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Geometric phase gate without dynamical phases

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A general scheme for an adiabatic geometric phase gate is proposed which is maximally robust against parameter fluctuations. While in systems with SU(2) symmetry geometric phases are accompanied by dynamical phases and are thus not robust, we show that in the more general case of an $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ symmetry it is possible to obtain a nonvanishing geometric phase without dynamical contributions. The scheme is illustrated for a phase gate using two systems with dipole-dipole interactions in external laser fields which form an effective four-level system.

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A major challenge for the practical implementation of fault-tolerant scalable quantum computing [1] is the requirement of single-bit and quantum gate operations approaching a fidelity of unity up to one part in 10^4 [2]. As any unitary operation on physical qubits involves the interaction with external systems, either a very precise control or a clever design is needed to make such operations insensitive to small variations of external parameters. Since a universal set of qubit operations includes arbitrary single-bit rotations, the qubit system must depend sensitively on at least one continuous external parameter, e.g., some laser phase. It is therefore clear that not all elements of quantum computation can be made robust with respect to external parameters. However apart from single-bit phase rotations, this should be possible. This has led to several proposals for adiabatic quantum computation [3] using geometric or Berry phases [4]. The choice of *adiabatic* evolutions removes the sensitivity to many external parameters at the expense of slower operations. Employing furthermore *geometric phases* which depend only on the geometry of the path followed in parameter space makes the operations tolerant with respect to parameter fluctuations. The problem of geometric quantum computation is that nonvanishing geometric phases are in general associated with nonvanishing dynamical phases. The latter need to be compensated in some way, which destroys the potential robustness. For example, in the proposal of Jones *et al.* for geometric quantum computation in nuclear magnetic resonance, dynamical phases are compensated by spin-echo techniques [5]. In the ion-trap system of Duan *et al.* [6] dynamical phase shifts arise due to unavoidable ac-Stark shifts which have the same strength as the couplings used for the implementation of the phase gate. Also in the recent proposal of Garcia-Ripoll and Cirac [7] to perform single- and two-bit operations with unknown interaction parameters, dynamical phases arise in the individual steps of the operation and the interaction parameters must be controlled over the whole cycle in order for these phase to add up to zero.

We here show that it is possible to obtain a nonzero geometric phase under conditions of exactly vanishing dynamical contributions at all times when going from an SU(2) system to a case with $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ symmetry. The simplest nontrivial representation of the $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ symmetry is a four-level scheme with a tripod coherent coupling [6,8,9]. An appropriate adiabatic rotation in parameter space creates phase shifts of π or $\pi/2$. This approach is then applied to a two-particle system with dipole-dipole interaction to construct a phase gate.

When a system undergoes a *cyclic evolution* the wave function of the system remembers its motion in the form of a phase factor. As first noted by Berry [4], one can distinguish two contributions to the phase acquired: a dynamical part and a geometrical part. Berry showed that when the Hamiltonian of the system depends on a set of parameters which evolve *adiabatically* along a closed curve in the parameter space, then the state vector that corresponds to a simple nondegenerate eigenvalue develops a phase which depends only on this curve. Unlike its dynamical counterpart, the geometrical or Berry phase does not depend on the duration of the interaction. The notion of a Berry phase was generalized to the case of degenerate levels by Wilczek and Zee [10] which involve non-Abelian operations and to general cyclic evolutions by Aharonov and Anandan [11].

Due to their intrinsically robust nature, geometric phases are attractive for quantum computation. Their application however face a problem. As has been demonstrated by Robbins and Berry [12] for a spin-J particle in a slowly and cyclically changing magnetic field $\mathbf{B}$,

$$H = \hbar\boldsymbol{\Omega}\cdot\hat{\mathbf{J}}, \quad \boldsymbol{\Omega} = \mu_B \mathbf{B}/\hbar, \tag{1}$$

the angular momentum state $|m\rangle$ (in the direction of $\mathbf{B}$) attains not only a geometric phase but also a dynamical one. To see this, let us assume that $\boldsymbol{\Omega}$ makes a cyclic evolution in parameter space as shown in Fig. 1.

Starting with a magnetic-field orientation in the z direction, $\boldsymbol{\Omega}$ is successively rotated around the y, z, and x axes by $\pm\pi/2$. The total solid angle in parameter space is then $\pi/2$. In principle, any other angle can be obtained, but this particular path can be implemented in a very robust way. The initial state vector is assumed to be an eigenstate of Eq. (1) with eigenvalue $E_m = m\hbar\Omega$, i.e., $|\psi_0\rangle = |J, m_z = m\rangle$. An adiabatic cyclic evolution then leads to a state

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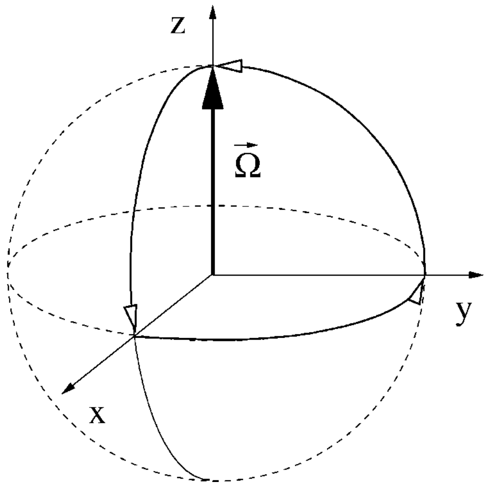

FIG. 1. Cyclic evolution of the magnetic field $\mathbf{\Omega}\sim\mathbf{B}$ from z to x to y and back to the z direction to generate geometric phase of $\pi/2$.

$$|\psi\rangle = e^{-i(\gamma_g+\gamma_d)}|\psi_0\rangle, \tag{2}$$

where γ_d is the dynamical phase

$$\gamma_d = \int dt \frac{E_m(t)}{\hbar}. \tag{3}$$

The geometrical phase can be obtained from a parallel transport of the adiabatic eigenstate following the rotation of the magnetic field

$$U_g|\psi_0\rangle = e^{-i(\pi/2)\hat{J}_x} e^{-i(\pi/2)\hat{J}_z} e^{-i(\pi/2)\hat{J}_y}|\psi_0\rangle = e^{-i(\pi/2)\hat{J}_z}|\psi_0\rangle, \tag{4}$$

i.e.,

$$\gamma_g = \frac{\pi}{2}m. \tag{5}$$

One recognizes that an elimination of the dynamical phase implies in general a vanishing geometric phase, since a zero eigenstate of H at $t=0$ is also an eigenstate of U_g with eigenvalue 1.

In order to have a nonvanishing geometric phase and at the same time a vanishing dynamical one, the state $|\psi_0\rangle$ must be an eigenstate of the initial Hamiltonian with eigenvalue 0 but should not be an eigenstate of U_g with eigenvalue 1. This can be achieved if we consider a Hamiltonian with SU(2) $\otimes$SU(2) symmetry rather than the simple SU(2) symmetry of Eq. (1):

$$H = \hbar\mathbf{\Omega}\cdot[\hat{\mathbf{J}}^{(1)} - \hat{\mathbf{J}}^{(2)}]. \tag{6}$$

Here the angular moments fulfill the commutation relations

$$[\hat{J}_i^{(n)}, \hat{J}_j^{(n)}] = i\varepsilon_{ijk}\hat{J}_k^{(n)}, \quad [\hat{J}_i^{(1)}, \hat{J}_j^{(2)}] = 0, \tag{7}$$

which is equivalent to the Lorentz group. Starting again with $\mathbf{\Omega}(0)=(0,0,\Omega)$ and in an eigenstate of Eq. (6), we find that the condition for a vanishing dynamical phase reads

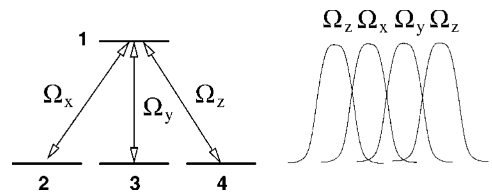

FIG. 2. Left: Tripod coupling scheme representing Hamiltonian (6) with SU(2)$\otimes$SU(2) dynamical symmetry. Right: Pulse sequence to generate geometric phase without dynamical phase.

$$H|\psi_0\rangle = \hbar\Omega[\hat{J}_z^{(1)} - \hat{J}_z^{(2)}]|\psi_0\rangle = 0, \tag{8}$$

while the geometric phase is determined by

$$e^{-i\gamma_g}|\psi_0\rangle = e^{-i(\pi/2)(\hat{J}_z^{(1)}+\hat{J}_z^{(2)})}|\psi_0\rangle. \tag{9}$$

Since $[\hat{J}_z^{(1)}+\hat{J}_z^{(2)}, \hat{J}_z^{(1)}-\hat{J}_z^{(2)}]=0$ there is the possibility of finding a zero-eigenvalue state of H with a geometric phase equal to a multiple of $\pi/2$. The existence of such a state depends on the realization of the Hamiltonian (6), in particular on the total spins $J^{(1)}$ and $J^{(2)}$.

It is easy to see that the simplest possible realization is that of two half spins, i.e., $J^{(1)}=J^{(2)}=1/2$. If we take into account that $\mathbf{\Omega}=\{\Omega_x,\Omega_y,\Omega_z\}$ has three independent components, the $J^{(1)}=J^{(2)}=1/2$ realization of Eq. (6) corresponds to a four-level scheme coupled by three coherent fields. The spin operators can then be expressed, e.g., by the following 4×4 matrices:

$$\hat{J}_x^{(1)} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \quad \hat{J}_x^{(2)} = \frac{1}{2}\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix},$$

$$\hat{J}_y^{(1)} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \quad \hat{J}_y^{(2)} = \frac{1}{2}\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix},$$

$$\hat{J}_z^{(1)} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{J}_z^{(2)} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

With this the Hamiltonian (6) has the matrix form

$$H = \begin{bmatrix} 0 & \Omega_x & \Omega_y & \Omega_z \\ \Omega_x & 0 & 0 & 0 \\ \Omega_y & 0 & 0 & 0 \\ \Omega_z & 0 & 0 & 0 \end{bmatrix}, \tag{10}$$

which represents the tripod scheme introduced in Ref. [8] and shown in Fig. 2, which has been discussed in the context of a non-Abelian geometric phases in Refs. [6,9]. The rotation of $\mathbf{\Omega}$ in parameter space from the z direction via x to y and back to z corresponds to a sequence of pulses $\Omega_z \to \Omega_x$

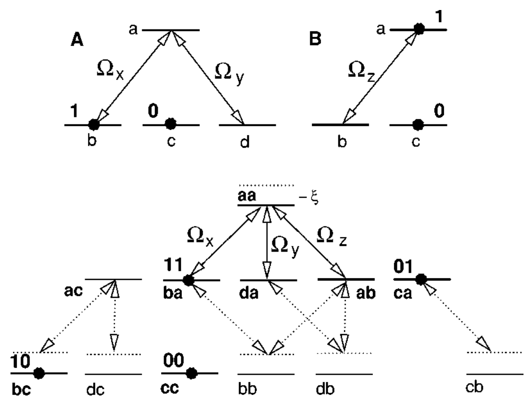

FIG. 3. System of two interacting spin systems. Top: Spectrum without dipole-dipole interaction. Ω_x, Ω_y, and Ω_z denote coherent couplings. Bottom: Spectrum of two-particle states with dipole-dipole interaction of strength ξ. Frequencies of coherent fields are chosen such that the states $|\mathbf{ba}\rangle$, $|\mathbf{da}\rangle$, $|\mathbf{ab}\rangle$, and $|\mathbf{aa}\rangle$ form a closed tripod system. Logical states 00, 01, 10, and 11 are indicated.

$\rightarrow\Omega_y\rightarrow\Omega_z$ also shown in Fig. 2. The adiabatic rotation in parameter space with solid angle $\pi/2$ can be implemented in a very robust way by this pulse sequence. The only requirements are sufficiently long pulses for adiabaticity and overlap of only consecutive pulses to guarantee a solid angle of $\pi/2$. Actual shape, precise timing, and amplitude of the pulses are irrelevant.

The orthogonal zero-eigenvalue states of the interaction Hamiltonian at the initial time $(\hat{J}_z^{(1)}-\hat{J}_z^{(2)})|\psi\rangle=0$ are obviously levels $|2\rangle$ and $|3\rangle$. Their coherent superpositions $(1/\sqrt{2})\times[|3\rangle\pm i|2\rangle]$ are also eigenstates of $\hat{J}_z^{(1)}+\hat{J}_z^{(2)}$ with eigenvalues ±1. After the sequence of four pulses these states acquire a geometric phase shift of $\gamma_{\rm g}=\mp\pi/2$, while there is no dynamical phase shift. Applying the pulse sequence twice leads to a geometric phase of $\gamma_{\rm g}=\pi$. The latter is also true if the initial state is level $|2\rangle$. Although $|2\rangle$ is not an eigenstate of U_g it is an eigenstates of U_g^2 with eigenvalue -1. The last case can also be understood in a much simpler way. If initially state $|2\rangle$ is populated, the double pulse sequence $\Omega_z\rightarrow\Omega_x\rightarrow\Omega_y\rightarrow\Omega_z\rightarrow\Omega_x\rightarrow\Omega_y\rightarrow\Omega_z$ corresponds to three successive population transfers via stimulated Raman adiabatic passage [13]. In this process the state vector is rotated according to $|2\rangle\rightarrow-|4\rangle\rightarrow+|3\rangle\rightarrow-|2\rangle$, thus acquiring a robust phase shift of π.

The use of geometric phases in the tripod scheme of Fig. 2 for single-qubit operations was considered by Duan, Cirac, and Zoller in Ref. [6]. We now show that a similar scheme can be used to create a robust phase gate between two qubits. Here it is important that all states that will be temporarily occupied are energetically degenerate to avoid uncontrollable phase shifts if the interaction parameters are not precisely known. To this end we consider two systems with a level structure as shown in Fig. 3 with an interaction of the form

$$H=H_0+H_{dd}+H_{int}, \tag{11}$$

$$H_0=\mu\sum_{i=1,2}|a\rangle_{ii}\langle a|, \tag{12}$$

$$H_{dd}=-\hbar\xi|a\rangle_{AA}\langle a|\otimes|a\rangle_{BB}\langle a|, \tag{13}$$

$$H_{int}=\hbar\Omega_x(|a\rangle_{11}\langle b|+\text{H.c.})+\hbar\Omega_y(|a\rangle_{11}\langle d|+\text{H.c.})+\hbar\Omega_z(|a\rangle_{22}\langle b|+\text{H.c.}). \tag{14}$$

ξ represents the strength of the interaction between the spins in the internal state $|a\rangle$. An interaction Hamiltonian of type (14) could, e.g., be realized with a pair of atoms with states $|a\rangle$ being Rydberg levels with a large permanent dipole moment [14]. Also a realization in NMR systems [15] is feasible. In each of the two systems a single qubit is encoded as indicated in Fig. 3:

$$[|0\rangle,|1\rangle]_A\equiv[|c\rangle,|b\rangle]_A,\quad[|0\rangle,|1\rangle]_B\equiv[|c\rangle,|a\rangle]_B.$$

In order to implement a robust phase gate, the interaction shall generate a geometric phase shift of π of state $|ba\rangle$ without dynamical phases at all times.

In the lower part of Fig. 3 the dressed-state energy diagram of the system corresponding to the free Hamiltonian H_0 and the dipole-dipole interaction H_{dd} are shown. Without coherent coupling, i.e., $\Omega_i\equiv0$ the qubit states $|ca\rangle$ and $|ba\rangle$ will acquire a phase $e^{-i\mu t}$, while the other two states $|cc\rangle$ and $|bc\rangle$ remain constant. The phase $e^{-i\mu t}$ is without consequence, since it can easily be compensated by local operations if the energy splitting μ is known very well. No precise knowledge of the interaction parameters Ω_i is required for this. It is also sufficient to know the dipole-dipole shift ξ only approximately in order to tune the fields close to resonance with $|aa\rangle$. If ξ is sufficiently large the coupling between the states $|ba\rangle$, $|da\rangle$, and $|ab\rangle$ with the lower-lying states $|db\rangle$ and $|bb\rangle$ can be disregarded. In this case there is a tripod coupling with energetically degenerate states $|ba\rangle$, $|da\rangle$, and $|ab\rangle$ and a sequence of overlapping pulses $\Omega_z\rightarrow\Omega_x\rightarrow\Omega_y\rightarrow\Omega_z\rightarrow\Omega_x\rightarrow\Omega_y\rightarrow\Omega_z$ will lead to a geometric phase shift $|ba\rangle\rightarrow-|ab\rangle\rightarrow|da\rangle\rightarrow-|ba\rangle$. Since the states $|ba\rangle$, $|da\rangle$, and $|ab\rangle$ are degenerate and the interaction H_{int} has $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ symmetry, there is no dynamical phase at any time of the process. Therefore temporal fluctuations of the field amplitudes and the dipole-dipole shift will not affect the phase gate. It should be noted that the degeneracy of the lower states in the tripod scheme is important however, since the exact time which the system spends in the three states depends on the details of the interaction and is in general not known precisely.

As can be seen from Fig. 3 the off-resonant coupling of $|ba\rangle$, $|da\rangle$, and $|ab\rangle$ with the lower states $|db\rangle$ and $|bb\rangle$ can give rise to real transitions and ac-Stark dynamical phases. To avoid real transitions into the lower manifold of states $|\xi|\gg|\Omega_i^{\max}|$ is required. The ac-Stark induced phase shift is negligible if $|\Omega_i^{\max}|^2T/|\xi|\ll1$, where T is the characteristic time of the process. These conditions combined read

$$\frac{|\Omega_i^{\max}|}{|\xi|}\ll\frac{|\Omega_i^{\max}|^2}{|\xi|}T\ll1. \tag{15}$$

It should be noted that the dipole-dipole shift ξ is here an independent parameter and can in principle be chosen very large without affecting the resonant couplings. This is in contrast to the proposal of Ref. [6] where the ac-Stark shifts

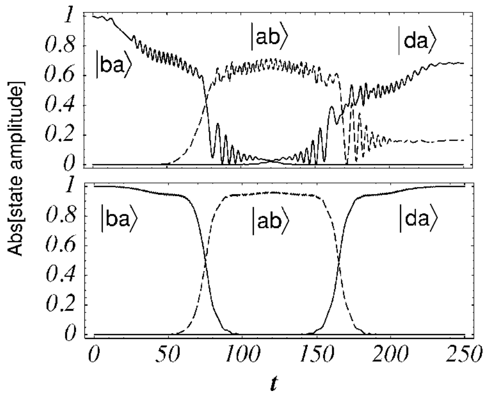

FIG. 4. Absolute value of state amplitude for $|ba\rangle$, $|ab\rangle$, and $|da\rangle$ obtained from numeric solution of full problem with sequence of time-delayed Gaussian pulses $\Omega_z \to \Omega_x \to \Omega_y \to \Omega_z$ (half cycle of phase gate). $\Omega_i = \alpha \exp[-(t-t_i)^2/(2T^2)]$ with $\alpha=1$, $T=20$, $t_x=100$, $t_y=140$, and $t_z=50$, resp., $t_z=190$; upper curve, $\xi=1$; lower curve, $\xi=4$.

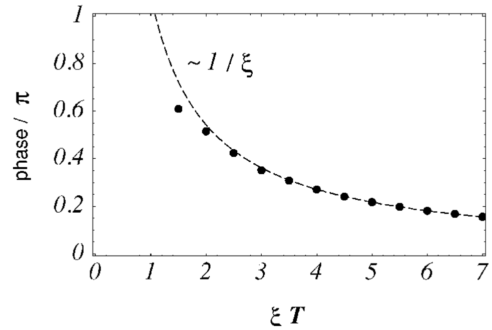

FIG. 5. Final phase of state $|da\rangle$ for pulse sequence of Fig. 4 and growing values of ξ. Dots represent numerical results, dashed curve is a $1/\xi$ fit.

cannot be neglected and need to be compensated.

To illustrate conditions (15), we have numerically calculated the amplitudes of states $|ba\rangle$, $|da\rangle$, and $|ab\rangle$ as well as the phase of the target state $|da\rangle$ for half of the phase-gate pulse sequence, i.e., $\Omega_z \to \Omega_x \to \Omega_y \to \Omega_z$. The results are shown in Figs. 4 and 5. Ideally, i.e., if Eq. (15) is perfectly fulfilled, there is a complete state transfer from $|ba\rangle$ to $|da\rangle$ with zero phase change. Figure 4 shows that a dipole-dipole shift slightly larger than the peak Rabi frequency is sufficient to suppress real transitions into other states. As can be seen from Fig. 5 the ac-Stark induced phase shifts scale only as $(\xi T)^{-1}$ and thus larger values of $|\xi|$ are needed to neglect them. Nevertheless it can be seen that it is always possible to choose sufficiently large values of ξ to obtain a purely geometric phase.

In the present paper we have shown that it is possible to obtain a nonvanishing geometric phase of multiples of $\pi/2$ with an exactly vanishing dynamical phase. For this it is necessary to consider systems with an $SU(2) \otimes SU(2)$ symmetry rather than just SU(2). The simplest nontrivial representation of this symmetry corresponds to a four-level system with a tripod coherent coupling. We have shown that a tripod coupling among two-qubit states can be implemented using a pair of coherently driven particles with dipole-dipole interaction. With this it is possible to design a geometric phase gate. Due to the absence of dynamical contributions to the phase and the geometric nature of the phase shift, the quantum gate is maximally robust against parameter fluctuations.

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