Mediated entanglement and correlations in a star network of interacting spins

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We investigate analytically a star network of spins, in which all spins interact exclusively and continuously with a central spin through Heisenberg *XX* couplings of equal strength. We find that the central spin correlates and entangles the other spins at zero temperature to a degree that depends on the total number of spins. We find that the entanglement mediating capability of the central spin depends on the evenness or oddness of this number. In the limit of an infinite collection of spins, the difference between entanglement and correlations in terms of divisibility among multiple parties is clearly demonstrated. We also show that with a significant probability one can maximally entangle any two noncentral spins by measuring all the other spins (a process related to the recently introduced notion of localizable entanglement). This probability depends on the evenness and oddness of the total number of spins and remains substantial even for an infinite collection of spins. We show how symmetric multiparty states for optimal sharing and splitting of entanglement can be obtained as ground states of this system using a magnetic field. These states can then be mapped on to flying qubits for transmission to distant parties. We discuss a number of advantages of this mode of generation and distribution of entanglement over other standard methods.

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For a long-time spin correlations in one-dimensional (1D) chains and higher dimensional lattices of interacting spins have been a subject of extensive interest [1,2]. Recently, the same systems have been studied from the point of view of truly *quantum correlations* or entanglement [3–14]. However, lattices of various dimensions are not the *only* physical systems whose fabrication is possible with current technology. It thus becomes interesting to extend the above line of research on entanglement in spin systems to other than spin chains. In particular, various technologies have evolved which can make any member of an array of qubits (systems isomorphic to spin $1/2$) interact with any other member [15–17]. These arrays have been developed with the ultimate aim of quantum computation in mind. However, much before a full-fledged quantum computer is developed, which requires both large arrays and controllable interactions, we will have small arrays with untunable (fixed) interactions. This is true because it is typically difficult to tune interactions in certain implementations of quantum computation [18] and generally difficult to have a large array of qubits in any implementation. With small arrays with untunable (fixed) interactions, it becomes possible to visualize structures of interacting spins which do not fall into the category of lattices in various dimensions. One very simple structure that one can imagine, is a *spin star*, as opposed to the extensively studied *spin chains*. In such a spin star, there is a preferred spin, which we call the *central spin* which interacts with *all* the other spins. All the noncentral spins (which we will call the *outer spins*), on the other hand, do not directly interact among themselves. The structure is depicted in Fig. 1 in which 0 depicts the central spin. The spins $1-5$ interact only with the central spin and not with each other. The architecture is analogous to the star distribution networks used in communications. To our knowledge, not just entanglement and correlations, but also the statistical mechanics of such a structure remains unexplored (we have recently become aware of work in the same geometry of qubits being carried out independently by another group, though they use a different measure to quantify the entanglement between qubits [19]). We show that the ground state of this configuration is an interesting multiparticle entangled state, symmetric in the outer spins. If interactions between qubits can be made truly longrange [20], then this structure could be used for entanglement distribution between several distant parties, where spins shared by distant parties interact directly. Even before such long-distance interactions become feasible, we can use the star configuration as a *source* of interesting multiparticle entangled states. The multiparticle entangled states of spins in the star configuration can be mapped onto flying qubits such as photons for distribution to distant parties. We point out some advantages of this mode of generation and

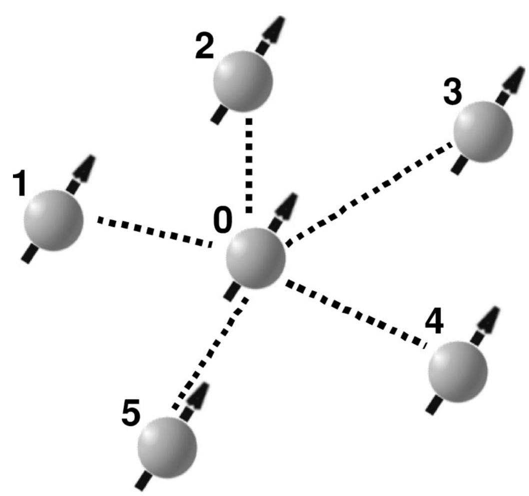

FIG. 1. This figure depicts the star configuration of spins. The spin labeled 0 is the central spin, which interacts with spins 1–5 around it.

distribution of entanglement over standard methods.

Our work can also be regarded as a part of the continued increase of interest in the study of entanglement between quantum systems placed in the vertices of various graphs showing some type of symmetry [21]. In our case the symmetry is the exchange symmetry of the outer spins. We will also show that not only is the ground state of our system an interesting multiparticle entangled state, but it also allows the creation of maximally entangled states between any two outer spins with a significant probability. This uses the recently introduced notion of localizable entanglement [22], in which all the other spins are measured to entangle any two spins of interest. Interestingly, the probability of creating a maximally entangled state between two outer spins also depends on the evenness and oddness of the total number of spins and remains significant even for an infinite collection of spins.

The star configuration with couplings of equal strength has many symmetry properties due to its invariance under the exchange of any two outer spins and we solve it exactly in the case of an Heisenberg *XX* interaction. The *XX* model was intensively investigated for spin chains by Lieb, Schultz, and Mattis [2] and has been realized in recent years as an effective Hamiltonian in some systems [16,17]. We find that the central spin *mediates* correlations and entanglement between the outer spins at zero temperature to a degree that depends on the total number of spins in the spin star. As expected, the entanglement goes down on average with the increase of the total number of spins. However, it shows oscillations with the evenness or oddness of this number. This means that the entanglement mediating capability of the central spin can sometimes increase with the addition of an extra outer spin. This contrasts the naive expectation that in a star network, addition of an extra outer spin in the network is only expected to make it harder for the central spin to mediate entanglement. In the limit of a large number of the outer spins, the model also illustrates a crucial difference between entanglement and correlations, when the mediated entanglement is vanishing but substantial spin ordering (correlation function $\geqslant 1/2$) is present in the X and Y directions. We also show that we can apply a magnetic field to our system to obtain multiparty states for optimal symmetric splitting [23] and optimal symmetric sharing [24] of entanglement as the ground state and as a simple derivative of the ground state, respectively.

The Hamiltonian which describes our system is given by

$$H = \mathcal{J}\left(\sigma_{0x}\sum_{\text{outer}}\sigma_{ix} + \sigma_{0y}\sum_{\text{outer}}\sigma_{iy}\right), \tag{1}$$

where the summation over "outer" refers to the outer spins, σ_{ix} and σ_{iy} denote the σ_x and σ_y Pauli operators for the ith outer spin and σ_{0x} and σ_{0y} denote the σ_x and σ_y Pauli operators for the central spin. It can be shown that $J_x=(1/2)\Sigma_{\text{outer}}\,\sigma_{ix}$, $J_y=(1/2)\Sigma_{\text{outer}}\,\sigma_{iy}$, and $J_z=(1/2)\Sigma_{\text{ring}}\,\sigma_{iz}$ obey the standard angular-momentum commutation relations (we have taken $\hbar=1$). This implies that the outer spins *collectively behave* as a single spin with spin operator $\mathbf{J}=\hat{\boldsymbol{i}}J_x+\hat{\boldsymbol{j}}J_y+\hat{\boldsymbol{k}}J_z$. It can be shown that $J^2=J_x^2+J_y^2+J_z^2$ commutes with H. It will also help to define the total angular-momentum operator $\boldsymbol{F}=(1/2)\boldsymbol{\sigma}_0+\mathbf{J}$, where $\boldsymbol{\sigma}_0=\hat{\boldsymbol{i}}\sigma_{0x}+\hat{\boldsymbol{j}}\sigma_{0y}+\hat{\boldsymbol{k}}\sigma_{0z}$ and it can be shown that the z component, F_z obeys $[H,F_z]=0$, $[J^2,F_z]=0$. Therefore simultaneous eigenstates of H, J^2, and F_z can be constructed.

It is convenient to recast the Hamiltonian in Eq. (1) using the raising and lowering operators $\sigma_\pm=(\sigma_x \pm i\sigma_y)$ and $J_\pm=(1/2)\Sigma_{\text{outer}}\,\sigma_\pm$ as

$$H = \mathcal{J}(\sigma_{0+}J_- + \sigma_{0-}J_+). \tag{2}$$

The above Hamiltonian thus represents a resonant interaction between a spin 1/2 and a higher spin (with operator $\mathbf{J}$) system. Such a system is readily analyzed (note the similarity of the above Hamiltonian with the Jaynes-Cummings Hamiltonian [25]) and has eigenstates of the form

$$\frac{1}{\sqrt{2}}(|0\rangle|j,m\rangle \pm |1\rangle|j,m-1\rangle), \tag{3}$$

where the first ket in each term denotes the central spin ($|0\rangle$ and $|1\rangle$ stand for the $|-1/2\rangle$ and $|1/2\rangle$ spin states of the central spin), and the second ket is an eigenstate of J^2, j is the quantum number associated with eigenstates of J^2 [eigenvalue of J^2 is $j(j+1)$], and m is the quantum number for J_z. This state is an eigenstate of F_z with eigenvalue $m-1/2$. Equation (3) is valid for $m=j$ to $m=-j+1$. There are also two additional states where only one of the terms exist: $|1\rangle|j,j\rangle$, because $|0\rangle|j,j+1\rangle$ does not exist, and similarly $|0\rangle|j,-j\rangle$.

As the angular momentum J comes from the ensemble of outer spins, there is degeneracy in j due to the different possible orientations of these spins. Let the number of outer spins be N. In general, there are ${}^{N}C_r - {}^{N}C_{r-1}$ ways of obtaining $j=(N-2r)\frac{1}{2}$, with allowed values of r ranging from $r=0$ to $r=N/2$ if N is even, or $r=(N-1)/2$ if N is odd.

To conclude this brief introduction to the eigenstates of this system, the energy eigenvalues for the star-spin system are given by

$$E = \pm\,\mathcal{J}\sqrt{(j+m)(j-m+1)}. \tag{4}$$

Our attention now turns to the properties of this model which are useful for sharing entanglement between different spins. To begin with we study the ground state. Assuming $\mathcal{J}$ positive, E is minimized when j has its maximum possible value and m has its minimum absolute value. For the case N odd, the lowest energy is when $m=\frac{1}{2}$, i.e., the eigenstate:

$$|\Psi_G\rangle_{\text{Odd}} = (1/\sqrt{2})(|0\rangle|N/2,1/2\rangle - |1\rangle|N/2,-1/2\rangle),$$

and if N is even, then in fact the ground state is degenerate because there are two states with the lowest possible energy, when $m=0$ or $m=1$:

$$|\Psi_G\rangle_{\text{Even1}} = \frac{1}{\sqrt{2}}(|0\rangle|N/2,0\rangle - |1\rangle|N/2,-1\rangle),$$

$$|\Psi_G\rangle_{\text{Even2}} = \frac{1}{\sqrt{2}}(|0\rangle|N/2,1\rangle - |1\rangle|N/2,0\rangle).$$

The reason for the above difference between N even and N odd is that the two cases lead to an integral and half integral value of j, respectively. When j is half integral, $m=\pm 1/2$ is allowed and gives an unique ground state. For j integral, the $0,-1$ and $1,0$ form two distinct j,m pairs to combine with the central spin-1/2 particle to give two degenerate ground states.

To compute entanglement and correlations, it is useful to have expressions in terms of the states of the individual outer spins for these ground states. Let $|0\rangle$ and $|1\rangle$ stand for the $|-1/2\rangle$ and $|1/2\rangle$ spin states of any outer spin. For N odd, the state $|N/2,1/2\rangle$ is an equal superposition of all states with $(N+1)/2$ ones and $(N-1)/2$ zeros with no relative phase between them. The state $|N/2,-1/2\rangle$ is the same type of state with $(N-1)/2$ ones and $(N+1)/2$ zeros. For example, for $N=3$, these are the familiar W states [26] given by

$$|3/2,1/2\rangle = \frac{1}{\sqrt{3}}[|011\rangle + |101\rangle + |110\rangle],$$

$$|3/2,-1/2\rangle = \frac{1}{\sqrt{3}}[|100\rangle + |010\rangle + |001\rangle].$$

There are similar expressions for the ground state for N even. The $|N/2,0\rangle$ state is an equal superposition of all states with an equal number of zeros and ones, with no relative phase between the superposed states. $|N/2,\pm 1\rangle$ is the same type of state with $N/2\pm 1$ ones.

Given the ground state, we are able to calculate the entanglement between any two outer spins in this state (i.e., at zero temperature). The symmetry of the problem implies that the entanglement will be the same between *any* two outer spins. Since H and F_z commute, the only nonzero elements of the reduced density matrix ρ for any two spins are $\langle 00|\rho|00\rangle=v, \langle 01|\rho|01\rangle=w, \langle 10|\rho|10\rangle=x, \langle 11|\rho|11\rangle=y, \langle 01|\rho|10\rangle=z=\langle 10|\rho|01\rangle^*$ [3]. For such density matrices, a measure of entanglement called the concurrence [27] is given by $C=2\max[|z|-\sqrt{vy},0]$ [3]. For N odd, the concurrence comes out as

$$C = 2\max\{1/2N,0\} = 1/N.$$

For the case of N even, where there are two ground states, a similar procedure is followed, except that the reduced density matrix is now described as an equal mixture of the two states. This gives the concurrence as

$$C = 2\max\{1/2N - 1/(2N^2-2N),0\} = 1/N - 1/(N^2-N).$$

Thus the entanglement goes to zero as $N\to\infty$, which is expected, as the entanglement is *mediated* by the central spin. The total entangling capability (and thereby mediating capability) of the central spin is divided among a larger number of outer spins as N becomes larger. However, on going from an even N to an odd $N+1$ number of outer spins, the concurrence rises from $1/N-1/(N^2-N)$ to $1/(N+1)$. Therefore as a consequence of the degeneracy in the ground state for even N, resulting in a mixed density matrix, the concurrence oscillates as N increases with amplitude $2/[N(N-1)(N+1)]$. On application of a magnetic field in the $+Z$ direction, the state $(|0\rangle|N/2,0\rangle-|1\rangle|N/2,-1\rangle)$ becomes the *nondegenerate* ground state for even N and the oscillations in entanglement disappear. Even though the above oscillations in entanglement with N disappear, we will show below that the individual (nondegenerate) eigenstates obtained by application of a magnetic field show curious oscillations with N in a different *type* of entanglement.

The ground state can be used to maximally entangle any two outer spins using the following simple protocol. The spin in the z direction of all spins except the two to be entangled is measured. This procedure stems from the idea of 'localizable entanglement' presented by Verstraete *et al.* in Ref. [22]. Knowing the outcomes of the measurements, it can be determined whether the two spins are maximally entangled. For N odd the probability of successfully obtaining a maximally entangled state is calculated by first considering the ground state in the $\{|0\rangle,|1\rangle\}$ basis.

$$\frac{1}{\sqrt{2}}\left(|0\rangle\frac{1}{\sqrt{^{N}C_{(N+1)/2}}}\left[\sum_{i}^{^{N}C_{(N+1)/2}}\text{Perm}_i\left|\frac{N-1}{2}|0\rangle\text{'s and}\frac{N+1}{2}|1\rangle\text{'s}\right\rangle\right] + |1\rangle\frac{1}{\sqrt{^{N}C_{(N+1)/2}}}\left[\sum_{i}^{^{N}C_{(N+1)/2}}\text{Perm}_i\left|\frac{N+1}{2}|0\rangle\text{'s and}\frac{N-1}{2}|1\rangle\text{'s}\right\rangle\right]\right),$$

where "Perm_i" is used to cycle through all possible kets with the given number of $|0\rangle$ and $|1\rangle$ states for the outer spins. Consider selecting out the two spins which are *not* being measured,

$$\frac{1}{\sqrt{2}}\Bigg(|0\rangle\frac{1}{\sqrt{{}^{N}C_{(N+1)/2}}}\Bigg[(|01\rangle+|10\rangle)\times\Bigg(\sum_{i}^{{}^{N-2}C_{(N-1)/2}}\mathrm{Perm}_i\Bigg|\frac{N-3}{2}|0\rangle\text{ ' s and}\frac{N-1}{2}|1\rangle\text{ ' s}\Bigg\rangle\Bigg)+|00\rangle(\text{other terms})+|11\rangle(\text{other terms})\Bigg]+|1\rangle\frac{1}{\sqrt{{}^{N}C_{(N+1)/2}}}(\text{similar terms})\Bigg).$$

A maximal entangled state is given for an outcome corresponding to each other permutation in the bracket next to $(|01\rangle+|10\rangle)$ in the equation above. There are ${}^{N-2}C_{(N-1)/2}$ such permutations. This is the case whether the central spin is measured to be zero or one. Therefore the probability of success is

$$p_{mes}=2\times 2^{N-2}C_{(N-1)/2}\left(\frac{1}{\sqrt{2}\sqrt{{}^{N}C_{(N+1)/2}}}\right)^2=\frac{1}{2}+\frac{1}{2N}. \quad (5)$$

A similar calculation gives the probability of success for even N (for even N it has been assumed that the degeneracy has been lifted by a B field)

$$p_{mes}=\frac{1}{2}+\frac{1}{2N}-\frac{1}{2N(N-1)} \quad (6)$$

of a successful outcome is plotted in Fig. 2. Thus, using LOCC operations (local operations and classical communications) only, any two outer spins can be maximally entangled with probability greater than 1/2.

Some aspects of the above entangling scheme are noteworthy. First is the fact that maximally entangled states between any pair of outer spins can be produced, which can then be used for perfect quantum communications between the parties holding these spins. The fact that this can be done by a series of single spin measurements in only one basis (namely by measuring spin in the z direction) imparts significant advantages to this mechanism of entanglement distribution over other methods. These will be discussed later in detail. The second fact to note that there are oscillations with even and odd N in the probability of successful creation of a maximally entangled state between two outer spins. As we have already lifted the degeneracy which happens for even N, the oscillations are now related to the nature of entanglement in the individual pure ground states for even and odd N (it cannot be a result of mixing). The third important fact is that the probability of successfully maximally entangling is finite (namely equal to 1/2) even for an infinite collection of outer spins. This again has advantages for the distribution of entanglement.

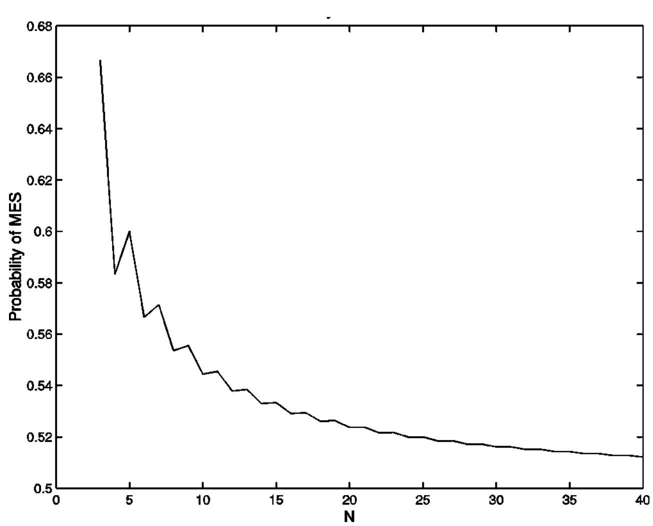

FIG. 2. This figure plots the probability of obtaining an maximally entangled state between two outer spins by projective measurements on all the other spins as N increases.

The correlation functions for the ground state in a star network are also interesting. The $\langle\sigma_{1z}\sigma_{2z}\rangle$ correlations follow the same pattern as the entanglement, but

$$\langle\sigma_{1x}\sigma_{2x}\rangle=\frac{1}{2}+\frac{1}{2N}\quad\text{for odd }N,$$

$$\langle\sigma_{1x}\sigma_{2x}\rangle=\frac{1}{2}+\frac{1}{2N}-\frac{1}{2N(N-1)}\quad\text{for even }N.$$

We note in particular the nonvanishing nature of the correlations in the *large* N limit. The solitary central spin imposes spin order in the X direction (so that $\langle\sigma_{1x}\sigma_{2x}\rangle=1/2$), even when there are an infinite number of outer spins to order. The same result holds for $\langle\sigma_{1y}\sigma_{2y}\rangle$. This straightforward consequence of the interaction with the central spin means that this system provides an effective way of imposing order *simultaneously* in the X and Y directions for an infinite collection of spins. This result also highlights a crucial difference between entanglement and correlations: while a finite dimensional quantum system cannot be individually entangled to each member of an infinite collection of systems, it can indeed be correlated individually to each of them. The nonvanishing aspect of the correlations are also very interesting for a specific reason. In Ref. [22], it has been shown that the highest correlation between two spins is a lower bound on the localizable entanglement obtained from the state. Thus the nonvanishing of $\langle\sigma_{1x}\sigma_{2x}\rangle$ in the $N\to\infty$ limit immediately implies that it should be possible to produce (or localize) an entanglement of magnitude at least $\langle\sigma_{1x}\sigma_{2x}\rangle$ by measuring the spins apart from 1 and 2. This is precisely the procedure that we have described in the two preceding paragraphs.

We now show that the application of a magnetic field allows us to change the ground state to

$$|\alpha\rangle=\frac{1}{\sqrt{2}}(|0\rangle|N/2,-N/2+1\rangle-|1\rangle|N/2,-N/2\rangle)=\frac{1}{\sqrt{2}}(|0\rangle\{|000\cdots1\rangle\}-|1\rangle|000\cdots0\rangle), \quad (7)$$

where $\{|000\cdots1\rangle\}$ is a normalized state that is an equal superposition of all states with only one $|1\rangle$ with no relative phase between the superposed components. This state has the special significance that the concurrence between the central spin and each of the outer spins is $1/\sqrt{N}$, which is the *maximum* consistent with symmetric splitting of the total entanglement of one qubit with a collection of N qubits among the N qubits [23]. To our knowledge, this is the first identification of the canonical state $|\alpha\rangle$ as the ground state of an interacting spin system. Once the ground state $|\alpha\rangle$ is generated, if the central spin is measured and found to be in the state $|0\rangle$, the rest of the spins are projected onto the state $\{|000\cdots1\rangle\}$. The central spin can now be removed to make the state dynamically steady (except for decoherence and spontaneous decay effects). This state has the property that the concurrence between any two spins is $2/N$, which is the *maximum* possible entanglement in a collection of N spins in which all pairs of spins are equally entangled [24]. While in Ref. [5], it was only conjectured that a state of the type $\{|000\cdots1\rangle\}$ could be made the ground state of the isotropic closed Heisenberg chain using a magnetic field, here we will rigorously prove the preparation of $|\alpha\rangle$ and thereby $\{|000\cdots1\rangle\}$.

To show that it is possible to make $|\alpha\rangle$ the ground state, we consider the energy eigenvalues $E = \pm\mathcal{J}\sqrt{(j+m)(j-m+1)}+(m-\frac{1}{2})B$ in a uniform magnetic field B in the $+Z$ direction (in which the eigenstates remain unchanged). When B becomes so high that the second term in E dominates, the relative ordering of the energy levels will be determined purely by m, with the ground state being the state with $m=-N/2$, i.e., $|\beta\rangle=|0\rangle|000\cdots0\rangle$. It is straightforward to show that $|\beta\rangle$ has an energy lower than $|\alpha\rangle$ for $B>\mathcal{J}\sqrt{N}$. Before $|\beta\rangle$ becomes the ground state, there is a range of B in which $|\alpha\rangle$ is the ground state. To prove this, we will have to show that the energy of $|\alpha\rangle$ will be less than that of all other states in a certain range. The value of B for which $|\alpha\rangle$ has the same energy as a general state described by j and m is given by $B=\mathcal{J}[\sqrt{N}-\sqrt{(j+m)(j-m+1)}]/[-(N/2-1)-m]$. B attains its largest value, when j is a maximum and m has it's most negative value. This happens when $j=N/2$ and $m=-(N/2)+2$ for which $B=\mathcal{J}\sqrt{N}[\sqrt{2(1-1/N)}-1]<\mathcal{J}\sqrt{N}$. Therefore, for a magnetic field with a range of $\mathcal{J}\sqrt{N}[\sqrt{2(1-1/N)}-1]<B<\mathcal{J}\sqrt{N}$, the state $|\alpha\rangle$ is the ground state.

We now describe how the spin star can be used to generate and distribute entanglement in a more effective way than the methods usually considered. We have already described how it can be used to generate the states $|\alpha\rangle$ and $|\beta\rangle$ for symmetric splitting and sharing of entanglement. Now we describe the benefits of generating the usual ground state $|\Psi_G\rangle_{\mathrm{Odd/Even1/Even2}}$. The spin star should be regarded as a solid-state template for the generation of these states. The XX interaction between electrons in quantum dots such as in the scheme proposed in Ref. [16] could be used to fabricate the spin–star. The location of the qubits (or spins) on the template are assumed to be *unchangeable* (i.e., the spin star is hard wired on a solid-state matrix) and the interactions between them are assumed to be "collectively" (but not individually) switchable. The desired ground state (among $|\Psi_G\rangle_{\mathrm{Odd/Even1/Even2}}$) is generated by switching on the spin–star interactions, cooling the system to its ground state and applying the appropriate magnetic fields. Having generated the required state, the interactions of the spin star are temporarily switched off, and the state of the outer spins is unitarily transferred to photons. It is assumed that the mapping of the state from the outer spins to the photons can take place at a much shorter time scale than the destruction of the states due to environmental decoherence. Each of these photons now fly off to a distinct distant user. In this way, the state $|\Psi_G\rangle_{\mathrm{Odd/Even1/Even2}}$ is now shared between the central spin and N distant photons, each of which belongs to a distinct user. At this stage, the distribution part of the protocol is assumed to be complete. When any two users want to perform perfect quantum communications using a maximally entangled state, all the other users measure their photons in the $\pm z$ basis, and the central spin is also measured in the $\pm z$ basis. As proved previously in this manuscript [Eqs. (5) and (6)], with a probability $p_{mes}\geqslant 1/2$, the photons of the two users who want to communicate will be projected on to a maximally entangled state. The above scheme is the one we advocate for the generation and distribution of entanglement. Though it may seem quite complicated at the first instance, there are many advantages of this method of generating and distributing entanglement over other, more standard methods. The advantages stem when we make three reasonable assumptions: (a) Any entanglement unused for a significant time will deteriorate due to decoherence and become useless. We assume, without loss of generality, that the entanglement ceases to remain useful after unit time. (b) In unit time only one pair among the ${}^NC_2$ pairs of users genuinely require a maximally entangled state to communicate. It is not imperative that this will indeed be the case in an arbitrary communication scenario, but the spin-star ground states give a genuine advantage in this special case. We should note, however, that in typical telephone networks, far less users use it at a time, than are connected by lines (there could be off-peak hours, for example). (c) We do not know *a priori* which two users would want to communicate. One has to maintain a "flexibility" in the method of distribution of the entangled state, so that any pair of users in the network can use entanglement for quantum communications if they intend to. Under the above assumptions, our method of generating and distributing entanglement is clearly better than the naive way of distributing entanglement, where each user in a network shares a maximally entangled state with every other user. In that way, $O(N^2)$ entangled states would have to be generated and distributed. If only a single pair of parties genuinely require to communicate using entanglement in unit time, then the generation and distribution of such a resource is wasteful (wasteful because the entanglement of the unused ones deteriorate anyway after unit time). This leaves us to compare the method advocated here with other, less naive methods.

When the use of entanglement per unit time by users is infrequent, then the most appropriate alternative way to the one advocated here would be to use entanglement swapping for constructing quantum telephone exchanges [28]. This scheme is shown in Fig. 3(a) for four users. Each user shares a maximally entangled state (shown as solid lines connecting particles) with the central exchange (shown as the box with solid lines). When any two users intend to communicate, an appropriate Bell state measurement is performed between two specific qubits at the exchange. This measurement is shown by the box with dotted line. This projects the qubits possessed by the intended users to a maximally entangled

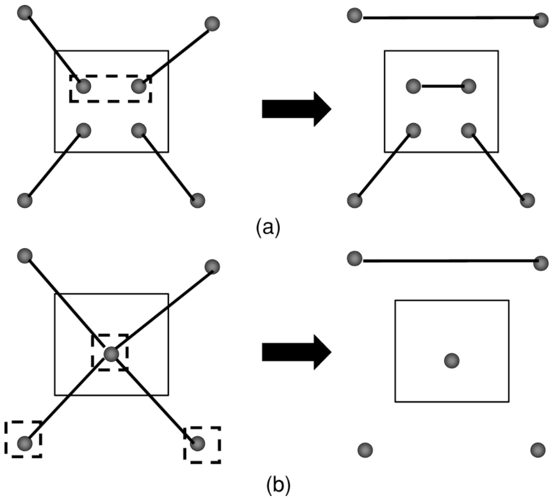

FIG. 3. This figure compares entanglement distribution between pairs of distant users using the scheme of the current paper with a more standard alternate scheme. Part (a) of the figure depicts entanglement distribution using quantum telephone exchanges based on entanglement swapping. Part (b) depicts entanglement distribution using the ground states of the spin star (the scheme proposed in the current paper).

state with unit probability as depicted on the right-hand side of Fig. 3(a). Let us now compare this scheme with our scheme, which is depicted in Fig. 3(b). The state $|\Psi_G\rangle_{\text{Even1/Even2}}$ for $N=4$ is shown in the left-hand part of Eq. (3) as a four edge graph of bold lines connecting particles. The particle at the center is a spin, while those with the distant users are photons. When any two users intend to communicate, the qubits (photons) held by all the other users and the central spin will have to be measured in the $\pm z$ basis. These single-particle measurements are shown by dashed boxes in the left-hand part of Fig. 3(b). These measurements can connect the intended users with a maximally entangled state as depicted in the right-hand part of Fig. 3(b) (with a significant probability of success). The advantages of our scheme are: (a) Only single-particle measurements are involved, as opposed to Bell state measurements of the alternative method, where two particle measurements are involved. (b) Only measurements in a specific (namely $\pm z$) basis are involved, and as a consequence, there is no need for local rotations of the states of the qubits. Such rotations are compulsory for the Bell state measurement of the alternative scheme. (c) The state $|\Psi_G\rangle_{\text{Even1/Even2}}$ is generated by collective switching of all the interactions in the spin star at the center. The Bell state measurement of the alternative scheme requires switching the interaction between specific pairs of spins at the center without switching the interaction between any of the other pair of spins. This can be very difficult in practice because all the qubits at the central exchange would be quite close to each other. For a group of spins close to each other, it is always easier to apply a global external field to switch on all interactions at once. In particular, we assume an easy to realize situation where the position of the spins are frozen in a solid-state matrix and the interactions between them are hard wired. It is very easy to realize a spin-star geometry under such a situation and then switch all interactions in the Hamiltonian given by Eq. (1) on at once by applying a global field to the matrix. In the same type of physical system, it is very difficult to switch on any specific pair of interactions without switching on any of the others (this would require, for example, an external field which addresses that specific pair while avoiding the others). Thus the alternative scheme, which requires a Bell state measurement between an arbitrary pair of spins at the central exchange is considerably harder than the method advocated by us in this paper. We should, however, note that the probability of success in our case is $1/2 \leq p_{mes} < 1$. But this is a significant probability of success, and if one attempts to create the maximally entangled state between the intended users seven times (with a newly prepared $|\Psi_G\rangle_{\text{Odd/Even1/Even2}}$ state), one's probability of failure is already lower than 0.01.

At this point, we would also like to point out the general advantages of generating a certain entangled state as the ground state of a specific Hamiltonian as opposed to *dynamically* generating it using a sequence of qubits to interact in turn with each other. The main difficulty of the latter method is the carefully timed switching on and off of interactions (which may boil down to very carefully timed pulses on the system). When generating certain states as ground states of a system, one need not precisely time the switching on and off of interactions. One simply turns on the interactions at some time and cools the system to its ground state. Of course, if part of the state is to be carried off to a distance, then the mapping of states from spins to photons might require timed pulses. This will, however, be a common problem for any method which generates entangled states at a site and then distributes parts of it to a distance. Our method also, of course, allows the possibility of generating entanglement directly between distant parties, if interactions between the systems possessed by them can be made truly long range. Alternatively, if we require entanglement to be shared between parties which are physically close (particles possessed by them are well with in the range of interacting directly with each other), such as when neighboring quantum computers are to be networked, then the ground state of the spin star can be easily generated between them. Subsequent measurements can then connect any two of the computers with a maximally entangled state.

In this letter we have introduced and studied entanglement and correlations in a spin-star, an architecture of interacting spins which *cannot* be classified as a lattice in any dimension. It is physically realizable in various arrays of qubits designed for quantum computation. Testing for the entanglement and correlations predicted here would serve as a benchmark test for the functioning of arrays of qubits. Our spin star is a curious example of a spin system where multiparty states for optimal symmetric sharing and splitting of entanglement occur naturally as ground states. If spin–spin interactions can be extended to long distances, the spin star could be used for the distribution of entanglement. Exploration of the full statistical mechanics of a spin star would be interesting future work.

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[1] H. A. Bethe, Z. Phys. **71**, 205 (1931); J. Bonner and M. E. Fisher, Phys. Rev. **135**, A640 (1964); B. Kaufman and L. Onsager, Phys. Rev. **76**, 1244 (1949).

[2] E. H. Lieb, T. D. Schultz, and D. C. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).

[3] K. M. O'Conner and W. K. Wootters, Phys. Rev. A **63**, 052302 (2001).

[4] M. A. Nielsen, Ph.D thesis, University of New Mexico, 1998 (unpublished).

[5] M. C. Arnesen, S. Bose, and V. Vedral, Phys. Rev. Lett. **87**, 017901 (2001).

[6] X. Wang, Phys. Rev. A **64**, 012313 (2001); Phys. Lett. A **281**, 101 (2001); X. Wang, H. Fu, and A. I. Solomon, J. Phys. A **35**, 4293 (2002).

[7] D. Gunlycke, V. Kendon, V. Vedral, and S. Bose, Phys. Rev. A **64**, 042302 (2001).

[8] G. L. Kamta and A. F. Starace, Phys. Rev. Lett. **88**, 107901 (2002).

[9] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002); T. J. Osborne and M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).

[10] X. Wang, Phys. Rev. A **66**, 034302 (2002); X. Wang and P. Zanardi, Phys. Lett. A **301**, 1 (2002).

[11] I. Bose and E. Chattopadhyay, Phys. Rev. A **66**, 062320 (2002).

[12] A. Lakshminarayan and V. Subrahmanyam, Phys. Rev. A **67**, 052304 (2003).

[13] K. Audenaert *et al.*, Phys. Rev. A **66**, 042327 (2002).

[14] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).

[15] J. I. Cirac and P. Zoller, Nature (London) **404**, 579 (2000); B. E. Kane, *ibid.* **393**, 133 (1998); Y. Makhlin and G. Schön, *ibid.* **398**, 305 (1999).

[16] A. Imamoḡlu *et al.*, Phys. Rev. Lett. **83**, 4204 (1999).

[17] S. B. Zheng and G. C. Guo, Phys. Rev. Lett. **85**, 2392 (2000).

[18] X. Zhou *et al.*, Phys. Rev. Lett. **89**, 197903 (2002); S. C. Benjamin and S. Bose, *ibid.* **90**, 247901 (2003).

[19] T. Eggeling, R. F. Werner, and M. M. Wolf (unpublished).

[20] S. Mancini and S. Bose, e-print quant-ph/0111055.

[21] M. M. Wolf, F. Verstraete, and J. I. Cirac, e-print quant-ph/0307060.

[22] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. **92**, 027901 (2004).

[23] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000); W. Dür, *ibid.* **63**, 020303 (2001).

[24] M. Koashi, V. Buzek, and N. Imoto, Phys. Rev. A **62**, 050302 (2000).

[25] B. W. Shore and P. L. Knight, J. Mod. Opt. **40**, 1195 (1993).

[26] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[27] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[28] S. Bose, V. Vedral, and P. L. Knight, Phys. Rev. A **57**, 822 (1998).