Nuclear recoil effects in antiprotonic and muonic atoms

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Relativistic nuclear recoil effects are studied for antiprotonic and muonic atoms. The generalization of the Breit-Pauli Hamiltonian including vacuum polarization is presented. Previous treatments are corrected, and the result for the $2S_{1/2}$-$2P_{1/2}$ splitting in muonic hydrogen is updated.

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I. INTRODUCTION

The purpose of this work is the study of nuclear recoil effects for low-lying states of antiprotonic and muonic atoms. Since the ratio of the proton or muon mass to the nuclear mass is relatively large, particularly for light systems, these recoil effects significantly modify energy levels, fine and hyperfine structure. Several precise measurements of x-ray transitions were performed in light $\bar{p}$ atoms, such as $\bar{p}$-H, $\bar{p}$-D [1,2], $\bar{p}$-He [3], but also in $\bar{p}$-O [4] and $\bar{p}$-Pb [5]. The comparison of theoretical predictions with the experimentally measured transition energies gives us information on the strong interaction shifts in antiprotonic atoms and thus on the low-energy $\bar{p}$-nucleon interaction expressed in terms of the scattering length [6]. On the other hand, experiments with muonic atoms give information on the electromagnetic properties of the nucleus. Particularly interesting is the ongoing measurement of the $2P$-$2S$ splitting in μH-muonic hydrogen [7], from which we hope to obtain an accurate value of the proton charge radius. At present, the uncertainty in the proton charge radius limits the accuracy of QED tests with the hydrogen atom [8]. In this work we generalize the Breit interaction to include vacuum polarization in the relativistic and nonrelativistic framework. It has been studied extensively among others in Refs. [9–13]. We claim that some previous treatments of nuclear recoil effects were not complete, and the difference will be visible when high-precision measurements of transition frequencies in antiprotonic atoms become available. Finally, we present an improved value for the combined recoil and vacuum polarization corrections in μH.

II. RECOIL EFFECTS IN ANTIPROTONIC ATOMS

Nuclear recoil effects have been investigated in detail for normal atoms, mostly in the context of hyperfine splitting and isotope shifts. The most advanced and precise results have been obtained for hydrogenic systems, for a recent review see Ref. [14]. This is because very high accuracy measurements of the $1S$-$2S$ transition in hydrogen and deuterium [15] have been performed in the last few years. The theoretical treatment is based on the QED. Its application to bound states is a nontrivial task and has been under development since the early beginning of QED. While in the nonrelativistic limit recoil effects are included in the reduced mass treatment, the relativistic approach at the level of the Dirac equation is much more complicated [16]. First of all, there is no unique Hamiltonian to describe the relativistic electron in the field of a moving nucleus. The Dirac Hamiltonian with the reduced mass μ gives energy levels which are valid only in order $(Z\,\alpha)^2$, i.e., in the nonrelativistic limit. A more accurate treatment is the Dirac-Breit Hamiltonian with the nonrelativistic kinetic energy of the nucleus with a mass M [16],

$$H = \vec{\alpha}\cdot\vec{p} + \beta m - \frac{Z\alpha}{r} + \frac{\vec{p}^2}{2M} - \frac{Z\alpha}{2M}\frac{\alpha^i}{r}\left(\delta^{ij} + \frac{r^i r^j}{r^2}\right)p^j. \quad (1)$$

This Hamiltonian can be easily generalized to the many electron case [17] and gives energy levels which are accurate up to order $(Z\alpha)^4$. Higher order in $Z\alpha$ corrections are not accounted for by the Breit interaction. They can be derived only on the basis of QED theory. The general formula for an arbitrary mass ratio m/M and nuclear charge Z is not known. For systems with large Z, only the first-order correction in the mass ratio is important. The correct expression which is nonperturbative in $Z\alpha$ was derived in Ref. [18]. Although quite complicated, it was used in the numerical and analytical calculation of recoil effects in hydrogenlike atoms. For electronic atoms, corrections of order $(m/M)^2$ tend to be negligible, and can be treated in the nonrelativistic approximation. However, for light muonic and antiprotonic atoms they are no longer negligible and a different approach should be developed. The positronium atom serves as a good example. Here, all recoil effects have been obtained up to order $m\alpha^6$ [19] and a similar calculation can be performed for excited states of the p-$\bar{p}$ system when precise experimental results become available [20].

We return now to the Dirac-Breit Hamiltonian with the aim of describing recoil effects in heavy antiprotonic atoms. The nuclear spin-dependent interactions have been considered among others by Borie in Ref. [12], by Pilkuhn and Schlaile in Ref. [13], and by us in Ref. [21]. Here, we do not study hyperfine structure, so nuclear spin-dependent terms are neglected and we assume that the nucleus is spinless. The first modification of Eq. (1) to describe antiprotonic atoms is the inclusion of the anomalous magnetic moment, which is large for antiprotons, namely $\kappa = 1.792\,847\,34$. The interaction with the electromagnetic field is changed to

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$$\gamma^\mu A_\mu(q) \to \left(\gamma^\mu + \frac{i\kappa}{2m}\sigma^{\mu\nu}q_\nu\right)A_\mu(q), \tag{2}$$

which results in a correction to the Hamiltonian of the form

$$\delta H = \frac{e\kappa}{2m}(i\vec{\gamma}\cdot\vec{E} - \beta\vec{\Sigma}\cdot\vec{B}) \tag{3}$$

and leads to the following Dirac-Breit Hamiltonian with the anomalous magnetic moment

$$H = \vec{\alpha}\cdot\vec{p} + \beta m - \frac{Z\alpha}{r} - \frac{i\kappa}{2m}\frac{Z\alpha}{r^3}\vec{\gamma}\cdot\vec{r} + \frac{\vec{p}^2}{2M} - \frac{Z\alpha}{2M}\frac{\alpha^i}{r}\left(\delta^{ij} + \frac{r^i r^j}{r^2}\right)p^j + \frac{\kappa}{2mM}\beta\vec{L}\cdot\vec{\sigma}\frac{Z\alpha}{r^3}. \tag{4}$$

Since we assumed here a point nucleus, this Hamiltonian is not well defined. It is known that even the first line of Eq. (4) leads to unphysical solutions for any value of $Z\alpha$. We have studied this Hamiltonian numerically and draw the conclusion that the inclusion of finite nuclear size is necessary for any angular momentum l. Since the nuclear size is larger than the $\bar{p}$ Compton wavelength, its inclusion is necessary anyway. While the finite-size modification of the Coulomb interaction is obvious, it is less obvious how to modify the Breit interaction. The same problem appears with vacuum polarization, and the solution is the following. We assume that $\bar{p}$ is a pointlike particle and for a moment, the nucleus is also a pointlike particle. The vacuum polarization modifies the photon propagator on the scale of the electron Compton wavelength. Dispersion relations allow one to write the modified propagator as an integral over a photon mass,

$$\frac{g_{\mu\nu}}{k^2} \to \frac{\alpha}{\pi}\int_2^\infty d\rho\frac{2}{3\rho}\sqrt{1-\frac{4}{\rho^2}}\left(1+\frac{2}{\rho^2}\right)\frac{g_{\mu\nu}}{k^2 - m_e^2\rho^2}. \tag{5}$$

Thus, we can derive the effective interaction, which accounts for the exchange of a massive photon in the Coulomb gauge in the no retardation limit ($k^0=0$),

$$G_{00}(\vec{k}) = -\frac{1}{\vec{k}^2 + \tilde{\rho}^2}, \qquad G_{ij}(\vec{k}) = \frac{1}{\vec{k}^2+\tilde{\rho}^2}\left(\delta_{ij} - \frac{k_i\,k_j}{\vec{k}^2+\tilde{\rho}^2}\right), \tag{6}$$

which has the Fourier transform

$$G_{00}(\vec{r}) = -\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+\tilde{\rho}^2} = -\frac{e^{-\tilde{\rho}r}}{4\pi r}, \tag{7}$$

$$G_{ij}(\vec{r}) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+\rho^2}\left(\delta_{ij} - \frac{k_ik_j}{\vec{k}^2+\tilde{\rho}^2}\right)e^{i\vec{k}\cdot\vec{r}} = \frac{e^{-\tilde{\rho}r}}{8\pi r}\left[\delta^{ij} + \frac{r^ir^j}{r^2}(1+\tilde{\rho}r)\right], \tag{8}$$

where we introduce the notation $\tilde{\rho}=m_e\rho$. We analyze now in detail the vacuum polarization modification of the Dirac-Breit Hamiltonian with the anomalous magnetic moment in Eq. (4). The Coulomb potential, the third term in Eq. (4), becomes

$$-\frac{Z\alpha}{r} \to -\frac{Z\alpha}{r} - \frac{\alpha}{\pi}\int_2^\infty d\rho\frac{2}{3\rho}\sqrt{1-\frac{4}{\rho^2}}\left(1+\frac{2}{\rho^2}\right)\frac{Z\alpha}{r}e^{-\rho m_e r} \equiv V(r). \tag{9}$$

The fourth term in Eq. (4) corresponds to the coupling of the anomalous magnetic moment to the electric field, and thus is proportional to the derivative of V,

$$-\frac{i\kappa}{2m}\frac{Z\alpha}{r^3}\vec{\gamma}\cdot\vec{r} \to -\frac{i\kappa}{2m}\frac{V'}{r}\vec{\gamma}\cdot\vec{r}. \tag{10}$$

The last two terms of Eq. (4) come from the magnetic interaction between $\bar{p}$ and the nucleus. In momentum representation and with a fixed value of $\tilde{\rho}$ it is ($\vec{q}=\vec{p}\,'-\vec{p}$)

$$-\frac{Ze^2}{q^2+\tilde{\rho}^2}\left(\delta^{ij} - \frac{q^iq^j}{q^2+\tilde{\rho}^2}\right)\left(\alpha^i - \frac{i\kappa}{2m}\varepsilon^{ilk}q^l\beta\Sigma^k\right)\frac{(p^j+p_j')}{2M}. \tag{11}$$

The term with the anomalous magnetic moment,

$$\frac{Ze^2}{q^2+\tilde{\rho}^2}\frac{i\kappa}{2m}\varepsilon^{ilk}q^l\beta\,\Sigma^k\,\frac{p^i}{M}, \tag{12}$$

in the position representation is proportional to the derivative of V,

$$\frac{\kappa}{2mM}\beta\,\vec{\Sigma}\cdot\vec{L}\frac{V'}{r}. \tag{13}$$

The part of Eq. (11) which does not depend on κ in position space takes the form, with the help of Eq. (8),

$$-\frac{Z\alpha}{4M}\alpha^i\frac{e^{-\tilde{\rho}r}}{r}\left(\delta^{ij} + \frac{r^ir^j}{r^2}(1+\tilde{\rho}r)\right)p^j + \text{H.c.} \tag{14}$$

The integration with respect to ρ is performed using Eq. (9) and leads to

$$\frac{1}{4M}\alpha^i\left(\delta^{ij}V - \frac{r^ir^j}{r}V'\right)p^j + \text{H.c.} \tag{15}$$

The sum of terms in Eqs. (9), (10), (13), and (15) together with the potential-independent terms of Eq. (4) gives the following Breit Hamiltonian for the interaction of $\bar{p}$ with the spinless nucleus:

$$H = \vec{\alpha}\cdot\vec{p} + \beta m + V - \frac{i\kappa}{2m}\frac{V'}{r}\vec{\gamma}\cdot\vec{r} + \frac{\vec{p}^2}{2M} + \frac{\kappa}{2mM}\frac{V'}{r}\beta\vec{L}\cdot\vec{\sigma} + \frac{\alpha^i}{4M}\left[\left(\delta^{ij}V - \frac{r^ir^j}{r}V'\right)p^j + p^j\left(\delta^{ij}V - \frac{r^ir^j}{r}V'\right)\right]. \tag{16}$$

It is interesting to note that vacuum polarization effects on the Breit interaction can be expressed in terms of the static potential V. If the nucleus is a finite-size particle, but still spinless, it effectively leads to the modification of the photon propagator. Therefore, one can write a similar spectral representation of a photon propagator in Coulomb gauge as in Eq. (6), which leads to the conclusion that Eq. (16) still holds if the potential V includes the finite nuclear size. However, one has to keep in mind that the strong interaction shift, neglected here, is much more significant than the finite charge radius. Moreover, if $\bar{p}$ is close to the nucleus, annihilation of $\bar{p}$ is a dominating effect. Therefore Eq. (16) is valid for sufficiently large angular momenta. Equation (16) differs from the previously published [10,11] results by several details. The differences mostly come from the fact that the massless Coulomb gauge in the no retardation limit was used in these former works. When the vacuum polarization and the finite size are neglected, Eq. (16) has been further transformed by Grotch and Yennie in Ref. [16] to incorporate as much as possible recoil effects into the reduced mass of the system and some additional term. The effects beyond reduced mass have been treated perturbatively. This approach becomes quite complicated when vacuum polarization is included, see, for example, Ref. [10], and for this reason we think that the direct numerical solution of Hamiltonian in Eq. (16) is simpler and more convenient.

Our principal interest is, however, recoil effects in light systems where both particles, the antiproton and the nucleus, are treated on an equal footing. Recoil effects neglecting vacuum polarization were considered by Chraplyvy in Ref. [22] and Barker and Glover in Ref. [23], where the Fouldy-Wouthuysen transformation was applied to the relativistic two-body Breit interaction. A simple derivation, which we follow here, relies on the nonrelativistic expansion of the one-photon scattering amplitude [24]. Using Coulomb gauge for massive photons of Eq. (6) one obtains the following Breit-Pauli Hamiltonian with the vacuum polarization and the finite nuclear size,

$$H = \frac{p^2}{2}\left(\frac{1}{m} + \frac{1}{M}\right) + V + \delta H, \tag{17}$$

$$\begin{aligned}\delta H = &-\frac{p^4}{8}\left(\frac{1}{m^3} + \frac{1}{M^3}\right) + \frac{1+2\kappa}{8m^2}\nabla^2 V \\ &+ \left(\frac{1+2\kappa}{4m^2} + \frac{1+\kappa}{2mM}\right)\frac{V'}{r}\vec{L}\cdot\vec{\sigma} + \frac{1}{2mM}\nabla^2 \\ &\times\left[V - \frac{1}{4}(rV)'\right] + \frac{1}{2mM}\left[\frac{V'}{r}L^2 + \frac{p^2}{2}(V - rV') \right. \\ &\left. + (V - rV')\frac{p^2}{2}\right].\end{aligned} \tag{18}$$

Note that the apparent asymmetry in $\bar{p}$ and the nucleus comes from the assumption that the nucleus is spinless. For this reason, there are no nuclear spin operators nor nuclear Darwin terms in Eq. (18). This Hamiltonian differs slightly from that obtained by Borie in Ref. [11] by the presence of V' instead of $-V/r$ in last two terms. For the point nucleus and without vacuum polarization it coincides with the Breit-Pauli Hamiltonian. Higher-order QED effects involve Lamb-shift-like corrections which have been studied in detail for electronic atoms. However, the case of $\bar{p}$ atoms has not been investigated. We assume here that the angular momentum is different from 0, as it is in the experimental conditions, and additionally neglect vacuum polarization. The self-energy of the $\bar{p}$, self-energy of the nucleus, the single and double exchange of transverse photons give the Bethe logarithm and an additional recoil term [25],

$$\begin{aligned}\delta E = &-\frac{(Z\alpha)^5}{\pi n^3}\frac{4}{3}\ln k_0(n,l)\left(\frac{1}{Z}\frac{\mu^3}{m^2} + 2\frac{\mu^3}{mM} + Z\frac{\mu^3}{M^2}\right) \\ &- \frac{(Z\alpha)^5}{\pi}\frac{\mu^3}{mM}\frac{7}{6}\left\langle\frac{1}{r^3}\right\rangle.\end{aligned} \tag{19}$$

There are no further corrections at order $m(Z\alpha)^5$; however the analogous calculations including vacuum polarization have not yet been performed.

III. RECOIL EFFECTS IN MUONIC ATOMS

The treatment of muonic atoms differs from antiprotonic atoms due to the different mass and anomalous magnetic moment. The muon Compton wavelength $\lambda\!\!\!^{-} = 1.867$ fm is comparable to the nuclear size. This means that use of the anomalous magnetic moment on the level of the Dirac equation is limited, and in a proper approach one should consider a complete muon self-energy, as in the case of electronic atoms. Therefore, for muonic atoms we put in Eq. (16) $\kappa = 0$ and obtain a Dirac-Breit Hamiltonian including vacuum polarization and finite-size effects,

$$\begin{aligned}H = \vec{\alpha}\cdot\vec{p} + \beta m + V + \frac{\vec{p}^2}{2M} + \frac{\alpha^i}{4M}\left[\left(\delta^{ij}V - \frac{r^i r^j}{r}V'\right)p^j \right. \\ \left. + p^j\left(\delta^{ij}V - \frac{r^i r^j}{r}V'\right)\right].\end{aligned} \tag{20}$$

For small nuclear masses, a more appropriate treatment relies on the Breit-Pauli Hamiltonian, as in the case of antiprotonic atoms. Here, we also set $\kappa = 0$ and following Ref. [21] the anomalous magnetic moment is included as a part of the Lamb shift. Moreover, the inclusion of the finite nuclear size in the Breit interaction in Eq. (16) does not account properly for this effect. This is because the nuclear size is of order of the muon Compton wavelength, and the nonretardation approximation ($k^0 = 0$) used to derive Eq. (16) is no longer valid. In the more accurate approach one considers finite-size effects separately and the leading correction beyond the nonrelativistic r^2 term is given by the forward-scattering amplitude; for details see Ref. [21]. With these approximations the Breit-Pauli Hamiltonian for muonic atoms with a spinless nucleus becomes

$$H = \frac{p^2}{2}\left(\frac{1}{m} + \frac{1}{M}\right) + V + \delta H, \tag{21}$$

$$\delta H = -\frac{p^4}{8}\left(\frac{1}{m^3}+\frac{1}{M^3}\right)+\frac{1}{8m^2}\nabla^2 V+\left(\frac{1}{4m^2}+\frac{1}{2mM}\right)\frac{V'}{r}\vec{L}\cdot\vec{\sigma} + \frac{1}{2mM}\nabla^2\left[V-\frac{1}{4}(rV)'\right] + \frac{1}{2mM}\left[\frac{V'}{r}L^2+\frac{p^2}{2}(V-rV')+(V-rV')\frac{p^2}{2}\right], \quad (22)$$

where V, as given in Eq. (9) includes Coulomb and vacuum polarization potentials for the point nucleus. The calculation of $\nabla^2 V$ in the above is a little troublesome. The potential V behaves at small radius r as

$$V(r) \approx -\frac{Z\alpha}{r}+\frac{2\alpha}{3\pi}\frac{Z\alpha}{r}\left[\ln(m_e r)+\gamma+\frac{5}{6}\right]+O(r^0), \quad (23)$$

therefore the calculation of ∇^2 should be performed in the sense of a Schwartz distribution with a trial function f,

$$\int d^3r f(r)\nabla^2 V(r) \equiv \int d^3r V(r)\nabla^2 f(r). \quad (24)$$

We turn now to muonic hydrogen and obtain the relativistic vacuum polarization correction to $2P_{1/2}$-$2S_{1/2}$ splitting in muonic hydrogen μH. This has been obtained by one of us (K.P.) in Ref. [21], however incorrectly due to a computational mistake. This correction is given by the matrix element of the Hamiltonian δH in Eq. (22) with and without the vacuum polarization:

$$\langle\phi|\delta H|\phi\rangle_{\text{no vp}} = 0.0575 \text{ meV},$$
$$\delta E = \langle\phi|\delta H|\phi\rangle_{\text{vp}} - \langle\phi|\delta H|\phi\rangle_{\text{no vp}} = 0.0169 \text{ meV}, \quad (25)$$

where we used here the analytical approach of Ref. [21], and physical constants are taken from Ref. [26]. The former, incorrect result for δE was 0.0594 meV which makes a significant difference. We correct here also a few other works [14,27] which employed this result. The new improved result for the theoretical prediction of the $2P_{1/2}$-$2S_{1/2}$ splitting in μH, based on the work [27], is

$$E(2P_{1/2}\text{-}2S_{1/2}) = 206.042(3) - r^2 5.2256 + r^3 0.0363 = 202.182(108)\text{meV}, \quad (26)$$

with $r=0.862(12)$fm.

IV. SUMMARY

We have investigated recoil effects in antiprotonic and muonic atoms. The formulas obtained can be used for high-precision determination of energy levels. Due to the new antiproton source and decelerator planned at GSI (Darmstadt, Germany) [28], precise measurements with antiprotonic atoms seem to be feasible. What we can learn from comparison of theoretical predictions with experiments, apart about from testing QED in a yet unexplored region, are for example, low energy $\bar{p}$-p, $\bar{p}$-n interactions and properties of nuclei, such as distribution of neutrons [29], which are not easily accessible with other techniques.

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