Quantum-information entropies of the eigenstates and the coherent state of the Pöschl-Teller potential

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The position and momentum space information entropies, of the ground state of the Pöschl-Teller potential, are exactly evaluated and are found to satisfy the bound obtained by Beckner, Bialynicki-Birula, and Mycielski. These entropies for the first excited state, for different strengths of the potential well, are then numerically obtained. Interesting features of the entropy densities, owing their origin to the excited nature of the wave functions, are graphically demonstrated. We then compute the position space entropies of the coherent state of the Pöschl-Teller potential, which is known to show revival and fractional revival. Time evolution of the coherent state reveals many interesting patterns in the space-time flow of information entropy.

I. INTRODUCTION

Information entropy plays a crucial role in a *stronger* formulation of the uncertainty relations. The information theoretic uncertainty relations were first conjectured by Everett [1] and Hirschman [2] in 1957, and proved by Bialynicki-Birula and Mycielski, and independently by Beckner [3]. From the general properties of the Fourier transform, it was proved that $S_{pos} + S_{mom} \geqslant 1 + \ln \pi$. Here, $S_{pos} = -\int dx |\psi(x)|^2 \ln |\psi(x)|^2$ and $S_{mom} = -\int dp |\tilde{\psi}(p)|^2 \ln |\tilde{\psi}(p)|^2$ are the position and momentum space entropies, respectively. In a D-dimensional space, the right-hand side of the above inequality contains a multiplicative factor D. The above equation, known in the literature as Beckner, Bialynicki-Birula and Mycielski (BBM) inequality, captures the physical fact that localized $|\psi(x)|^2$ leads to a diffused $|\tilde{\psi}(p)|^2$ and vice versa. It should be emphasized that, though S_{pos} and S_{mom} are individually unbounded, their sum is bounded from below. It is interesting to point out that, the above mentioned inequality was discovered by Everett in the context of *many worlds interpretation* of quantum mechanics. A framework for deriving uncertainty relations of the above type, between general dynamical variables, not necessarily canonically conjugate ones, have been given recently [4–9]. A more general formulation of information theoretic uncertainty relations, which incorporates a pair of arbitrary quantum measurements, including both projective and single measurements have also been given [10].

The need to go beyond the Heisenberg type uncertainty relations arises because of the fact that the key assumption in its derivation, of finite dispersions for the two canonically conjugate variables, fails to hold for certain quantal problems. A few examples for which these dispersions are patently divergent are discussed in Refs. [11,12].

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These entropies lead to new and stronger version of the Heisenberg uncertainty relations. Using a variational inequality, relating entropy and standard deviation for an arbitrary one dimensional variable A [3,13,14]: $S(A) \leqslant \frac{1}{2} + \ln(\sqrt{2\pi}\Delta A)$ and the BBM inequality, one can derive Heisenberg type uncertainty relations. Apart from the above-mentioned shortcomings, these relations, are also not very reliable, when conjugate variables are discrete and the corresponding Hilbert space is finite dimensional. These entropies have been quite useful for characterizing quantum entanglement, since the von Neumann entropy $S = -\mathrm{Tr}\{\hat{\rho}\ln\hat{\rho}\}$, for the reduced density operators, is often difficult to calculate. For the sake of completeness, it should be mentioned that there exist various type of entropy functionals in the literature to characterize different classes of functions [15]. Various properties of quantum mechanical entropy and its classical counterparts have been elucidated in Refs. [16–18]. As is clear, the single-particle distribution densities, measuring the spread of the wave functions in coordinate and momentum spaces, define their respective entropies. Interestingly, in the density functional theory of Hohenberg and Kohn, the single particle densities also completely characterize a many-body system [19].

The analytical determination of position and momentum space entropies have been carried out only for a few quantum mechanical systems. For the simple harmonic oscillator, the entropies were exactly calculated for the ground state, in both, coordinate and momentum space, for which the BBM inequality is saturated [20]. For an arbitrary state, the entropies were determined approximately, using asymptotic values of the entropy of the orthogonal Hermite polynomials. The entropy integrals $\int P_n^2(x) \ln P_n^2(x) d\mu(x)$, for several orthogonal polynomials P_n's having suitable measures, have been recently studied, from which the asymptotic expressions for the information entropies for large values of n, have been obtained for D dimensional harmonic oscillator and Coulomb problems [21]. Information entropy of neutral atoms [22,23], in the Thomas-Fermi theory, also manifests in a universal form, analogous to the one given in Ref. [3]. Infor-
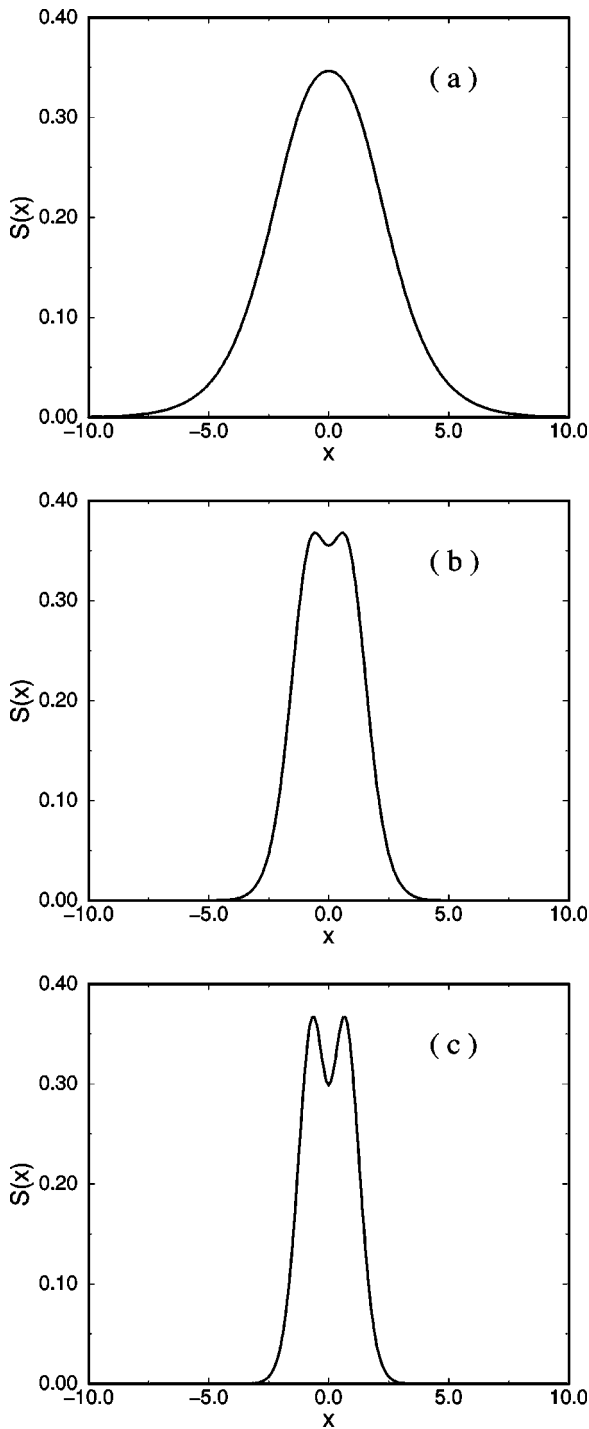
FIG. 1. Plots of the position space entropy densities for the ground state of hyperbolic Pöschl-Teller potential for (a) $n=1$, (b) $n=3$, and (c) $n=5$.

mation entropies in various contexts, e.g., mathematical physics, mathematics, information theory, chemical physics and other areas of physics, have been extensively analyzed in recent times [12,24–39].

The present article is devoted to the study of the information entropies of the Pöschl-Teller (PT) family of potentials. These potentials widely appear in the analysis of soliton bearing nonlinear equations, e.g., Bose-Einstein condensates

and in quantum problems on curved background [40–42]. In the following section, we first consider the hyperbolic PT potential and evaluate the position and momentum space entropies exactly for the ground state, for a range of potential strengths. For the first excited state, we calculate these entropies, numerically, which is shown to satisfy the bound obtained by Bialynicki-Birula and Mycielski. Some interesting features of the entropy densities are then graphically demonstrated. In the third section, we compute the position space entropy densities of the coherent states of the trigonometric PT potential, which exhibits revival and fractional revival, due to interference effects. Under time evolution these densities reveal interesting patterns in the space-time flow of information entropy. We conclude in the last section after pointing out various future directions of work in this area.

II. INFORMATION ENTROPY FOR PÖSCHL-TELLER SYSTEMS

We begin with the Schrödinger equation for hyperbolic Pöschl-Teller potential, which is reflectionless and admits n bound states for integer values of n (in units $\hbar=2m=1$),

$$\left[-\frac{d^2}{dx^2} - \frac{n(n+1)}{4}\mathrm{sech}^2\left(\frac{x}{2}\right)\right]\psi^{(n)}(x) = E\psi^{(n)}(x). \quad (1)$$

The normalized ground state eigenfunction is given by

$$\psi_0^{(n)}(x) = \frac{1}{\sqrt{2B\left(\frac{1}{2},n\right)}}\mathrm{sech}^n\left(\frac{x}{2}\right), \quad (2)$$

where $B\left(\frac{1}{2},n\right)$ is the beta function.

Using the definition of position space entropy, after a lengthy but straightforward calculation, we obtain the analytical expression:

$$S_{pos} = -(2n-1)\ln 2 + \ln B\left(\frac{1}{2},n\right) + 2n[\Psi(2n) - \Psi(n)], \quad (3)$$

where Ψ is the digamma function. For $n=1$ and 2, the position space entropy has the values $S_{pos}=2$ and $\frac{10}{3}-\ln 6$, respectively.

The corresponding momentum space entropies can be evaluated by first obtaining the momentum space ground state wave functions, which are the Fourier transforms of the corresponding position space wave functions:

$$\psi_0^{(n)}(p) = A\ 2^n B\left(\frac{n}{2} + ip, \frac{n}{2} - ip\right). \quad (4)$$

For $n=1$, $\psi_0(p)=\sqrt{\pi/2}\,\mathrm{sech}(\pi p)$ and the S_{mom} can be easily evaluated,

$$S_{mom} = -\int_{-\infty}^{\infty} 2\psi_0^2(p)\ln\ \psi_0(p)dp = 2 - \ln(2\pi),$$
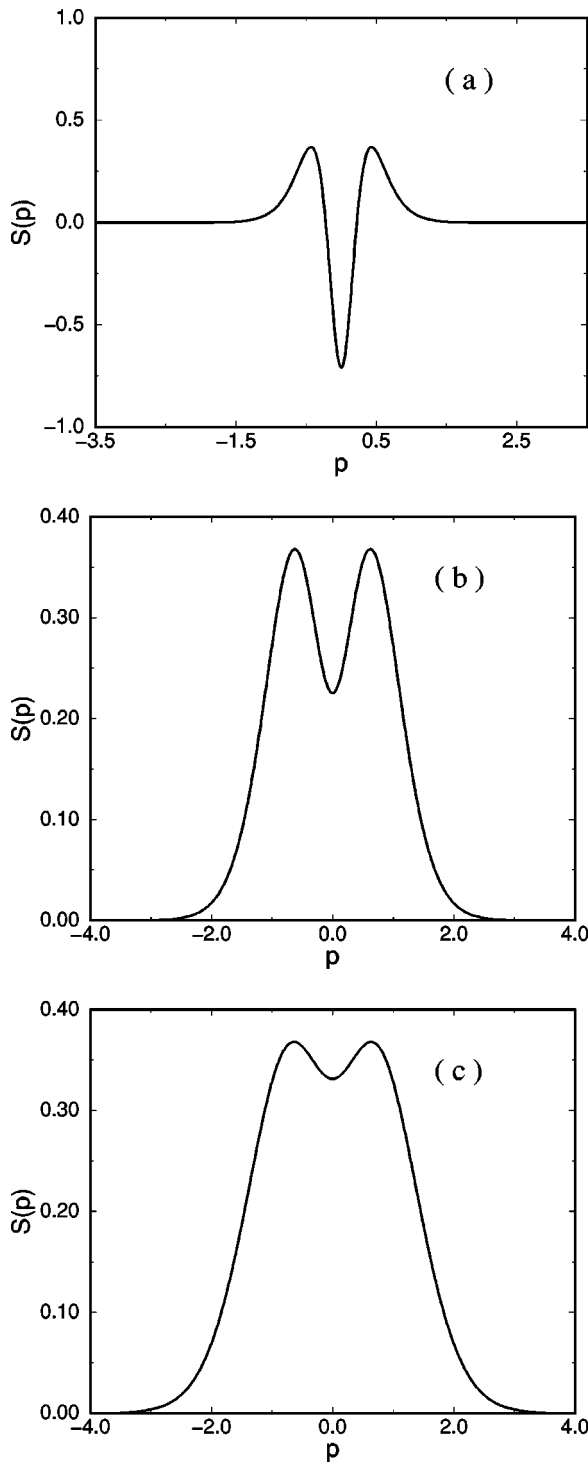
and the corresponding BBM inequality reads

FIG. 3. The plot depicting the variation of sum of position and momentum space entropies with respect to potential parameter n.

crease the value of the parameter n; exactly contrary behavior is observed in their momentum space counterparts. As seen in Fig. 3, for the ground state, as n increases, the BBM inequality tends to be saturated. Physically, for increasing n, the depth of the potential increases and it increasingly resembles the oscillator potential, which saturates the above inequality.

We now proceed to evaluate the position space entropies of the first excited state. The corresponding wave function for the potential $V(x)$ reads

$$\psi_1^{(n)}(x) = N \operatorname{sech}^{n-1}\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right) \quad \text{for} \quad n \geq 2, \quad (5)$$

where $N = 1/\sqrt{2\left[B\left(\frac{1}{2}, n-1\right) - B\left(\frac{1}{2}, n\right)\right]}$ is the normalization constant.

For $n=2$, $S_{pos} = 2.23472$ and for general n the behavior of the position space information entropy is depicted in Fig. 2. Table I, depicts the BBM inequality for the first excited state as a function of n. One sees that as the value of n increases the sum of the entropies tends towards a saturation value higher than the ground state value.

III. ENTROPY DENSITIES FOR COHERENT STATES OF THE TRIGONOMETRIC PÖSCHL-TELLER POTENTIAL

Quantum systems with eigenspectra depending quadratically on a quantum number n are known to show revival and partial revivals in time evolution of corresponding wave packets. These quantum carpet structures have been studied quite extensively [43]. There have been suggestions to use the revival structure for obtaining a factorization algorithm [44]. The possibility of realizing PT type of potentials in atomic systems such as BEC, through optical means, makes the study of time evolution of these systems more interesting [45]. It should be pointed out that perturbation of BEC on a soliton or cnoidal wave type solitary train background are known to satisfy the hyperbolic or trigonometric PT

FIG. 2. Plots of the momentum space entropy densities for the ground state of hyperbolic Pöschl-Teller potential for (a) $n=1$, (b) $n=3$, and (c) $n=5$.

$$S_{pos} + S_{mom} = 4 - \ln 2\pi \geq 1 + \ln(\pi).$$

For higher values of n, evaluation of momentum space entropies is quite cumbersome, instead we plot entropy densities for both position and momentum space. As shown in Figs. 1 and 2, it is interesting to notice that, the position entropy-density plots, develop a dip at its peak as we in-
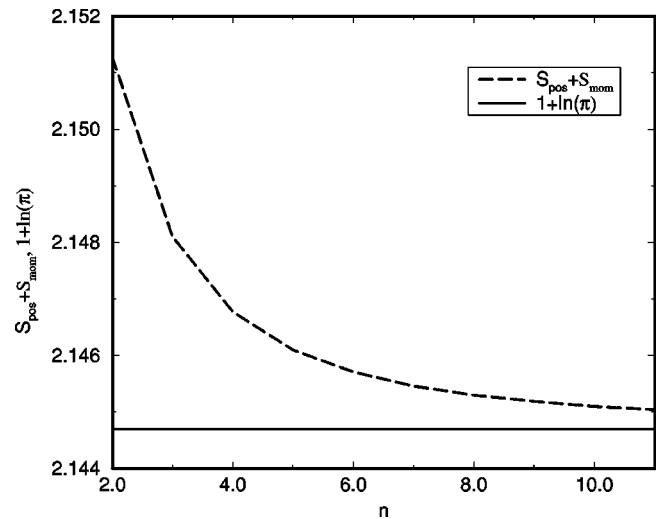
TABLE I. Table for BBM inequality for the first excited state of the hyperbolic Pöschl-Teller potential.

n	S_{pos}	S_{mom}	$S_{pos}+S_{mom}$	$1+\ln\pi$	n	S_{pos}	S_{mom}	$S_{pos}+S_{mom}$	$1+\ln\pi$
2	2.23472	0.722555	2.95728	2.1447	8	1.0971	1.63508	2.73217	2.1447
3	1.7988	1.0384	2.8372	2.1447	9	1.02621	1.70025	2.72646	2.1447
4	1.56242	1.22799	2.7904	2.1447	10	0.96409	1.7579	2.72199	2.1447
5	1.40082	1.36474	2.76556	2.1447	11	0.908807	1.80958	2.71839	2.1447
6	1.27825	1.47193	2.75018	2.1447	12	0.859009	1.85643	2.71544	2.1447
7	1.1796	1.56013	2.73973	2.1447	13	0.81371	1.89926	2.71297	2.1447

Schrödinger equations [41]. In the following we study the time evolution of the information entropy density for an annihilation operator coherent state of the trigonometric PT potential [46]. The fact that, coherent structure like laser is an annihilation operator eigenstate and the coherent manipulation of atoms, possibly with optical means is being increasingly considered seriously, may make these analyses useful. The trigonometric case has been chosen deliberately, since it has an infinite number of bound states as compared to the hyperbolic one, which makes the construction of the coherent states straightforward.

We consider here the Hamiltonian of symmetric Pöschl-Teller (SPT) potential (in the units $\hbar=2m=1$),

$$H = -\frac{d^2}{dy^2} + \frac{\alpha^2\rho(\rho-1)}{\cos^2(\alpha y)}, \tag{6}$$

with eigenvalues and eigenfunctions, in the variable $x=\sin(\alpha y)$,

$$E_n^{SPT} = \alpha^2(n+\rho)^2,$$

$$\psi_n^{SPT}(x) = \left[\frac{\alpha(n!)(n+\rho)\Gamma(\rho)\Gamma(2\rho)}{\sqrt{\pi}\Gamma(\rho+1/2)\Gamma(n+2\rho)}\right]^{1/2}(1-x^2)^{\rho/2}C_n^\rho(x). \tag{7}$$

Recently based on a dynamical SU(1,1) algebra, an annihilation operator coherent state, was constructed for this system: $K_-|\gamma> = \gamma|\gamma>$, here K_- is the annihilation operator of the SU(1,1) algebra [46]. The coordinate space realization of this coherent state is given by

$$\tilde{\chi}_{SPT}(x,\gamma) = N(\gamma)^{-1}\sum_{n=0}^{\infty}\left[\frac{\Gamma(2\rho)\Gamma(\rho+1/2)\sqrt{\pi}}{\alpha(n!)(n+\rho)\Gamma(\rho)\Gamma(2\rho+n)}\right]^{1/2}$$
$$\times \gamma^n \psi_n^{SPT}(x). \tag{8}$$

As already pointed out in the beginning of this section, the quadratic nature of the spectra of SPT potential leads to the possibility of revival and fractional revival in this quantum system due to subtle interference effects. Keeping in mind, the fact that in the realistic situations the complete span of the wave functions may not be available, we study the time evolution of position space entropy densities for
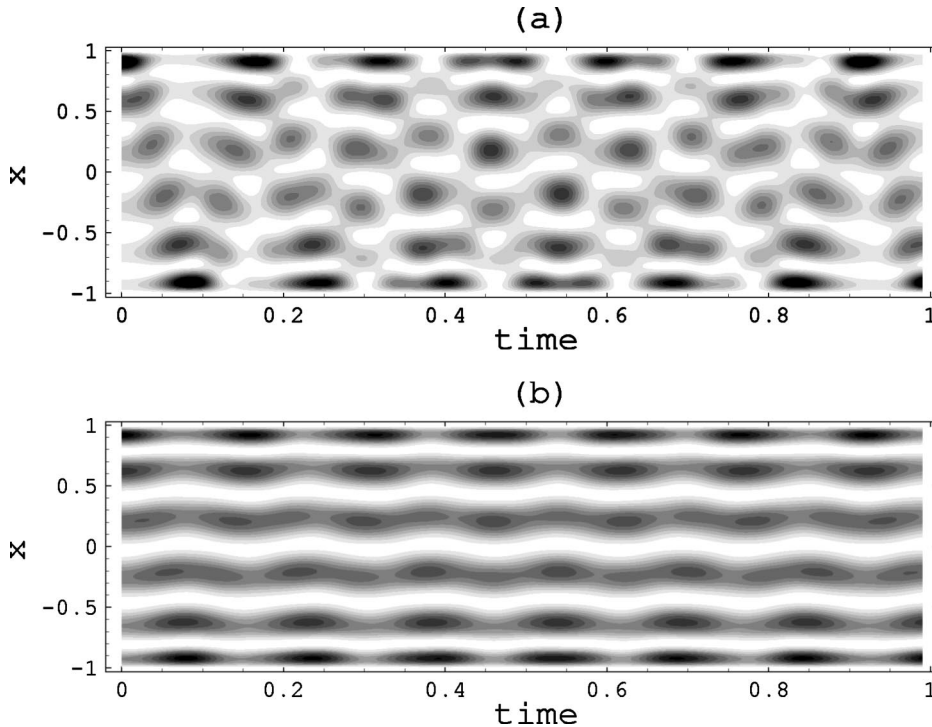
(a)

(b)

FIG. 4. Contour plots, depicting time evolution of position space entropy densities for the coherent states trigonometric Pöschl-Teller potential for (a) $n=5, \gamma=10$ and (b) $n=5, \gamma=30$. Darkness displays a low and brightness a high functional value.
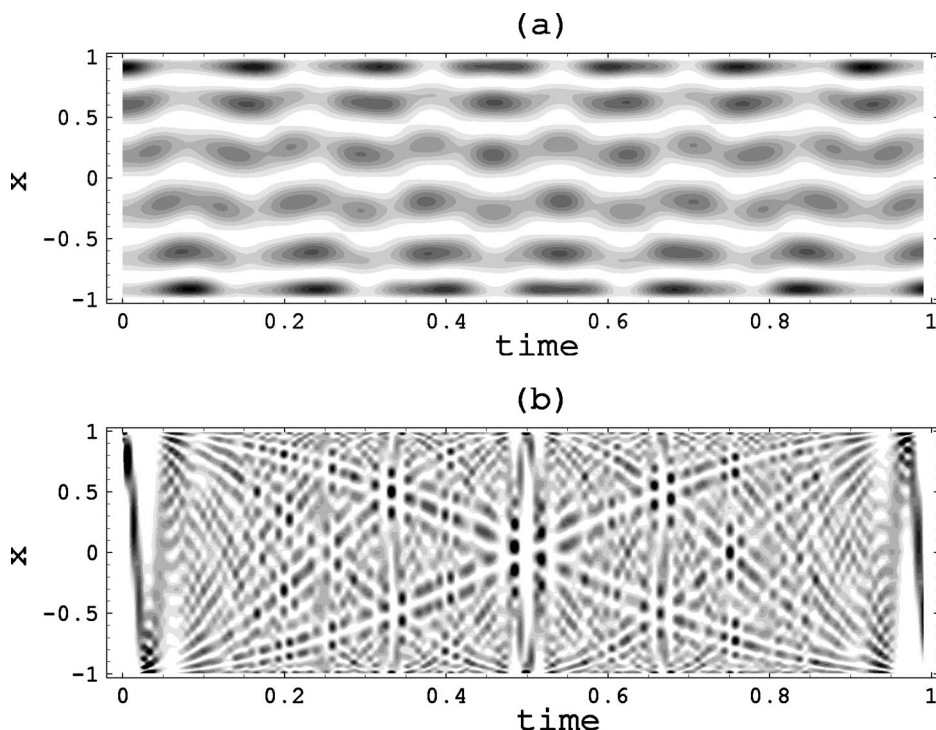
(a)

(b)

FIG. 5. Contour plots, depicting time evolution of position space entropy densities for the coherent states trigonometric Pöschl-Teller potential for (a) $n=5, \gamma=15$ and (b) $n=30, \gamma=15$. Darkness displays a low and brightness a high functional value.

various values of n, i.e., the number of states, interfering and constituting the resultant coherent wave packet. It should be mentioned that the time evolution of this entropy density arises due to the time evolution of the individual states that constitute the coherent state. Since the entropy density of the coherent state at any time t is calculated by taking the modulus square of these states, it depends on time. The effect of change of coherence parameter γ on the same is also analyzed. These are depicted in the Figs. 4 and 5. One finds dramatic changes in the carpet structure of the entropy densities in space and time. One observes rich tapestrylike structures, where one can manipulate the valleys and ridges of the entropy density in space and time. It is interesting to observe that, as we increase the value of coherence parameter γ, keeping n fixed at some value, the various ridges come close together and form a continuous structure. These patterns become sharper for the higher values of n.

IV. CONCLUSIONS

In conclusion, we have studied the information entropies of a class of quantum systems belonging to the Pöschl-Teller family of potentials. Exact results, for the position space entropies of the ground and first excited states of the hyperbolic Pöschl-Teller potential were obtained, for a range of potential strengths. The expression for momentum space entropy

was obtained analytically for the ground state and numerically computed for the first excited state. It was found that, these entropies satisfy the Beckner, Bialynicki-Birula and Mycielski inequality. The entropy densities for the above cases were depicted graphically, for demonstrating the entropy distribution in the well. For the trigonometric case, after investigating the entropies associated with the eigenstates, we studied the time evolution of entropy density for the coherent state [46]. The intricate carpet structure shows the richness of this quantum system, which needs to be explored further. It should be noted that, coherent states are being envisaged for the storage of quantum information. Pöschl-Teller potential manifests in quantum problems on curved background [42], as also in nonlinear integrable models with soliton solutions like, KdV equation [40]. In light of this, the physical relevance of the information entropies computed here, needs further study. We hope to come back to some of these questions in future.

ACKNOWLEDGMENTS

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