, including quantum cryptography [3] and teleportation [4]. For any quantum algorithm operating on pure states, the presence of multipartite (many-qubit) entanglement is necessary to achieve an exponential speedup over classical computation [5]. Therefore the ability to control entangled states is one of the basic requirements for constructing quantum computers.

In this paper, we introduce a suitable method to characterize the stability of pairwise entanglement in quantum computation, by considering the echo of an initially maximally entangled pair of qubits; namely, we assume that our quantum computer is initially in the state  $|\psi_0\rangle = |\Phi_B\rangle \otimes |\chi\rangle$ . Here, the first two qubits are prepared in a maximally entangled Bell state ( $|\Phi_B\rangle$ ), while the other  $n_q - 2$  qubits are set in a pure state ( $|\chi\rangle$ ), and they are completely disentangled from the Bell pair ( $n_q$  denotes the total number of qubits in the quantum computer). The state  $|\psi_0\rangle$  first evolves according to the given quantum algorithm, described by the unitary evolution operator  $\hat{U}$ . Then we invert the sequence of quantum gates that implement this algorithm, that is, we apply  $\hat{U}^\dagger$ . In the ideal case we would reconstruct the initial state, since  $\hat{U}^\dagger \hat{U} |\psi_0\rangle = |\psi_0\rangle$ . However, due to noise and imperfections, the initial state  $|\psi_0\rangle$  is not exactly recovered. In particular, the

first two qubits are no longer in a Bell state, and therefore their pairwise *entanglement echo* is reduced. Conversely, this pair of qubits becomes entangled with the other qubits, thus generating multipartite entanglement. In this paper, we study numerically the attenuation of the pairwise entanglement echo in a quantum computer implementing an efficient quantum algorithm which simulates quantum chaotic dynamics. We point out that the entanglement echo simulations discussed in the following are close in spirit to the spin-echo experiments in many-body quantum systems in the presence of perturbations [6].

We study the entanglement echo for the quantum algorithm simulating the sawtooth map dynamics [2]. The sawtooth map is a periodically driven dynamical system, described by the Hamiltonian

$$H(\theta, n, \tau) = \frac{n^2}{2} - \frac{k(\theta - \pi)^2}{2} \sum_{j=-\infty}^{+\infty} \delta(\tau - jT), \quad (1)$$

where  $(n, \theta)$  are conjugated action-angle variables ( $0 \leq \theta < 2\pi$ ). The time evolution  $\tau \rightarrow \tau + T$  of this system is classically described by the map

$$\bar{n} = n + k(\theta - \pi), \quad \bar{\theta} = \theta + T\bar{n}, \quad (2)$$

where the bars denote the variables after one map iteration. By rescaling  $n \rightarrow p = Tn$ , one can see that classical dynamics depends only on the parameter  $K = kT$ . The classical motion is stable for  $-4 \leq K \leq 0$  and completely chaotic for  $K < -4$  and  $K > 0$ . The quantum evolution in one map iteration is described by the unitary operator  $\hat{U}$ :

$$|\bar{\psi}\rangle = \hat{U}|\psi\rangle = e^{-iT\hat{n}^2/2} e^{ik(\hat{\theta} - \pi)^2/2} |\psi\rangle, \quad (3)$$

where  $\hat{n} = -i\partial/\partial\theta$  (we set  $\hbar=1$ ). The classical limit is obtained by taking  $k \rightarrow \infty$  and  $T \rightarrow 0$ , keeping  $K = kT$  const. We study map (3) on the torus  $0 \leq \theta < 2\pi$ ,  $-\pi \leq p < \pi$ . With an  $n_q$  qubit quantum computer, we can simulate the quantum dynamics of the sawtooth map with  $N = 2^{n_q}$  levels, and we set  $T = 2\pi/N$ . Hence the angle variable  $\theta$  takes only  $N$  equidistant values in the interval  $0 \leq \theta < 2\pi$ , while the action vari-

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able  $n$  ranges from  $-N/2$  to  $N/2-1$  (corresponding to  $-\pi \leq p < \pi$ ). The effective Planck's constant of the quantum system is  $\hbar_{\text{eff}} \sim 1/N$  and the classical limit corresponds to  $n_q \rightarrow \infty (\hbar_{\text{eff}} \rightarrow 0)$  [2]. We focus on the case  $K=5$ , which corresponds to the chaotic regime.

It is convenient to simulate map (3) by means of the forward-backward Fourier transform between  $\theta$  and  $n$  representations. While the classical fast Fourier transform requires  $O(N \log_2 N)$  operations, an efficient quantum algorithm has been found [2], which uses the quantum Fourier transform and simulates Eq. (3) in  $O(n_q^2 = (\log_2 N)^2)$  elementary quantum gates per map iteration. Therefore this quantum algorithm is exponentially efficient with respect to any known classical algorithm. Moreover, all  $n_q$  qubits are used in an optimal way, that is, no extra work space qubits are required. In this way interesting physical phenomena, such as dynamical localization [7], cantori localization, and anomalous diffusion could be simulated already with less than ten qubits. Therefore the quantum sawtooth map represents an interesting testing ground for quantum computation, and it is important to understand the limits to the quantum computation of this model due to noise and imperfections.

To compute the entanglement echo, we start from the initial state

$$|\psi_0\rangle = |\Phi_B\rangle \otimes |\chi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |00\dots 0\rangle \quad (4)$$

and we perform a forward evolution of the quantum sawtooth map (3) up to time  $t=t_r$ , that is,  $\hat{U} = \hat{U}^{t_r}$  (the discrete time  $t = \tau/T$  denotes the number of map iterations). Then we compute the time-reversal evolution up to the *echo time*  $t_e = 2t_r$  [namely,  $\hat{U}^\dagger = (\hat{U}^\dagger)^{t_r}$ ]. Our algorithm can be decomposed into single-qubit Hadamard gates and two-qubit controlled-phase-shift gates [2]. In particular, the Hadamard gate can be written as  $\hat{\mathbf{n}}_H \cdot \boldsymbol{\sigma}$ , where  $\hat{\mathbf{n}}_H = (1/\sqrt{2}, 0, 1/\sqrt{2})$  and  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ ,  $\sigma_i$ 's being the Pauli matrices. Due to the imperfect control of the quantum system during the quantum computation, the initial state is not perfectly recovered. In this paper, we only deal with unitary errors, modeled by noisy gates. Such noise results, for example, from fluctuations in the duration of the pulses that implement a sequence of quantum gates. These fluctuations add a stochastic perturbation to the Hamiltonian that describes the evolution of the quantum computer. Unitary errors accumulate during the implementation of a quantum algorithm, which requires a sequence of many elementary quantum gates, and affect the performance of a quantum computer. We assume that errors tilt the rotation axis  $\hat{\mathbf{n}}_H$  by an angle randomly fluctuating in the interval  $[-\epsilon, \epsilon]$ . In the noisy controlled-phase-shift gates, random phases of amplitude inside the interval  $[-\epsilon, \epsilon]$  are added. We assume that the errors affecting two consecutive quantum gates are completely uncorrelated.

Since we consider unitary errors, the state  $|\psi(t)\rangle$  of the quantum computer at any time  $t$  is still a pure state. By tracing  $|\psi(t)\rangle$  over all the qubits, except those initially prepared in a Bell state, we obtain the reduced density matrix

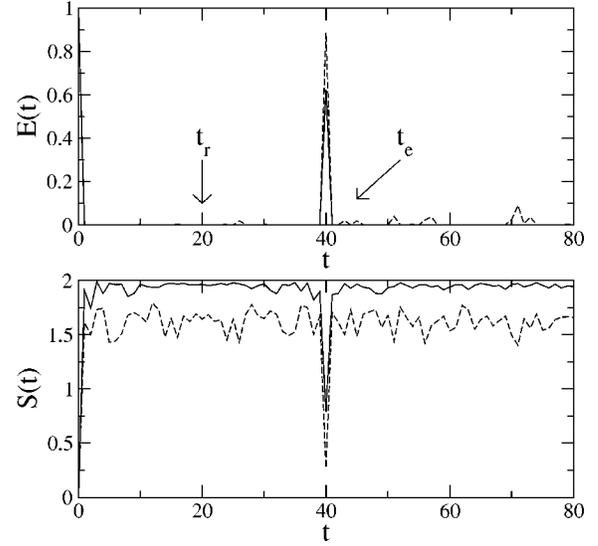


FIG. 1. Entanglement echo in a noisy quantum computer implementing the sawtooth map algorithm in the chaotic regime at  $K=5$ , with  $n_q=5$  qubits (dashed line) and  $n_q=8$  qubits (solid line), and perturbation strength  $\epsilon=10^{-2}$ . We start from the initial state (4), and, from  $t=0$  to  $t_r=20$  a forward evolution of the sawtooth map is applied. After that time, we invert the dynamics. The echo occurs at time  $t_e=2t_r=40$ . Top: entanglement of qubits 1 and 2. Bottom: entanglement between these two qubits and the other ones.

$\rho_{12}(t) = \text{Tr}_{3,\dots,n_q}[|\psi(t)\rangle\langle\psi(t)|]$ . Note that, in general, qubits 1 and 2 are no longer disentangled from the other qubits of the quantum computer, and therefore  $\rho_{12}$  is a mixed state. We evaluate the entanglement of formation  $E(t)$  of the state  $\rho_{12}$  following Ref. [8]. First of all we compute the concurrence, defined as  $C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$ , where  $\lambda_i$ 's are the square roots of the eigenvalues of the matrix  $R = \rho_{12} \tilde{\rho}_{12}$ , in decreasing order. Here  $\tilde{\rho}_{12}$  is the spin-flipped matrix of  $\rho_{12}$ , and it is defined by  $\tilde{\rho}_{12} = (\sigma_y \otimes \sigma_y) \rho_{12}^* (\sigma_y \otimes \sigma_y)$  (note that the complex conjugate is taken in the computational basis  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ ). Once the concurrence has been computed, entanglement is obtained as  $E = h[(1 + \sqrt{1 - C^2})/2]$ , where  $h$  is the binary entropy function  $h(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ . We also compute the Von Neumann entropy  $S(t) = -\text{Tr}[\rho_{12}(t) \log_2 \rho_{12}(t)]$  of the reduced density matrix  $\rho_{12}$ . This quantity measures the entanglement between the qubits 1 and 2 and the other  $n_q-2$  qubits of the quantum computer. In particular, we compute the entanglement echo  $E(t_e)$  and the Von Neumann entropy  $S(t_e)$  at the echo time  $t_e$ .

A typical numerical simulation of the entanglement echo is shown in Fig. 1. The upper part shows the behavior of the pairwise entanglement of the two qubits initially prepared in a Bell state; in the lower part we plot the Von Neumann entropy of this two-qubit subsystem. The dynamics completely destroys the initial pairwise entanglement, which is partially recovered only at the echo time  $t_e$  [9]. Instead, the Von Neumann entropy quickly reaches the saturation value. These results can be understood as follows: in few map iterations ( $t \sim 1$ ), the chaotic dynamics transforms the initial state  $|\psi_0\rangle$  into an ergodic state. We can expand the state  $|\psi(t)\rangle$  at time  $t$  over the computational basis:  $|\psi(t)\rangle$

$= \sum_{\alpha_1, \dots, \alpha_{n_q}} c_{\alpha_1, \dots, \alpha_{n_q}}(t) |\alpha_1, \dots, \alpha_{n_q}\rangle$ , where  $\alpha_i = 0, 1$  for  $i = 1, \dots, n_q$ . Since  $|\psi(t)\rangle$  is ergodic, the coefficients  $c_{\alpha_1, \dots, \alpha_{n_q}}(t)$  have random phases and amplitudes  $|c_{\alpha_1, \dots, \alpha_{n_q}}| \sim 1/\sqrt{N}$  (to assure wave-function normalization). For an ergodic system, the reduced density matrix of a two-qubit subsystem is essentially diagonal. Indeed, the diagonal matrix elements are given by  $(\rho_{12})_{\alpha_1, \alpha_2; \alpha_1, \alpha_2} = \sum_{\alpha_3, \dots, \alpha_{n_q}} |c_{\alpha_1, \dots, \alpha_{n_q}}|^2$ , and their value is  $\approx 1/4$ , since they are given by the sum of  $N/4$  positive terms, whose value is  $\sim 1/N$ . The off-diagonal matrix elements of  $\rho_{12}(t)$  are instead given by the sum of  $N/4$  terms of amplitude  $1/N$  and random phases. Hence their value is  $O(1/\sqrt{N})$ . For such a nearly diagonal density matrix  $\rho_{12}(t)$ , the entanglement of formation can be analytically computed and we find that  $E=0$ . This means that chaotic dynamics quickly destroys the entanglement of any two-qubit subsystem, as shown in Fig. 1. Under the hypothesis that the wave function is ergodic, it is also possible to compute analytically the Von Neumann entropy  $S$  of the qubits 1 and 2. After averaging over noise realizations, one has  $S \approx 2 - 8/(N \ln 2)$  [10,11], in good agreement with our numerical data. This value is close to the maximum possible entropy of the two-qubit subsystem,  $S_{\max} = 2$ . Therefore the chaotic dynamics mimics a decoherence process for a subsystem of the quantum computer, say for qubits 1 and 2. Indeed the density matrix  $\rho_{12}$  becomes essentially diagonal, and the reduced Von Neumann entropy  $S(\rho_{12})$  is maximized. It is interesting to note that, in the presence of noise, it is possible to invert the quantum dynamics after long times (much longer than the times for relaxation to statistical equilibrium) and recover the initial out of equilibrium state. This is a clear demonstration of the stability of the quantum motion in contrast to the high instability of the classical chaotic motion [12].

We now focus on the stability of the entanglement echo at time  $t_e$ . The numerically computed entanglement echo and Von Neumann entropy at the echo time  $t_e$  are shown, for different noise strengths in Figs. 2 and 3, respectively. Figure 2 shows that noisy gates attenuate the entanglement echo, and, if the time  $t_e$  is long enough, completely destroy it. This can be explained by noticing that unitary errors transform the echo state into a state which becomes closer to an ergodic state as  $t_e$  increases. As we have seen above, an ergodic state of the whole quantum computer implies a pairwise entanglement  $E=0$  [13].

The decay of the entanglement echo can be understood by considering that each noisy gate transfers a probability of order  $\epsilon^2$  from the ideal state to all other states. Since there are no correlations between consecutive noisy gates, the population of the initial state decays exponentially, and we can write the echo state as follows:

$$|\psi(t_e)\rangle \approx e^{-R\epsilon^2 n_g t_e / 2} |\psi_0\rangle + \sum_{\alpha \neq \alpha_A, \alpha_B} a_\alpha(t_e) |\alpha\rangle, \quad (5)$$

where  $R$  is a constant to be determined numerically,  $n_g t_e$  is the total number of gates required to perform the echo experiment ( $n_g = 3n_q^2 + n_q$  being the number of gates per map iteration), and the sum runs over all the states  $|\alpha\rangle$

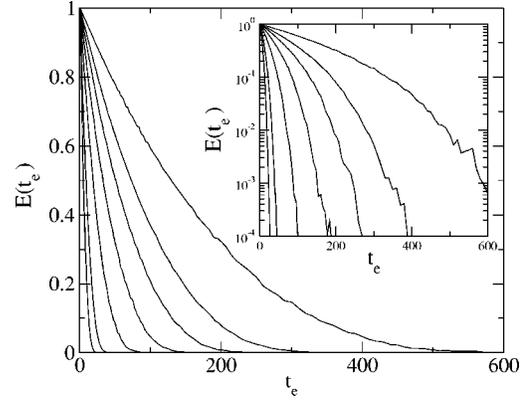


FIG. 2. Attenuation of the entanglement echo of a Bell pair in the sawtooth map, at  $K=5$ ,  $n_q=7$ , and, from right to left,  $\epsilon=7.5 \times 10^{-3}, 10^{-2}, 1.2 \times 10^{-2}, 1.5 \times 10^{-2}, 2 \times 10^{-2}, 3 \times 10^{-2}, 4 \times 10^{-2}$ . Here and in the following figures data are averaged over 400 runs with different noise realizations. Inset: semilogarithmic plot of the same curves.

$= |\alpha_1 \dots \alpha_{n_q}\rangle$  of the computational basis, except for the two states involved in the initial wave vector  $|\psi_0\rangle$  ( $|\alpha_A\rangle = |000\dots 0\rangle$  and  $|\alpha_B\rangle = |110\dots 0\rangle$ ). As the quantum algorithm for the sawtooth map simulates a complex dynamics, it is reasonable to assume that the coefficients  $a_\alpha$  have random signs and amplitudes of the order of  $\sqrt{(1 - e^{-R\epsilon^2 n_g t_e}) / (N-2)}$  [to assure that  $|\psi(t_e)\rangle$  has unit norm]. Given ansatz (5) for the echo wave function, the entanglement echo  $E(t_e)$  can be computed analytically. It turns out that the concurrence  $C(t_e)$  is

$$C(t_e) = \max\left[\frac{1}{2}(3e^{-R\epsilon^2 n_g t_e} - 1), 0\right]. \quad (6)$$

For  $R\epsilon^2 n_g t_e \ll 1$ , we obtain

$$C(t_e) \approx 1 - \frac{3}{2} R\epsilon^2 n_g t_e \quad (7)$$

and

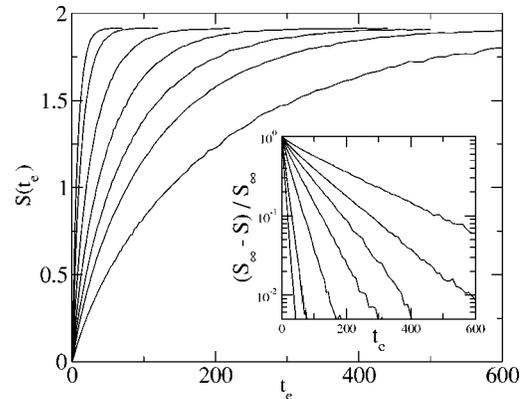


FIG. 3. Von Neumann entropy of the two initially entangled qubits as a function of the echo time  $t_e$ , with the same parameter values as in the previous figure. Inset: approach to the saturation value  $S_\infty$  for the same curves.

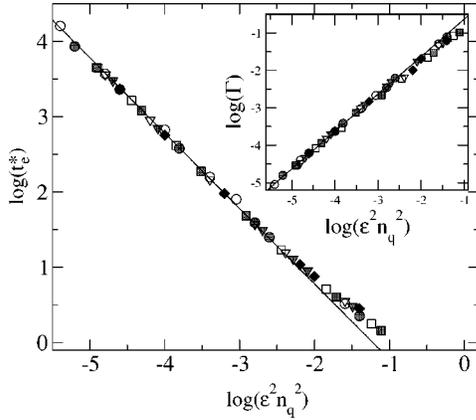


FIG. 4. Time scale  $t_e^*$  for the decay of the entanglement echo in the sawtooth map at  $K=5$ , for different strengths and number of qubits,  $n_q=4$  (empty circles), 5 (filled circles), 6 (empty squares), 7 (filled squares), 8 (empty triangles), 9 (filled triangles), and 10 (diamonds). Straight line:  $t_e^*=A/n_q^2\epsilon^2$ , with the fitting constant  $A \approx 6.04 \times 10^{-2}$ . Inset: rate  $\Gamma$  of the approach to equilibrium of the Von Neumann two-qubit reduced entropy. The straight line gives  $\Gamma=B\epsilon^2 n_q^2$ , with  $B \approx 2.34$ . Logarithms are decimal.

$$E(t_e) \approx 1 - \frac{3}{2 \ln 2} R \epsilon^2 n_g t_e. \quad (8)$$

Therefore the entanglement echo is stable up to time scale

$$t_e \propto 1/(\epsilon^2 n_g) \propto 1/(\epsilon^2 n_q^2). \quad (9)$$

This theoretical estimate is confirmed by Fig. 4, in which we plot the characteristic time scale  $t_e^*$  for the decay of the entanglement echo, defined by the condition  $E(t_e^*)=c$  (we take  $c=0.9$ ). It is clearly seen that  $t_e^* \propto n_q^{-2} \epsilon^{-2}$ . Therefore noisy gates degrade the entanglement echo after a number  $n_e^*$  of elementary gates which is *independent of the number of qubits*:

$$n_e^* = n_g t_e^* \propto \epsilon^{-2}. \quad (10)$$

The Von Neumann entropy  $S(t_e)$  of the reduced two-qubit subsystem at the echo time is shown in Fig. 3. It saturates, for sufficiently long echo times, to the value  $S_\infty \approx 2 - 8/(N \ln 2)$ , as expected for an ergodic state of the quantum

computer. We can compute analytically the approach to equilibrium from Eq. (5), that gives, for  $R\epsilon^2 n_g t_e \gg 1$ ,

$$\frac{S(t_e) - S_\infty}{S_\infty} \approx -\frac{3}{4 \ln 2} \exp(-2R\epsilon^2 n_g t_e). \quad (11)$$

This theoretical prediction is borne out by our numerical data shown in Figs. 3 and 4. The inset of Fig. 3 shows that the approach to the asymptotic value is exponential. Indeed, we have  $S_\infty - S(t_e) \propto e^{-\Gamma t_e}$ . In Fig. 4 (inset) we plot the rate  $\Gamma$  for different number of qubits and perturbation strengths. We see that  $\Gamma \propto \epsilon^2 n_q^2$ .

It is interesting to compare the entanglement echo decay with the decay of the fidelity, which is the usual tool used to characterize the stability of quantum computation [1,2]. The fidelity  $f$  at time  $t_e$  is defined as  $f(t_e) = |\langle \psi(t_e) | \psi_0 \rangle|^2$ , and Eq. (5) implies that

$$f(t_e) \approx \exp(-R\epsilon^2 n_g t_e). \quad (12)$$

This is in agreement with our numerical data (not shown here). Therefore the decay of the fidelity, the decay of the entanglement echo, and the approach to equilibrium for the reduced Von Neumann entropy take place *in the same time scale*  $\propto 1/(\epsilon^2 n_g)$  [14].

In summary, we have proposed a suitable method, the entanglement echo, to study the stability of entanglement under perturbations. We have shown that noise destroys the entanglement of a pair of qubits and produces entanglement between these two qubits and the other qubits of the quantum computer. We point out that, since the entanglement can be measured experimentally in an efficient way [15], entanglement echo experiments analogous to the numerical simulations discussed in this paper could be implemented in quantum processors with a small number of qubits (4–10) and a few hundreds of gates. These experiments are close to present capabilities [16–18] and would bring new insights in our understanding of the limits to quantum computation due to decoherence and imperfections.

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