Disentanglement and decoherence by open system dynamics

P. J. Dodd and J. J. Halliwell
Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
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The destruction of quantum interference, decoherence, and the destruction of entanglement both appear to occur under the same circumstances. To address the connection between these two phenomena, we consider the evolution of arbitrary initial states of a two-particle system under open system dynamics described by a class of master equations which produce decoherence of each particle. We show that all initial states become separable after a finite time, and we produce the explicit form of the separated state. The result extends and amplifies an earlier result of Diósi. We illustrate the general result by considering the case in which the initial state is an Einstein-Podolsky-Rosen state (in which both the positions and momenta of a particle pair are perfectly correlated). This example clearly illustrates how the spreading out in phase space produced by the environment leads to certain disentanglement conditions becoming satisfied.

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I. INTRODUCTION

Much effort has recently been devoted to understanding the properties of entangled quantum states. This effort is largely driven by the emerging field of quantum computation, and in particular, the desire to manipulate entangled states in a practically useful way. However, another reason why the study of entanglement is of interest concerns the question of emergent classicality from quantum theory. Entanglement represents the possibility of correlations which are greater than those anticipated in classical theories. Hence, any account of emergent classicality must explain how entanglement is lost. The explanation of this is in fact reasonably simple and is closely related to decoherence, the destruction of interference. The purpose of this paper is to discuss the destruction of entanglement in some simple systems and its connection to decoherence.

A state of a bipartite system is said to be separable (or disentangled) if it may be written in the form

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \tag{1}$$

where $p_i \geq 0$ [1]. Such a state describes essentially classical correlations and can never violate Bell's inequalities. However, it turns out to be surprisingly difficult to determine, in general, whether a state may be written in this form. Peres [2] made the very useful observation that a separable ρ remains a density operator under the operation of partial transpose (transposition of one subsystem only). Hence, a necessary condition for separability is that density operator properties are preserved under partial transpose. This condition was shown by Horodecki to be sufficient in the case of 2×2 and 2×3 dimensions [3], but generally not otherwise. In the case of continuous variables the Peres-Horodecki condition has a useful expression in terms of Wigner functions, where Eq. (1) becomes

$$W(p_1,x_1,p_2,x_2) = \sum_i p_i W_i^A(p_1,x_1) W_i^B(p_2,x_2). \tag{2}$$

The Peres-Horodecki condition is then that $W(p_1,x_1,p_2,x_2)$ remains a Wigner function under $p_2 \to -p_2$. It has been shown by Simon that the condition is both necessary and sufficient for the case when the Wigner function of a bipartite system is Gaussian [4]. Duan *et al.* [5] considered a different necessary and sufficient condition for the separability of bipartite Gaussian states based on the variances of a class of pairs of commuting Einstein-Podolsky-Rosen-like (EPR-like) operators (of which $x_1 - x_2$ and $p_1 + p_2$ are an example). (See Ref. [6] for a discussion of the connection between these conditions.) There are undoubtedly more conditions for Gaussian states.

A closely related idea is that of entanglement-breaking maps. This is a map Φ acting on a subsystem A such that $(1_A \otimes \Phi)(\Gamma)$ is separable for all choices of state Γ on $A \otimes B$, and for all finite-dimensional choices of B. A theorem given by Horodecki and Shor then states that a map Φ is entanglement breaking if and only if it has the Holevo form,

$$\Phi(\rho) = \sum_k R_k \mathrm{Tr}(F_k \rho), \tag{3}$$

where F_k are a set of positive-operator-valued measures and R_k are a set of density operators which are independent of ρ [7–9]. Note that this result refers to the dynamics of one of the subsystems only. What is particularly interesting about this type of map is that they naturally appear in the open system master equations of the type frequently used in decoherence studies. In particular, Diósi [10] has recently considered the open system dynamics described by the master equation

$$\dot{\rho} = \frac{i}{\hbar}[H,\rho] - \frac{D}{\hbar^2}[x,[x,\rho]]. \tag{4}$$

This equation describes a particle coupled to a heat bath in the limit of high temperature and negligible dissipation, and is the simplest equation used to describe decoherence. If we write the solution to this equation as

$$\rho_t = \Phi(\rho_0), \tag{5}$$

then Diósi has shown that this map becomes entanglement breaking after sufficient time has passed for the P function to positive [10]. The time taken for this to happen is typically very short and is essentially the same as the time scale required for decoherence. Although what is particularly interesting is that complete disentanglement occurs after a finite time, unlike decoherence which, in the usual view of it, is asymptotic in time. (See also the similar result by Diósi for spin systems in Ref. [11].)

It is not hard to see that both decoherence and disentanglement tend to be produced under the same circumstances. A simple illustration of that fact is as follows. Consider first a one-particle system in an initial superposition state

$$|\Psi\rangle = |\psi\rangle + |\phi\rangle \tag{6}$$

and thus with density operator

$$\rho = |\psi\rangle\langle\psi| + |\psi\rangle\langle\phi| + |\phi\rangle\langle\psi| + |\phi\rangle\langle\phi|. \tag{7}$$

Suppose now it is subject to evolution according to the master equation (4), with solution written in the form of Eq. (5). It is generally known that, if the initial state (6) is a superposition of localized position states, evolution according to Eq. (4) tends to kill the off-diagonal terms. That is, we have

$$\Phi(|\psi\rangle\langle\phi|) \approx 0 \tag{8}$$

after a typically very short time. This means that the density operator becomes essentially indistinguishable from the evolution of the mixed initial state,

$$\rho' = |\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|. \tag{9}$$

This is the simplest account of decoherence of a single particle coupled to an environment.

Now suppose we consider a two-particle system in the entangled state,

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle + |\phi_1\rangle \otimes |\phi_2\rangle \tag{10}$$

with density operator

$$\rho = |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2| + |\psi_1\rangle\langle\phi_1| \otimes |\psi_2\rangle\langle\phi_2| + |\phi_1\rangle\langle\psi_1| \otimes |\phi_2\rangle\langle\psi_2| + |\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2|. \tag{11}$$

If we now let both particles evolve according to the dynamics $\Phi\otimes\Phi$, we find that once again the off-diagonal terms go away, so for example,

$$\Phi(|\psi_1\rangle\langle\phi_1|) \approx 0 \tag{12}$$

and the density operator becomes indistinguishable from that obtained by the initial state,

$$\rho = |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2| + |\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2|, \tag{13}$$

which is separable. Hence, the mechanism that destroys interference also destroys entanglement.

Another way to see why coupling to an environment will destroy entanglement is to appeal to the fact that the property of entanglement has an exclusive quality [12]. Suppose A is entangled with B. Then if B becomes entangled with a third party C it diminishes its entanglement with A. Hence an environment coupling to one or both of the two particles in an entangled state will cause one or both of them to become entangled with the environment, thereby diminishing their entanglement with each other.

The aim of this paper is to investigate the destruction of entanglement through interacting with an environment, extending and elaborating the earlier result of Diósi [10,11] and others [13,14].

In Sec. II we consider the dynamics of a particle coupled to an environment, concentrating on dynamics of the form (4). This is reasonably standard material but we write it in a form which is most useful for studying disentanglement. In Sec. III, we consider the evolution of bipartite systems under the dynamics of Sec. II. We show that an arbitrary initial state achieves the explicitly separated form (2) after finite time. In Sec. IV, we consider the evolution of the EPR state in the presence of an environment. This simple example gives a clear picture of how the various separability conditions come to be satisfied as a result of interacting with the environment. We summarize and conclude in Sec. V.

II. EVOLUTION IN THE PRESENCE OF AN ENVIRONMENT

Before considering the evolution of entangled states in the presence of a thermal environment, it is useful to consider first the simplest case of a single particle coupled to a thermal environment in the limit of high temperature and negligible dissipation, with no external potential. The master equation (4) for the density matrix $\rho(x,y)$ is

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m}\left(\frac{\partial^2\rho}{\partial x^2} - \frac{\partial^2\rho}{\partial y^2}\right) - \frac{D}{\hbar^2}(x-y)^2\rho, \tag{14}$$

where $D=2m\gamma kT$. In the Wigner representation, the corresponding Wigner function

$$W(p,x) = \frac{1}{2\pi\hbar}\int d\xi e^{-(i/\hbar)p\xi}\rho\left(x+\frac{1}{2}\xi, x-\frac{1}{2}\xi\right) \tag{15}$$

obeys the equation

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + D\frac{\partial^2 W}{\partial p^2}. \tag{16}$$

Following Ref. [15], this equation may be solved in the form

$$W_t(p,x) = \int dp_0 dx_0 K(p,x,t|p_0,x_0,0) W_0(p_0,x_0), \tag{17}$$

where $K(p,x,t|p_0,x_0,0)$ is the Wigner function propagator, and is given by

$$K = \exp[-\alpha(p-p_{cl}) - \beta(x-x_{cl}) - \epsilon(p-p_{cl})(x-x_{cl})], \tag{18}$$

where p_{cl} and x_{cl} denote the classical evolution from p_0, x_0 to time t,

$$p_{cl} = p_0, \quad x_{cl} = x_0 + \frac{p_0 t}{m}. \tag{19}$$

(For convenience, we ignore exponential prefactors unless necessary.) The coefficients α, β, and ϵ are given by

$$\alpha = \frac{1}{Dt}, \quad \beta = \frac{3m^2}{Dt^3}, \quad \epsilon = -\frac{3m}{Dt^2}. \tag{20}$$

In fact, the general form of the propagator (18) can be used to describe the most general type of linear dynamics—arbitrary environment temperatures, non-negligible dissipation and the inclusion of a harmonic oscillator potential—for suitable choices of α, β, ϵ, and p_{cl}, x_{cl} [15]. More general dynamics are considered in another paper [16].

With the simple change of variables $x_0 \to x_0 - p_0 t/m$ we may write,

$$W_t(p,x) = \int dp_0 dx_0 \exp[-\alpha(p-p_0) - \beta(x-x_0) - \epsilon(p-p_0) \times (x-x_0)] W_0'(p_0,x_0), \tag{21}$$

where

$$W_0'(p_0,x_0) = W_0(p_0,x_0 - p_0 t/m). \tag{22}$$

This simple transformation is a linear canonical transformation, which corresponds to a unitary transformation of the initial state, so W_0' is still a Wigner function. For decoherence and disentanglement, the important aspects of the evolution are contained in the convolution with the exponential function. Following Diósi and Kiefer [17], it is now very useful to introduce the notation

$$\mathbf{z} = \begin{pmatrix} p \\ x \end{pmatrix} \tag{23}$$

and also to introduce a class of Gaussian phase space functions,

$$g(\mathbf{z};C) = \exp\left(-\tfrac{1}{2}\mathbf{z}^T C^{-1}\mathbf{z}\right). \tag{24}$$

The 2×2 matrix C is positive definite and $|C|$ denotes its determinant. The phase space function $g(\mathbf{z};C)$ is a Wigner function if and only if

$$|C| \geq \frac{\hbar^2}{4}. \tag{25}$$

(This is essentially the uncertainty principle.) A useful result is the simple convolution property,

$$\int d^2z g(\mathbf{z}_1 - \mathbf{z};C) g(\mathbf{z} - \mathbf{z}_2;B) = g(\mathbf{z}_1 - \mathbf{z}_2, C+B). \tag{26}$$

We can use these Gaussians to compute smeared Wigner functions by convolution,

$$\widetilde{W}(\mathbf{z}) = \int d^2z' g(\mathbf{z} - \mathbf{z}';C) W(\mathbf{z}'). \tag{27}$$

Then it follows that the smeared Wigner function will be positive if and only if Eq. (25) holds. This is because the smeared Wigner function is then equal to the overlap of two Wigner functions, for which we have the result

$$\int dpdx \quad W_{\rho_1}(p,x) W_{\rho_2}(p,x) = \frac{1}{2\pi\hbar}\mathrm{Tr}(\rho_1\rho_2), \tag{28}$$

which is clearly always positive. For example, the Q function, which is always positive, is obtained in this way by smearing with a minimum uncertainty Wigner function. In this notation the propagation (21) of the Wigner function is

$$W_t(\mathbf{z}) = \int d^2z' g(\mathbf{z} - \mathbf{z}';A) W_0'(\mathbf{z}'). \tag{29}$$

For the free particle without dissipation considered above we have

$$A = Dt\begin{pmatrix} 2 & t/m \\ t/m & 2t^2/3m^2 \end{pmatrix} \tag{30}$$

and therefore

$$|A| = \frac{D^2 t^4}{3m^2}, \tag{31}$$

showing that the Wigner function tends to spread out with time.

Using the above description of the dynamics, it is straightforward to compute the variances of x and p after a time t. They are

$$(\Delta p)_t^2 = 2Dt + (\Delta p)_0^2, \tag{32}$$

$$(\Delta x)_t^2 = \frac{2}{3}\frac{Dt^3}{m^2} + (\Delta p)_0^2 \frac{t^2}{m^2} + \frac{2}{m}\sigma(x,p) + (\Delta x)_0^2, \tag{33}$$

where

$$\sigma(x,p) = \tfrac{1}{2}\langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle - \langle \hat{x}\rangle\langle \hat{p}\rangle \tag{34}$$

evaluated in the initial state.

III. EVOLUTION OF BIPARTITE STATES IN THE PRESENCE OF AN ENVIRONMENT

Consider now the case of a two-particle system in an initially entangled state. The two particles are not coupled to each other, but are each separately coupled to a thermal environment, as described in the preceding section.

For our two-particle system, the Wigner evolution equation for the Wigner function $W(p_1,x_1,p_2,x_2) = W(\mathbf{z}_1,\mathbf{z}_2)$ is

$$\frac{\partial W}{\partial t} = -\frac{p_1}{m}\frac{\partial W}{\partial x_1} - \frac{p_2}{m}\frac{\partial W}{\partial x_2} + D\frac{\partial^2 W}{\partial p_1^2} + D\frac{\partial^2 W}{\partial p_2^2}. \tag{35}$$

This equation may again be solved using propagators [with the unitary part removed, as in Eqs. (21) and (22)]. The two-particle Wigner function at time t is then given by

$$W_t(\mathbf{z}_1,\mathbf{z}_2) = \int d^2z_1' d^2z_2' g(\mathbf{z}_1 - \mathbf{z}_1';A) g(\mathbf{z}_2 - \mathbf{z}_2';A) W_0'(\mathbf{z}_1',\mathbf{z}_2'), \tag{36}$$

where the matrix A is given by Eq. (30). We will show that this evolves into the explicitly separable form (2) after sufficient time has elapsed. The key idea is to write the matrix A in the propagator as

$$A = A_{1/4} + B, \tag{37}$$

where $A_{1/4}$ is the matrix of variances for a minimum uncertainty Wigner function (whose explicit form will be given below). We use the convolution property (26) to write

$$g(\mathbf{z}_1 - \mathbf{z}_1'; A) = \int d^2\overline{z_1} g(\mathbf{z}_1 - \overline{\mathbf{z}_1}; B) g(\overline{\mathbf{z}_1} - \mathbf{z}_1'; A_{1/4}) \tag{38}$$

and similarly with $g(\mathbf{z}_2 - \mathbf{z}_2'; A)$. On the right-hand side, the second Gaussian is a Wigner function because $|A_{1/4}| = \hbar^2/4$. The first Gaussian with variance matrix B will also be a Wigner function if and only if

$$|B| = |A - A_{1/4}| \geq \frac{\hbar^2}{4}. \tag{39}$$

We may now write the two-particle Wigner function at time t as

$$W_t(\mathbf{z}_1, \mathbf{z}_2) = \int d^2\overline{z_1} d^2\overline{z_2} g(\mathbf{z}_1 - \overline{\mathbf{z}_1}; B) g(\mathbf{z}_2 - \overline{\mathbf{z}_2}; B) \widetilde{W}_0(\overline{\mathbf{z}_1}, \overline{\mathbf{z}_2}), \tag{40}$$

where

$$\widetilde{W}_0(\overline{\mathbf{z}_1}, \overline{\mathbf{z}_2}) = \int d^2 z_1' d^2 z_2' g(\overline{\mathbf{z}_1} - \mathbf{z}_1'; A_{1/4}) g(\overline{\mathbf{z}_2} - \mathbf{z}_2'; A_{1/4}) W_0'(\mathbf{z}_1', \mathbf{z}_2') \tag{41}$$

and is positive, as explained above. This is the desired result. The Wigner function at time t is of the separable form (2) as long as the matrix B satisfies Eq. (39). This is because the two Gaussians in Eq. (40) are Wigner functions if Eq. (39) holds, and the smeared Wigner function $\widetilde{W}_0$ is positive and corresponds to the term p_i in Eq. (2).

We may also compare with the closely related result of Diósi [10], who showed that a one-particle system achieves the entanglement-breaking form (3) after finite time, and hence cannot be entangled with anything else. Tracing over particle 2 in Eq. (40), we obtain

$$W_t(\mathbf{z}_1) = \int d^2 z_1 g(\mathbf{z}_1 - \overline{\mathbf{z}_1}; B) \widetilde{W}_0(\overline{\mathbf{z}_1}), \tag{42}$$

where $\widetilde{W}_0$ is the smeared Wigner function of the one-particle system. This is clearly the Wigner transform of an expression of the form (3) because first, the Gaussian $g(\mathbf{z}_1 - \overline{\mathbf{z}_1}; B)$ is a class of Wigner functions independent of the initial state so corresponds to R_k, and second, $\widetilde{W}_0$ is a Q function so is of the desired form $\mathrm{Tr}(F_k \rho)$ with F_k taken to be a coherent state projector. Hence we completely agree with Diósi's result. However, this result takes Diósi's result further in that it gives the explicitly separated form of the disentangled state.

Consider now the time scale on which the disentanglement condition (39) becomes satisfied, where the matrix A is given by Eq. (30) and $A_{1/4}$ is still to be chosen. Calculation of this time scale was reported (without explicit details) in Refs. [10,17], but we can extend this analysis in a small way. We also need to compare the calculation with a result of the following section, so we give some of the details.

Since the calculation involves a small numerical calculation, it is useful to define dimensionless variables $\overline{t}, \overline{p}, \overline{x}$, defined by

$$t = \left(\frac{\hbar m}{D}\right)^{1/2} \overline{t}, \tag{43}$$

$$p = (\hbar m D)^{1/4} \overline{p}, \tag{44}$$

$$x = \left(\frac{\hbar^3}{mD}\right)^{1/4} \overline{x}. \tag{45}$$

In these dimensionless variables, we now take the matrix $A_{1/4}$ to be

$$A = \begin{pmatrix} \sigma_0^2/\sqrt{2} & 1/2 \\ 1/2 & \sigma_0^{-2}/\sqrt{2} \end{pmatrix}. \tag{46}$$

References [10,17] made the particular choice $\sigma_0^2 = \sqrt{2}$ on the grounds that this gives a particularly robust evolution for certain sets of initial states [18]. For the moment, we leave it general. The condition (39) now becomes

$$\frac{1}{3}\overline{t}^3 - \frac{2}{3} s \overline{t}^2 + \overline{t} - \frac{1}{s} \geq 0, \tag{47}$$

where $s = \sigma_0^2/\sqrt{2}$. The time at which this condition becomes satisfied may be estimated by plotting the function in Maple for various values of s. For the choice $\sigma_0^2 = \sqrt{2}$ we have $s = 1$, and it is straightforward to then show that the the condition is satisfied for values of $\overline{t}$ greater than about 1.39. This means

$$t \geq 1.97 \left(\frac{\hbar m}{2D}\right)^{1/2}, \tag{48}$$

in agreement with Refs. [10,17]. (The factor of 2 is included because Refs. [10,17] used $D/2$ to denote what we denote by D.)

However, if we tune the value of σ_0 to make the time scale as short as possible, then numerical experiments show that the optimal value is about $s = 0.9$ which gives the slightly shorter time scale,

$$t \geq 1.95 \left(\frac{\hbar m}{2D}\right)^{1/2}. \tag{49}$$

This improvement is clearly insignificant, but it does, however, show that the Diósi-Kiefer choice is very nearly optimal.

IV. THE EPR STATE

Although the above results show disentanglement for general initial states, it is of interest to look in detail at a particular entangled state to see exactly how the entanglement goes away. We therefore consider the entangled state first introduced by Einstein, Podolsky, and Rosen [19]. Instead of the two canonical pairs p_1, x_1 and p_2, x_2 it is very useful to use instead the rotated coordinates,

$$X = x_1 - x_2, \quad K = \tfrac{1}{2}(p_1 - p_2), \tag{50}$$

$$Q = (x_1 + x_2), \quad P = \tfrac{1}{2}(p_1 + p_2). \tag{51}$$

This is a canonical transformation, with the new canonical pairs being K, X and P, Q. However, we also have the important relations,

$$[X,P] = 0, \quad [Q,K] = 0. \tag{52}$$

This simple observation is the basis of the EPR state, since it means we may choose a representation in which X and P are definite. In fact, EPR defined the state

$$\Psi(X,P) = \delta(X)\delta(P). \tag{53}$$

This state is not strictly normalizable. We therefore instead consider Gaussian states which may be made arbitrarily close to this state. In particular, we consider a Gaussian state, which, in the rotated coordinates has Wigner function

$$W(K,X,P,Q) = \exp\left(-\frac{K^2}{2\sigma_K^2} - \frac{X^2}{2\sigma_X^2} - \frac{P^2}{2\sigma_P^2} - \frac{Q^2}{2\sigma_Q^2}\right). \tag{54}$$

The widths σ_X and σ_P may be chosen to be arbitrarily small, but for this to be a Wigner function, the remaining widths must satisfy

$$\sigma_K^2\sigma_X^2 \geqslant \frac{\hbar^2}{4}, \quad \sigma_P^2\sigma_Q^2 \geqslant \frac{\hbar^2}{4}. \tag{55}$$

(Because the transformation from the original coordinates to rotated ones is a linear canonical transformation, it preserves Wigner function properties, so the conditions to be a Wigner function have the same form as in the original coordinates.)

As an aside, we remark that Bell has noted that Gaussian states have a positive Wigner function, and therefore have hidden variable interpretation, so cannot violate Bell's inequalities [20]. This perhaps suggests that they do not have any significant entanglement properties. This is, however, in terms of observables local in phase space quantities. It has been pointed out that there are other observables, in particular, the parity operator, in terms of which Gaussian states do violate Bell's inequalities [21]. Hence it is of interest to study entanglement of Gaussians.

In terms of the variables (50) and (51), the Peres-Horodecki [2–4] condition for disentanglement is the condition that $W(K,X,P,Q)$ must remain a Wigner function when P and K are interchanged, that is,

$$\widetilde{W}(K,X,P,Q) = \exp\left(-\frac{P^2}{2\sigma_K^2} - \frac{X^2}{2\sigma_X^2} - \frac{K^2}{2\sigma_P^2} - \frac{Q^2}{2\sigma_Q^2}\right) \tag{56}$$

is a Wigner function. The conditions for this to be a Wigner function are

$$\sigma_X^2\sigma_P^2 \geqslant \frac{\hbar^2}{4}, \quad \sigma_Q^2\sigma_K^2 \geqslant \frac{\hbar^2}{4}, \tag{57}$$

which is different to Eq. (55). In particular, the first of these relations will not be satisfied when σ_X and σ_P are chosen to be very small, which is the EPR case. This is generally the case for entangled states—they fail to remain Wigner functions under interchange of P and K because they are then too strongly peaked in phase space and violate the uncertainty principle.

To watch the destruction of entanglement, we will work with the disentanglement condition put forward by Duan *et al.* [5], rather than the Peres-Horodecki condition. They wrote down a class of necessary and sufficient conditions for a Gaussian state to be disentangled. In terms of dimensionless variables $\overline{K}, \overline{X}, \overline{P}, \overline{Q}$ [defined as in Eqs. (43)–(45)], one of those conditions is

$$(\Delta\overline{X})^2 + 4(\Delta\overline{P})^2 \geqslant 2. \tag{58}$$

This is clearly not satisfied for the EPR initial state (54), but it is of interest to see how it becomes satisfied under evolution in the presence of an environment. The key point here is that the condition (58) means that the state is reasonably spread out in phase space, and this phase space spreading is precisely what evolution in the presence of an environment produces, so we expect that the condition will become satisfied after a short period of time.

In the presence of an environment, the initial state will evolve according to Eq. (35). However, since the regularized EPR state (54) factors in terms of the rotated coordinates (50) and (51), it is more useful to use those, and in terms of them, the Wigner equation is

$$\frac{\partial W}{\partial t} = -\frac{2P}{m}\frac{\partial W}{\partial Q} - \frac{2K}{m}\frac{\partial W}{\partial X} + D\frac{\partial^2 W}{\partial P^2} + D\frac{\partial^2 W}{\partial K^2}. \tag{59}$$

Interestingly, the dynamics also factors in the rotated variables and is essentially the same as the single-particle dynamics. Using Eqs. (32) and (33), it is then easily seen that

$$(\Delta P)_t^2 = 2Dt + \sigma_P^2, \tag{60}$$

$$(\Delta X)_t^2 = \frac{8}{3}\frac{Dt^3}{m^2} + \frac{4}{m^2}\sigma_K^2 t^2 + \sigma_X^2. \tag{61}$$

Inserting these expressions in the condition (58), we find that, in dimensionless variables, it is satisfied as long as

$$\frac{8}{3}\overline{t}^3 + 4c\overline{t}^2 + 8\overline{t} + \frac{1}{4c} \geqslant 2. \tag{62}$$

Here, c is the dimensionless form of σ_K^2 and we have used the uncertainty principle to eliminate σ_X^2. The interesting case is that in which σ_X^2 is very small, but we cannot set it to zero because then σ_K^2 would be infinite. We have also set $\sigma_P^2 = 0$, which represents the most extreme case. (We can do this because σ_Q^2, its conjugate width, does not appear.)

The polynomial in Eq. (62) may be plotted using Maple for various values of c. We have found from these plots that for all values of c the condition becomes satisfied for values of $\bar{t}$ greater than about 0.19, that is, for

$$t \geqslant 0.27\left(\frac{\hbar m}{2D}\right)^{1/2}. \tag{63}$$

This is the time after which the EPR state becomes disentangled, according the condition (58). This time is actually a lot shorter than the time scale (49), hence is safely consistent with the general result. The time scale (49) is the time for *any* initial state to become disentangled whereas the time scale (63) is only for the EPR state. This actually indicates that the EPR state is perhaps not a very entangled state, at least in terms of phase space variables.

V. DISCUSSION

We have shown, in a variety of different ways, how open system dynamics destroys entanglement in a system consisting of two entangled particles. Entanglement is destroyed by the same mechanism that destroys interference. In particular, we have shown that, under a simple open system dynamics, any initial two-particle state achieves the explicitly disentangled form Eq. (2) after a short, finite time. We illustrated this general result with the particular case of the EPR state.

The dynamics employed here are those of the free particle coupled to a bath of oscillators, in the limit of negligible dissipation. It is of interest to extend the analysis to include dissipation, a potential, and also to include interactions between the particles which will then compete with the disentangling effects of the environment. This is considered in another paper [16].

As previously noted [10,11], it is striking that complete disentanglement is achieved in finite time, whereas the interference terms in the density matrix tend to take infinite time to completely go away. That is, one is inclined to say that complete decoherence takes an infinite time. Although note that the word "decoherence" is used in a variety of different ways.

This perhaps suggests that the exact vanishing of the off-diagonal terms in the density matrix is perhaps too strong a condition for determining when a quantum system is essentially classical. It is, for example, often suggested that positivity of the Wigner function is a useful condition characterizing quasiclassicality, because this then means that the one-particle system is a hidden variables theory. Significantly, the Wigner function typically becomes positive after a finite time in open system dynamics. Hence, a reasonable meaning to attach to the word decoherence is that it is the situation under which prediction for all variables may be described by a hidden variables theory. These ideas will be explored in future publications. See Ref. [22] for related discussions.

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