PHYSICAL REVIEW A **69**, 050303(R) (2004)

Entanglement-assisted alignment of reference frames using a dense covariant coding

E. Bagan, M. Baig, and R. Muñoz-Tapia
Grup de Física Teòrica & IFAE, Facultat de Ciències, Edifici Cn, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain

(Received 18 December 2003; published 14 May 2004)

We present a procedure inspired by dense coding, which enables a highly efficient transmission of information of a continuous nature. The procedure requires the sender and the recipient to share a maximally entangled state. We deal with the concrete problem of aligning reference frames or trihedra by means of a quantum system. We find the optimal covariant measurement and compute the corresponding average error, which has a remarkably simple close form. The connection of this procedure with that of estimating unitary transformations on qubits is briefly discussed.

DOI: 10.1103/PhysRevA.69.050303 PACS number(s): 03.67.Hk, 03.65.Fd, 03.65.Ta

Entanglement has long been recognized as a powerful resource in quantum communication. Teleportation [1] and dense coding [2], for instance, would not be possible without entanglement. Even when entanglement is not strictly necessary, one frequently runs across situations for which the use of entangled states, instead of plain product states, provides a significant improvement. Examples of this can be easily found in the literature. This Rapid Communication provides yet another interesting instance, which one could refer to as dense covariant coding.

Two interesting problems in quantum communication in which entanglement plays a fundamental role are those of sending the information that specifies (i) a direction in space, i.e., a unit vector $\vec{n}_1$, or (ii) three orthogonal ones (a trihedron) $\mathsf{n}=\{\vec{n}_1,\vec{n}_2,\vec{n}_3\}$. Whereas (i) has been extensively discussed in the literature [3–5], only recently significant attention [6–8] has been paid to (ii). It has been shown that quantum states can indeed be used to establish a common reference frame between two parties (Alice and Bob). Thus, for instance, atoms or a number of spins (throughout this Rapid Communication we use the word spin as synonym of spin-1/2 particle) can encode the relative orientation of two trihedra. The fidelity (or alternatively, the mean-square error per axis) of the optimal covariant communication protocol [where covariance refers to the set of signal states being the orbit of a group; SU(2) for the problem at hand] is now known for both finite and asymptotically large number N of copies of the messenger state.

In this Rapid Communication we show that the intensive use of entanglement yields a remarkable improvement over the approaches for aligning spatial frames discussed above. More specifically, suppose Alice and Bob share a maximally entangled state. Then, we will show that using a covariant protocol it is possible to establish a common reference frame with a mean-square error per axis given by $[1-\cos 2\pi/(N+3)]/3$, which behaves as $2\pi^2/(3N^2)$. This protocol bears a great similarity to dense coding as far as it uses entanglement in the same manner and provides a remarkable improvement in the transmission of information [9]. Dense coding has mainly been discussed for discrete signals. However, the information we are attempting to transmit has an intrinsically continuous nature: it refers to the relative orientation of Alice and Bob and, in some situations [3–7,10–12], such information cannot be codified by a series of bits. Indeed, a digital representation of an orientation has no meaning unless it is referred to a common reference frame. No such frame will be assumed to be known to both Alice and Bob unless otherwise stated, though we will use Bob's to simplify the mathematics. Hence, the messenger will have to be a quantum system with intrinsic orientation. More specifically, in this Rapid Communication we will consider a system of spins. (See Ref. [13] for another protocol of sending information without a shared reference frame.) The subject of this Rapid Communication is also related to the important issue of estimating a unitary operation on qubits [14]. We will come back to this point in the conclusions.

Suppose both Alice and Bob have a system of N spins; let us call $\mathcal{H}_A$ and $\mathcal{H}_B$ their respective Hilbert spaces (throughout this Rapid Communication subscripts A and B will always refer to Alice and Bob). Before they start their intergalactic journeys, they prepare a state of the form

$$|\Phi\rangle = \sum_j a_j|\Phi^j\rangle = \sum_j \frac{a_j}{\sqrt{d_j}}\sum_{m=-j}^{j}|jm\rangle_A|jm\rangle_B, \tag{1}$$

where j runs from zero to $N/2$ for N even (from $1/2$ to $N/2$ for N odd), $d_j=2j+1$ is the dimension of the representation $\mathbf{j}$ of SU(2), and $\Sigma_j\, a_j^2=1$. Also before departure, they lock the orientation of their systems of N spins to that of their respective spacecrafts. When they are far apart, they need to get aligned. Unfortunately, their classical computers crash and they cannot retrieve the information about the change of their relative orientation. At this point in time, the state of Alice's and Bob's spins is still given by Eq. (1) but $|jm\rangle_A$ and $|jm\rangle_B$ are now referred to Alice's and Bob's reference frames, respectively (in this presentation the words spacecraft and reference frame are synonyms). Relative to Bob's reference frame this state can be written as

$$|\Phi(g)\rangle \equiv U_A(g)\otimes \mathbb{I}_B|\Phi\rangle, \tag{2}$$

where $U_A(g)$ belongs to the direct sum of irreducible representations of SU(2) and g stands for the three Euler angles of the spacial rotation that takes Bob's reference frame into Alice's. With no other resource available, Alice sends her N spins to Bob, with the hope that he will retrieve from them the information they need. To do so, he is allowed to perform generalized collective measurements on both Alice's and his own spins, namely, on the state (2). Note that $|\Phi\rangle$ and $|\Phi(g)\rangle$ are maximally entangled in each $\mathbf{j}$. Note also that in Eq. (1) all of these representations appear only once, despite the fact that in the Clebsch-Gordan decomposition of $(\mathbf{1}/\mathbf{2})^{\otimes N}$ they may show up several times. We will show that $|\Phi\rangle$ is optimal for the problem at hand, provided a suitable choice of $a_j > 0$ is made [see Eq. (19) below].

The quality of the communication strategy can be quantified by the averaged Holevo's error [15]

$$\langle h\rangle = \sum_r \int dg\, h(g,g_r)p(r|g), \tag{3}$$

where $h(g,g') = \sum_{a=1}^{3} |\vec{n}_a(g) - \vec{n}_a(g')|^2$; $\mathsf{n}(g) = \{\vec{n}_1(g), \vec{n}_2(g), \vec{n}_3(g)\}$ defines the frame Alice is transmitting to Bob; $\mathsf{n}(g_r) = \{\vec{n}_1(g_r), \vec{n}_2(g_r), \vec{n}_3(g_r)\}$ defines the frame Bob guesses from the outcome r of his measurement; and dg is the invariant Haar measure of SU(2). Each one of these trihedra is labeled with the parameters g of the rotation which bring $\mathsf{n}_0 = \{\vec{x}, \vec{y}, \vec{z}\}$ into the desired orientation. $p(r|g)$ is the conditional probability of Bob obtaining the outcome r if Alice sends $\mathsf{n}(g)$. Note that $h(g,g')$ is related to the character χ_1 of the representation $\mathbf{1}$ of SU(2) through $h(g,g') = 6 - 2\chi_1(gg'^{-1})$. Hence, we just need to compute $\langle\chi_1\rangle$. From this, the square error per axis, to which we referred above, is $(3 - \langle\chi_1\rangle)/6$. Quantum mechanics tells us that the conditional probability is $p(r|g) = \langle\Phi(g)|O_r|\Phi(g)\rangle$, where $\{O_r\}$ is a complete set of positive operators such that $\sum_r O_r = \mathbb{I}$, namely, the elements of a positive operator valued measurement (POVM) in the whole subspace of $\mathcal{H}_A \otimes \mathcal{H}_B$ where the signal states belong. Recalling the invariance of the Haar measure, $dg = d(gg')$, we can write

$$\langle\chi_1\rangle = \sum_r \int dg\, \chi_1(g)|\langle\Phi(g)|\Psi_r\rangle|^2, \tag{4}$$

where

$$|\Psi_r\rangle\langle\Psi_r| \equiv U_A^\dagger(g_r)\otimes \mathbb{I}_B O_r U_A(g_r)\otimes \mathbb{I}_B. \tag{5}$$

This definition implicitly assumes that optimal POVM's can always be chosen to have rank 1 elements [16]. We claim that (a) the states of the form (1) are optimal if the positive coefficients a_j are properly chosen and (b) and for the optimal POVM one has

$$|\Psi_r\rangle = \sum_j |\Psi_r^j\rangle = \sum_j c_{jr}|\Phi^j\rangle; \quad \sum_r c_{jr}^2 = d_j^2. \tag{6}$$

To prove claim (a) we borrow from Ref. [14] some results concerning the estimation of an SU(2) transformation, in particular, that the optimal state can be chosen to be $|\Phi\rangle = \sum_j a_j|\Omega^j\rangle$, with

$$|\Omega^j\rangle = \frac{1}{\sqrt{d_j n_j}}\sum_{m=-j}^{j}\sum_{\alpha=1}^{n_j} |jm;\alpha\rangle_A|jm;\alpha\rangle_B \tag{7}$$

instead of Eq. (1). Here α labels the different n_j occurrences of $\mathbf{j}$ in the Clebsch-Gordan decomposition of $(\mathbf{1}/\mathbf{2})^{\otimes N}$. We next show that, as far as the evaluation of the maximal $\langle\chi_1\rangle$ (minimal error) is concerned, we need to consider each $\mathbf{j}$ only once. Let us define $v_\alpha^{(s)}, s=1,\ldots,n_j-1$, as the set of $n_j(n_j-1)$ complex numbers (which we may regard as the components of n_j-1 orthogonal unit vectors) such that $\sum_\alpha v_\alpha^{(r)*}v_\alpha^{(s)} = \delta_{rs}$ and $\sum_\alpha v_\alpha^{(s)} = 0$ [i.e., orthogonal to the n_j-dimensional vector $(1,1,\ldots,1)$]. We note that the states $|\Omega_{s,m}\rangle = \sum_\alpha v_\alpha^{(s)}|jm;\alpha\rangle|jm;\alpha\rangle$ satisfy $\langle\Omega^j|U_A(g)\otimes\mathbb{I}_B|\Omega_{s,m}\rangle = \sum_\alpha v_\alpha^{(s)}\mathfrak{D}_{mm}^{(j)}(g)/\sqrt{d_j n_j} = 0$ for all g, s, and m, where we have used that $\mathfrak{D}_{mm'}^{(j)}(g) = \langle jm;\alpha|U(g)|jm';\alpha\rangle$. Hence, $|\Omega^j\rangle$ effectively lives in only one of the irreducible representations $\mathbf{j}$ and it can be chosen as in Eq. (1) without any loss of generality.

To prove claim (b) we rewrite Eq. (4) as

$$\langle\chi_1\rangle = \sum_r\sum_{jl} a_j a_l \int dg\, \chi_1(g)\langle\Phi^j\tilde{\Phi}^l|\mathcal{U}(g)|\Psi_r^j\tilde{\Psi}_r^l\rangle, \tag{8}$$

where $|\Phi^j\tilde{\Phi}^l\rangle = |\Phi^j\rangle\otimes|\tilde{\Phi}^l\rangle$, the state $|\tilde{\Phi}^l\rangle$ is obtained by applying to $|\Phi^l\rangle$ time reversal only in $\mathcal{H}_A$ (analogous definitions hold for $|\Psi_r^j\tilde{\Psi}_r^l\rangle$), and $\mathcal{U} = U_A\otimes U_A\otimes\mathbb{I}_B\otimes\mathbb{I}_B$. By Schur's lemma, Eq. (8) is

$$\langle\chi_1\rangle = \frac{1}{3}\sum_r\sum_{jl} a_j a_l\, \mathrm{tr}_{\mathbf{1}}(\rho_r^j\otimes\tilde{\rho}_r^l), \tag{9}$$

where we have defined $\rho_r^j = \mathrm{tr}_B(|\Psi_r^j\rangle\langle\Phi^j|)$, $\tilde{\rho}_r^l = \mathrm{tr}_B(|\tilde{\Psi}_r^l\rangle\langle\tilde{\Phi}^l|)$, and $\mathrm{tr}_B(\mathrm{tr}_{\mathbf{1}})$ stands for the partial trace over $\mathcal{H}_B$ (over the representation $\mathbf{1}$ invariant subspace, i.e., $\mathrm{tr}_{\mathbf{1}}O = \sum_{m=-1}^{1}\langle 1m|O|1m\rangle$). Using the Schwarz inequality we obtain the bound

$$\mathrm{tr}_{\mathbf{1}}(\rho_r^j\otimes\tilde{\rho}_r^l) \leqslant \sqrt{\mathrm{tr}_{\mathbf{1}}(\rho_r^j\rho_r^{j\dagger}\otimes\mathbb{I}_l)\mathrm{tr}_{\mathbf{1}}(\mathbb{I}_j\otimes\tilde{\rho}_r^l\tilde{\rho}_r^{l\dagger})}, \tag{10}$$

where $\mathbb{I}_j$ $(\mathbb{I}_l)$ is the identity restricted to the representation $\mathbf{j}$ $(\mathbf{l})$ subspace. The equality holds if $|\Psi_r^j\rangle = c_{jr}|\Phi^j\rangle$ since this choice implies $\rho_r^j = c_{jr}\mathrm{tr}_B(|\Phi^j\rangle\langle\Phi^j|) = c_{jr}\mathbb{I}_j/d_j$. To obtain $\sum_r c_{jr}^2 = d_j^2$ one just has to trace Eq. (5) on each irreducible representation subspace.

With this information we can go back to Eq. (4) and cast it as

$$\langle\chi_1\rangle \leqslant \sum_r\int dg\, \chi_1(g)\left|\sum_j \frac{a_j c_{jr}}{d_j}\chi_j(g)\right|^2, \tag{11}$$

where we have used that $\langle\Phi^j|U_A^\dagger\otimes\mathbb{I}_B|\Phi^j\rangle = \chi_j(g)/d_j$. To get rid of the coefficients c_{jr}, note that

$$\sum_r c_{jr}c_{lr} \leqslant \sqrt{\sum_r c_{jr}^2}\sqrt{\sum_r c_{lr}^2} = d_j d_l. \tag{12}$$

The equality holds iff

$$c_{jr} = d_j\sqrt{c_r}, \tag{13}$$

where $\Sigma_r c_r = 1$. Hence

$$\langle \chi_1 \rangle \leqslant \int dg\ \chi_1(g) \left| \sum_j a_j \chi_j(g) \right|^2. \tag{14}$$

The group integral can be easily performed by recalling the Clebsch-Gordan series $\chi_j(g)\chi_l(g) = \Sigma_{k=|j-l|}^{j+l} \chi_k(g)$ and the orthogonality of the characters [17], namely, $\int dg \chi_j(g)\chi_l(g) = \delta_{jl}$. The result can be conveniently written as

$$\langle \chi_1 \rangle \leqslant 1 + \mathsf{a}^t \mathsf{M} \mathsf{a}. \tag{15}$$

Here $\mathsf{a}^t = (a_{N/2}, a_{N/2-1}, a_{N/2-2}, \dots)$ is the transpose of a and M is the tridiagonal matrix

$$\mathsf{M} = \begin{pmatrix} 0 & 1 & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & 0 & 1 & \\ & & 1 & 0 & 1 \\ 0 & & & 1 & \zeta \end{pmatrix}, \tag{16}$$

where $\zeta = -1(\zeta = 0)$ for N even (odd). One could also obtain Eq. (15) directly from Eq. (9) by simply noticing that $\mathrm{tr}_{\mathbf{1}}(\mathbb{I}_j \otimes \mathbb{I}_l) = 3$ if $j + l \geqslant 1 \geqslant |j - l|$ and it vanishes otherwise. The maximal value of the quadratic form in Eq. (15) is given by the largest eigenvalue of M. Its characteristic polynomial is $P_n(\lambda) = \det(\mathsf{M} + 2\lambda\mathbb{I})$, where n is the dimension of M, namely, $n = N/2 + 1 (n = N/2 + 1/2)$ for N even (odd). Note that we have defined the eigenvalues of M as $-2\lambda_s$, where λ_s, $s = 1, 2 \dots, n$, are the zeroes of $P_n(\lambda)$. The characteristic polynomials obey the simple recurrence relation

$$P_n(\lambda) = 2\lambda P_{n-1}(\lambda) - P_{n-2}(\lambda), \tag{17}$$

which is that of the Tchebychev polynomials [18], and the initial conditions are $P_0(\lambda) = 1$ and $P_1(\lambda) = 2\lambda + \zeta$. Hence, the solution is $P_n(\lambda) = U_n(\lambda) + \zeta U_{n-1}(\lambda)$, where $U_n(\cos\theta) = \sin[(n+1)\theta]/\sin\theta$ are the Tchebychev polynomials of the second kind. It is now straightforward to compute the largest eigenvalue of M. It can be written as $2\cos[2\pi/(N+3)]$ and, hence,

$$\langle \chi_1 \rangle_{\max} = 1 + 2\cos\frac{2\pi}{N+3}. \tag{18}$$

One can also verify that the corresponding eigenvector is

$$a_j = \frac{2}{\sqrt{N+3}} \sin\frac{(2j+1)\pi}{N+3}. \tag{19}$$

Equation (18) gives an upper bound of the actual $\langle \chi_1 \rangle_{\max}$. We need to show that this bound is indeed saturated by a covariant measurement. To do this, we just trace the conditions under which all the (Schwarz) inequalities used in the proof are saturated. Substituting in Eq. (5) the relation $|\Psi_r\rangle = \sqrt{c_r}\, d_j |\Phi^j\rangle$, which follows from Eqs. (6) and (13), we get

$$O_r = c_r \quad U_A(g_r) \otimes \mathbb{I}_B\, |\Psi\rangle\langle\Psi| U_A^\dagger(g_r) \otimes \mathbb{I}_B, \tag{20}$$

where $|\Psi\rangle = \Sigma_{j,m} \sqrt{d_j} |j, m\rangle_A |j, m\rangle_B$. But for a rescaling factor c_r, we see that the positive operators O_r are all obtained by rotating a fix reference state $|\Psi\rangle$. This exhibits the covariance of the scheme. An immediate choice that saturates the bound (18) is provided by the continuous POVM,

$$O(g) = U_A(g) \otimes \mathbb{I}_B\, |\Psi\rangle\langle\Psi| U_A^\dagger(g) \otimes \mathbb{I}_B. \tag{21}$$

Using Schur's lemma, we get $\int dg\, O(g) = \Sigma_j \mathbb{I}_A^j \mathbb{I}_B^j$, where $\mathbb{I}_A^j$ ($\mathbb{I}_B^j$) is the identity in Alice's (Bob's) representation $\mathbf{j}$ subspace. This is the identity in the Hilbert subspace to which all signal states $|\Phi(g)\rangle$ belong. Hence, the infinite set $\{O(g)\}$ is a POVM for these signal states.

A continuous POVM, such as Eq. (21), with infinitely many outcomes is not physically realizable. Hence, it is important to show that optimal POVMs with a finite number of outcomes do exist. The most straightforward way of obtaining a finite (though not necessarily minimal) POVM is by finding a finite set $\{g_r\}, r = 1, \cdots, n(J)$, of elements of SU(2) and positive weights $\{c_r'\}$ such that the orthogonality relation

$$\sum_{r=1}^{n(J)} c_r'\, \mathfrak{D}_{mm'}^{(j)}(g_r) \mathfrak{D}_{nn'}^{(l)*}(g_r) = C_J \frac{\delta^{jl} \delta_{mn} \delta_{m'n'}}{2j+1} \tag{22}$$

holds for all $j, l \leqslant J = N/2 + 1$, where $C_J = \Sigma_{r=1}^{n(J)} c_r'$. This discrete version of the standard orthogonality relations of SU(2) is only valid up to a maximal value J. The larger the J is, the larger the $n(J)$ that must be chosen. There are many solutions to these equations and we refer the reader to Ref. [7] for details. Once $\{g_r\}$ and $\{c_r'\}$ have been computed, we simply define $c_r = c_r'/C_J$ and obtain the desired finite POVM elements by substituting these values in Eq. (20). Equation (22) ensures that Schur's lemma will work for the finite set $\{g_r, c_r\}$, thus obtaining $\Sigma_r O_r = \Sigma_j \mathbb{I}_A^j \mathbb{I}_B^j$, as it should be.

Let us conclude by summarizing and commenting our results. We present a covariant (and, hence, very natural) scheme for transmitting continuous information efficiently through a quantum channel. It requires Alice and Bob to share an entangled state of the form (1). This state can be prepared with, e.g., a number of spins or two hydrogen atoms. We determine the coefficients —given in Eq. (19)— which enable Alice to communicate with the smallest error. The procedure is as simple as Alice locking her part of the system to her frame and sending it to Bob who performs a generalized covariant measurement on the whole Hilbert space. The error, defined in Eq. (3), is given by

$$\langle h \rangle_{\min} = 4\left(1 - \cos\frac{2\pi}{N+3}\right), \tag{23}$$

which follows from the relation $\langle h \rangle = 6 - 2\langle \chi_1 \rangle$. The corresponding asymptotic behavior is $\langle h \rangle_{\min} = 8\pi^2/N^2$. This is a striking improvement over any other previously known scheme. We also prove that the optimal measurements are covariant POVMs, which one can choose to be either continuous, Eq. (21), or to have a finite number of outcomes.

Our work bears a strong connection with Ref. [14], where the estimation of a unitary transformation on qubits is studied. This problem and that of aligning reference frames are formally the same. To be more concrete, let us assume Alice is given a black box that performs an unknown unitary operation on qubits (they do not need to be spins in this case) and she is asked to identify it. If she is allowed to apply the unknown operation N times, the best she can do is the following [14]: (a) prepare the $2N$-qubit state (1), (b) apply $u(g) \in \mathbf{1/2}$ over N qubits, which results in the state (2), and (c) perform the POVM whose elements are given in Eq. (21). Note that now all the states are referred to a unique reference frame, that of Alice (Bob does not play any role in this case). We must stress that this task cannot be performed unless both $|\Phi\rangle$ and the POVM elements can be referred to the same reference frame, which requires that the person who performs the measurement, if not Alice herself, must share a reference frame with her.

Another (minor) difference with respect to the alignment of frames concerns the figure of merit used in Ref. [14], which is the fidelity $F=|\mathrm{tr}[u(g)u^{\dagger}(g_r)]|^2/4=\chi^2_{1/2}(gg_r^{-1})/4$. Our results can be straightforwardly applied in this context because of the simple relation $\chi^2_{1/2}(g)=1+\chi_1(g)$. Hence, for instance, Eq. (18) implies that the optimal mean fidelity is

$$\overline{F} = \langle F\rangle = \frac{1}{2}\left(1+\cos\frac{2\pi}{N+3}\right), \tag{24}$$

whereas for large N one has $\overline{F}=1-\pi^2/N^2+\cdots$. This extends the results of Ref. [14] to arbitrary N.

Finally, we would like to point out that our approach resembles the so-called continuous dense coding introduced in Ref. [19], where the communication of a single phase—$U(1)$ group—was discussed. They found that dense coding can improve the channel capacity, but not always. This is an indication that the absolute optimal scheme for a phase [10] does not require bipartite entanglement, contrasting with our approach for SU(2), which always improves the efficiency of the communication.

We are grateful to A. Acín and E. Jané for helpful conversations. We acknowledge financial support from Spanish Ministry of Science and Technology under Project No. BFM2002-02588, CIRIT Project No. SGR-00185, and QUPRODIS working group EEC Contract No. IST-2001-38877.

[1] C. H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
[2] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[3] S. Massar and S. Popescu, Phys. Rev. Lett. **74**, 1259 (1995); R. Derka, V. Buzek, and A. K. Ekert, Phys. Rev. Lett. **80**, 1571 (1998); J. I. Latorre, P. Pascual, and R. Tarrach, *ibid.* **81**, 1351 (1998); N. Gisin and S. Popescu, *ibid.* **83**, 432 (1999); S. Massar, Phys. Rev. A **62**, 040101 (2000); A. Peres and P. F. Scudo, Phys. Rev. Lett. **86**, 4160 (2001); E. Bagan, M. Baig, and R. Munoz-Tapia, Phys. Rev. A **64**, 022305 (2001).
[4] E. Bagan *et al.*, Phys. Rev. A **85**, 5230 (2000); Phys. Rev. Lett. **63**, 052309 (2001).
[5] D. G. Fischer, S. H. Kienle, and M. Freyberger, Phys. Rev. A **61**, 032306 (2000); R. D. Gill and S. Massar, *ibid.* **61**, 042312 (2000); Th. Hannemann *et al.*, *ibid.* **65**, 050303 (2002); E. Bagan, M. Baig, and R. Munoz-Tapia, Phys. Rev. Lett. **89**, 277904 (2002).
[6] A. Peres and P. F. Scudo, Phys. Rev. Lett. **87**, 167901 (2001); N. H. Lindner, A. Peres, and D. R. Terno, Phys. Rev. A **68**, 042308 (2003).
[7] E. Bagan, M. Baig, and R. Munoz-Tapia, Phys. Rev. Lett. **87**, 257903 (2001).
[8] A. Peres and P. F. Scudo, J. Mod. Opt. **49**, 1235 (2002).
[9] G. M. D'Ariano, P. LoPresti, and M. G.A. Paris, Phys. Rev. Lett. **87**, 270404 (2001).
[10] H. M. Wiseman and R. B. Killip, Phys. Rev. A **56**, 944 (1997); **57**, 2169 (1998); D. W. Berry, H. M. Wiseman and J. K. Breslin, *ibid.* **63**, 053804 (2001); D. W. Berry and H. M. Wiseman, *ibid.* **65**, 043803 (2002).
[11] H. M. Wiseman, Phys. Rev. Lett. **75**, 4587 (1995); W. P. Smith *et al.*, *ibid.* **89**, 133601 (2002); M. A. Armen *et al.*, *ibid.* **89**, 133602 (2002).
[12] D. Bruss and C. Macchiavello, Phys. Lett. A **253**, 249 (1999).
[13] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Phys. Rev. Lett. **91**, 027901 (2003).
[14] A. Acin, E. Jane, and G. Vidal, Phys. Rev. A **64**, 050302 (2001); E. Jane, Ph.D. thesis, University of Barcelona.
[15] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[16] E. B. Davies, IEEE Trans. Inf. Theory **24**, 596 (1978).
[17] J. F. Cornwell, *Group Theory in Physics* (Academic Press, London, 1984).
[18] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).
[19] S. L. Braunstein and H. J. Kimble, Phys. Rev. A **61**, 042302 (2000).