Entropic uncertainty relations for the ground state of a coupled system

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There is a renewed interest in the uncertainty principle, reformulated from the information theoretic point of view, called the entropic uncertainty relations. They have been studied for various integrable systems as a function of their quantum numbers. In this work, focussing on the ground state of a nonlinear, coupled Hamiltonian system, we show that approximate eigenstates can be constructed within the framework of adiabatic theory. Using the adiabatic eigenstates, we estimate the information entropies and their sum as a function of the nonlinearity parameter. We also briefly look at the information entropies for the highly excited states in the system.

I. INTRODUCTION

The uncertainty relations, which express the inability to simultaneously measure the states of two noncommuting observables, form the cornerstone of quantum physics. For any pair of operators $\hat{A}$ and $\hat{B}$, in standard form [1], they are stated as

$$\Delta A \Delta B \geq \frac{1}{2} \langle \Psi | [\hat{A}, \hat{B}] | \Psi \rangle, \tag{1}$$

where ΔA and ΔB represent the dispersions in $\hat{A}$ and $\hat{B}$ and $[\hat{A}, \hat{B}]$ is the commutator. In recent years, there has been a revival of interest in the uncertainty relations reformulated from the standpoint of information theory, called the entropic uncertainty relations (EUR) [2]. For instance, the position-momentum uncertainty relation is formulated as follows; given an eigenstate of a quantum system, $\psi(q)$ and $\widetilde{\psi}(p)$, in configuration and momentum space representations, and if S_q and S_p represent their information entropies, then the entropic uncertainty relations can be written down as

$$S_q + S_p \geq S_{qp}, \tag{2}$$

where S_{qp} is the lower bound to the entropic sum or EUR. Here, the information entropy is defined as

$$S_{\vec{\xi}} = - \int_{-\infty}^{\infty} \rho(\vec{\xi}) \ln \rho(\vec{\xi}) d\vec{\xi}, \tag{3}$$

where $\rho(\vec{\xi}) = |\psi(\vec{\xi})|^2$ is the probability density. The information entropy is a measure of the spreading or localization of the given eigenstate. Apart from its intrinsic value, the reformulation also seeks to address some of the shortcomings in the standard statement of the uncertainty principle [2]. It also quantifies the uncertainty more accurately than the standard statement based on dispersions [3]. In particular, a good amount of work has focussed on obtaining the lower bounds S_{qp} in general and we mention the result obtained by Bialynicki-Birula and Mycielski [4],

$$S_q + S_p \geq D(1 + \ln \pi), \tag{4}$$

where D is the number of spatial dimensions of the system. Inspired by this general result, several authors have focussed on obtaining the lower bounds for EUR of quantum systems whose classical limit is integrable, like the particle in an infinite well [5], harmonic oscillator and hydrogen atom in one and higher dimensions [6], power-law wave packets and multivariate Cauchy-Lorentz distribution [7], and oscillating circular membrane [8]. Apart from the lower bounds, the values of EUR as a function of quantum numbers have been analyzed in detail in these series of papers. Recently, Dehesa *et al.* have shown that for the one-dimensional power-law potentials of the form $V(x) = x^{2k}$, where k is a positive integer, in the region of highly excited states ($n \gg 1$), the entropic sum goes as $\ln(2n)$ for all k [9].

On the other hand, the single-particle probability density $\rho(\vec{\xi})$ is also the quantity of fundamental interest in the density-functional theory and hence its characterization using information entropy as a measure for spreading has assumed special interest. In fact, treating atomic and molecular entropic sum was considered by Gadre [10] and he derived an approximate expression for the entropic sum within the Thomas-Fermi framework for neutral atoms. Now it is known from several empirical studies for atomic, molecular, and nuclear distributions that the entropic sum can be modeled as [11]

$$S \equiv c_1 + c_2 \ln N, \tag{5}$$

where c_1 and c_2 are constants and N is the number of electrons or nucleons, as the case may be. The functional form given above seems to be fairly universal for many-fermions in some mean interactions [11].

Thus, one branch of the work on EUR has focussed on quantum systems in the classically integrable limit whose eigenstates are analytically known. The other complementary branch has explored the complex atomic and molecular systems using a combination of approximate analytical and empirical methods. In these cases, the EURs have been obtained as a function of increasing quantum numbers. However, simple and chaotic model systems that bridge this divide
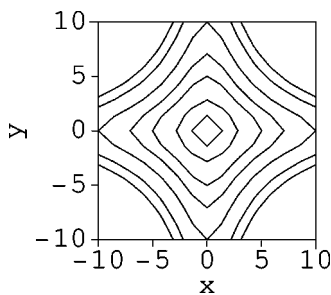
FIG. 1. The contours of the potential as a function of x and y (in arb. units) for $\alpha=0.2$ and $k_1, k_2=1$. Each contour represents a particular value of energy. For large values of energy, the potential develops "channels", as seen in the figure.

between the purely integrable and the complex many-body systems have not yet been considered. The main reason seems to be that, as yet, no straightforward analytical technique is available to determine their eigenstates. Thus, the main purpose of this paper is to show that using adiabatic technique approximate eigenstates can be constructed and further that the entropic sums can also be estimated. In contrast to earlier works, we study the EUR as a function of the nonlinearity parameter in the system. Thus, we hope to understand the effect of nonlinearity using information theoretic measures.

In general, the nonintegrable Hamiltonian systems are generic rather than an exception and model realistic physical systems. Their phase space presents a mixture of regular and irregular trajectories, whose properties change with a parameter. The evolution of the probability density as a function of the parameter reflects the influence of these classical objects. Thus it is important to look at the EUR upon variation of the nonlinearity parameter. Note that scale invariance of entropic sums leads to EURs that are independent of scaling parameters in an integrable system. But, in a nonintegrable system, the entropic sums could depend on the nonlinearity parameter.

II. MODEL HAMILTONIAN

In this work, we will consider the model Hamiltonian,

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \frac{k_1^2 x^2}{2} + \frac{k_2^2 y^2}{2} + \alpha x^2 y^2, \qquad (6)$$

whose potential is displayed in Fig. 1 and k_1, k_2, α are the parameters. The system is integrable for $\alpha=0$ and corresponds to a two-dimensional harmonic oscillator whose asymptotic (large quantum number) entropic sum was recently derived [6]. For a choice of $\alpha=0.05$ ($k_1=k_2=1$), the system displays classical chaos for energy $E>15$ [12]. The coupled oscillators are popular as models of chaos since they qualitatively mimic the complex dynamics in problems like the hydrogen atom in strong fields and generalized van der Waals potential [13]. For a generic chaotic system, most of its eigenstates could be thought of as random waves except for a small subset of "regular" states that are influenced by the nature of local classical dynam-

ics. Hence in this work, we will only focus attention, for most part, on the ground state and, briefly, on the regular states. We stress that even though the system is chaotic, we study only those eigenstates that are associated locally with regular classical structures. Ground state is preeminent because previous studies have shown that the ground state saturates the EUR inequality in some systems (e.g., harmonic oscillator) and often in complex systems, the focus is on the ground-state properties. Next, we obtain the ground-state entropies, as a function of α.

III. ADIABATIC THEORY

In the context of two degrees of freedom systems, if the frequency of oscillations between both the degrees of freedom differ vastly, then such a classical system becomes an ideal candidate for the adiabatic treatment. This is the well-known Born-Oppenheimer approximation in atomic physics [14]. The approach is to average over the "faster" degree of freedom and incorporate it into the Hamiltonian for the slower degree of freedom. In this adiabatic approach, the Hamiltonian is effectively integrable and can be quantized. This has been successfully implemented earlier to estimate the energies of the localized states in coupled oscillator systems [12,15]. For the Hamiltonian in Eq. (6), Certain and Moiseyev construct adiabatic eigenstates formally [12]. Prange et al. construct adiabatic eigenstates for a class of chaotic billiards [16]. For the ground state of Eq. (6), for $k_1=k_2=1$, there are no clearly identifiable "faster" and "slower" degrees of freedom and thus it might seem that the adiabatic approach is inappropriate. However, previous studies show that the adiabatic separation of variables, rather surprisingly, provides a good estimate for even the ground-state spectra [15]. The results in this paper confirm these previous findings but as yet a physical explanation is not clear.

A. Position eigenstates

In this section, we will obtain the ground state of the Hamiltonian in Eq. (6), for $\alpha \ll 1$, in the adiabatic approximation, closely following the technique in Refs. [12,15]. We assume that the y motion is faster and average over the fast motion by dropping terms that depend purely on x and p_x. This gives

$$H'(x;y) = \frac{p_y^2}{2} + \frac{k_2^2 y^2}{2} + \alpha x^2 y^2 = E'(y), \qquad (7)$$

where the variable x is treated as a parameter. This is just the harmonic oscillator whose frequency is

$$\Omega_x = \sqrt{k_2^2 + 2\alpha x^2} \qquad (8)$$

and for $\alpha \ll 1$, such that $k_2^2 \gg 2\alpha x^2$, then

$$\Omega_x \simeq \omega_x = k_2 + \frac{\alpha}{k_2} x^2. \qquad (9)$$

The ground state of $H'(x;y)$ for $\alpha \ll 1$ is

$$\phi(x;y) = \left(\frac{\omega_x}{\pi}\right)^{1/4} \exp(-\omega_x y^2/2). \qquad (10)$$

Now, the adiabatic Hamiltonian is obtained as

$$H_{ad} = \frac{p_x^2}{2} + \frac{k_1^2 x^2}{2} + \omega_x J_y, \qquad (11)$$

where J_y is the classical action corresponding to the faster motion. In terms of action-angle coordinates, we get

$$H_{ad} = J_x \sqrt{k_1^2 + 2\alpha J_y/k_2} + k_2 J_y. \qquad (12)$$

Note that semiclassical quantization of H_{ad}, which is also exact in this case, gives for the ground-state energy,

$$E_{ad} = \frac{1}{2}\sqrt{k_1^2 + \frac{\alpha}{k_2}} + \frac{k_2}{2}. \qquad (13)$$

The energy estimated by this formula is in good agreement with the computed energies for small α. Since we are interested only in the eigenstate, the last term in Eq. (12) can be omitted since it only serves to shift the energy scale. The ground state of H_{ad} is

$$\psi(x) = \left(\frac{\omega}{\pi}\right)^{1/4} \exp(-\omega x^2/2), \qquad (14)$$

where $\omega = \sqrt{k_1^2 + \alpha/k_2}$. The ground state of the Hamiltonian in Eq. (6), for $\alpha \ll 1$, in the adiabatic approximation, can be written down in its characteristic form as $\Psi(x,y) = \phi(x;y)\psi(x)$, so that,

$$\Psi(x,y) = \frac{(\omega\omega_x)^{1/4}}{\sqrt{\pi}} e^{-\omega_x y^2/2} e^{-\omega x^2/2}. \qquad (15)$$

Note that $\Psi(x,y)$ is a coupled eigenstate; the coupling provided by ω_x. It is correctly normalized. For $\alpha = 0$ it reduces to a two-dimensional harmonic oscillator wave function. The price we pay in the adiabatic approximation is that the eigenstate is not symmetric under $x \leftrightarrow y$ as required by the potential.

B. Momentum eigenstate

We calculate the eigenstate in the momentum representation by taking Fourier transform of the eigenstate given in Eq. (15). We perform the following integral:

$$\widetilde{\Psi}(p_x,p_y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(x,y)e^{ip_x x}e^{ip_y y}dx\,dy. \qquad (16)$$

By a straightforward transformation of variables, we can reduce it to an one-dimensional integral,

$$\widetilde{\Psi}(p_x,p_y) = \frac{\omega^{1/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \frac{e^{-\omega x^2/2}e^{-p_y^2/2\omega_x}e^{ip_x x}}{\omega_x^{1/4}}. \qquad (17)$$

At this point, in order to be able to do the integration, we approximate the integrand for $\alpha \ll 1$. We take

$$\exp\left(\frac{-p_y^2}{2\omega_x}\right) \simeq \exp\left(\frac{-p_y^2[1-(\alpha/k_2^2)]x^2}{2k_2}\right),$$

$$\omega_x^{-1/4} \simeq \frac{1}{k_2}\exp\left(-\frac{\alpha}{4k_2^2}x^2\right). \qquad (18)$$

Then, the integral reduces to a Fourier transform of a Gaussian,

$$\widetilde{\Psi}(p_x,p_y) = \frac{\omega^{1/4}}{\sqrt{2\pi}}\exp\left(\frac{-p_y^2}{2k_2}\right)\int_{-\infty}^{\infty} f(x)e^{ip_x x}\,dx, \qquad (19)$$

where

$$f(x) = \exp\left(\frac{-x^2}{2}\omega_p\right), \quad \omega_p = \beta - \frac{\alpha p_y^2}{k_2^3} \qquad (20)$$

and $\beta = \omega + \alpha/2k_2^2$. This integral can be performed and we obtain the momentum eigenstate as

$$\widetilde{\Psi}(p_x,p_y) = \frac{1}{\sqrt{\pi}}\left(\frac{\omega}{k_2}\right)^{1/4}\frac{\exp\left(-\frac{p_y^2}{2k_2}\right)\exp\left(-\frac{p_x^2}{2\omega_p}\right)}{\sqrt{\omega_p}}. \qquad (21)$$

This eigenstate is also of the product form characteristic of the adiabatic approximation, $\widetilde{\Psi}(p_x,p_y) = \chi_1(p_x;p_y)\chi_2(p_y)$, where χ_1 and χ_2 represent the eigenstates of fast and slow degrees of freedom. In contrast to the position eigenstate in Eq. (15), the ground state in momentum representation is normalized only to $O(\alpha)$. This is due to the approximations that were done in the course of obtaining the Fourier transform. As a limiting case, for $\alpha = 0$, it reproduces the two-dimensional (2D) momentum eigenstate.

IV. GROUND-STATE ENTROPY

A. Position entropy

Now it is possible to evaluate the information entropies with the eigenstates in Eqs. (15) and (21). The position density, from Eq. (15), is given by

$$\rho(x,y) = |\Psi(x,y)|^2 = \frac{\sqrt{\omega\omega_x}}{\pi}e^{-\omega_x y^2}e^{-\omega x^2}. \qquad (22)$$

Then, the information entropy can be evaluated as

$$S_q(\alpha,k_1,k_2) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(x,y)\ln \rho(x,y)dx\,dy$$

$$= 1 + \ln \pi - \ln \sqrt{\omega} - \frac{I_1}{\sqrt{\pi}}, \qquad (23)$$

where I_1 is the integral given by

$$I_1 = \int_{-\infty}^{\infty} e^{-x^2}\ln\left(k_2 + \frac{\alpha}{\omega k_2^2}x^2\right)dx. \qquad (24)$$

Though this integral can be exactly done, we expand the integrand to first order in α and arrive at the following expression for the entropy valid for $\alpha \ll 1$,
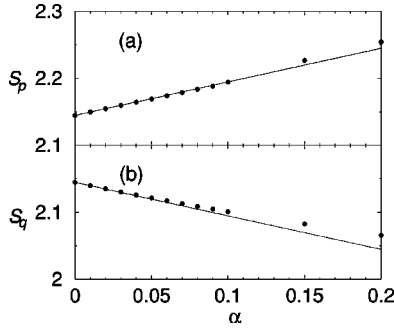
FIG. 2. Entropy for the ground state of the Hamiltonian in Eq. (6) with $k_1=k_2=1$ as a function of α. (a) Momentum entropy and (b) spatial entropy. Circles are numerical values of information entropies obtained without any approximation. Solid lines are theoretical curves of Eqs. (25) and (28).

$$S_q(\alpha,k_1,k_2) = 1 + \ln \, \pi - \ln \, \sqrt{k_1 k_2} - \frac{\alpha}{4}\frac{k_1+k_2}{k_1^2 k_2^2} + O(\alpha^2).$$

$$(25)$$

The spatial entropy decreases linearly with α. Setting $\alpha=0$, we recover the correct two-dimensional harmonic oscillator entropy. In Fig. 2(b) we show the spatial entropy ($k_1=k_2=1$) as a function of α. For comparison, we numerically estimate the entropies by diagonalizing the Hamiltonian in Eq. (6) in harmonic-oscillator basis set and a subsequent entropy calculation. There is a good agreement between the numerical results (dots) obtained without any approximation and the analytical formula (solid line). A linear fit to the numerical entropies gives a slope of -0.4533, not far from the theoretical result -0.5. The fitted slope closely approximates -0.5 as $\alpha \rightarrow 0$.

B. Momentum entropy

The momentum density $\widetilde{\rho}(p_x,p_y)$ from Eq. (21) is given by

$$|\widetilde{\Psi}(p_x,p_y)|^2 = \frac{1}{\pi}\sqrt{\frac{\omega}{k_2}} \frac{\exp\left(-\frac{p_y^2}{k_2}\right)\exp\left(-\frac{p_x^2}{\omega_p}\right)}{\omega_p}.$$

$$(26)$$

Once again, an entropy integral similar to Eq. (23) can be performed by Taylor expanding the denominator in the integrand to $O(\alpha)$. The final, rather cumbersome, expression turns out to be

$$S_p(\alpha,k_1,k_2) = \sqrt{\frac{\omega/\beta}{(1-c)}}\left[\frac{1}{2} - \ln\left(\frac{\sqrt{\omega/k_2}}{\pi \beta}\right) + \frac{(1-2c)}{2(1-c)}\right],$$

$$(27)$$

where we use the shorthand, $c=\alpha/2\beta k_2^2$. As a limiting case, we get for $\alpha=0$, the correct momentum entropy of 2D harmonic oscillator. To obtain an explicit expression, we expand to $O(\alpha)$ all the terms in the above equation. The result valid for $\alpha \ll 1$ is
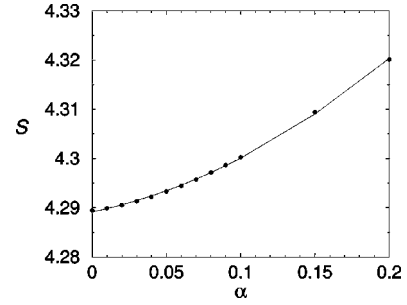

FIG. 3. Entropy sum for the ground state of the Hamiltonian in Eq. (6) with $k_1=k_2=1$ as a function of α. Circles are numerical values and solid line is the quadratic fit to the numerical data.

$$S_p(\alpha,k_1,k_2) = 1 + \ln \, \pi + \ln \, \sqrt{k_1 k_2} + \frac{\alpha}{4}\frac{k_1+k_2}{k_1^2 k_2^2} + O(\alpha^2).$$

$$(28)$$

This expression shows that the momentum entropy increases linearly with α. This is evident from Fig. 2(a) in which a good agreement is seen between the numerical and the analytical result. A linear fit gives the slope to be 0.4765, close to the expected value 0.5.

C. Entropic sum

From the entropy expressions in Eqs. (25) and (28), we obtain the sum of position and momentum entropies as

$$S(\alpha) = S_q + S_p = 2(1 + \ln \, \pi) + O(\alpha^2). \qquad (29)$$

We point some salient features of this result. To $O(\alpha)$, the entropic sum is invariant under change of α, but definitely satisfies the uncertainty inequality in Eq. (4). As expected, due to the scale invariance, it is also independent of the parameters k_1 and k_2. However, as revealed by numerical computations in Fig. 3, the significant contribution to entropic sum comes from terms of $O(\alpha^2)$, which could not be determined from the present method. The important result is that, for $\alpha \ll 1$, the entropies and their sum depend only on the parameter α, in fact on terms of $O(\alpha^2)$ or higher. Since our method does not yield information on the quadratic and higher-order α terms, we perform a fit to the numerical entropic sum. The equation of best fit, for the range of α considered, gives $S=4.289+0.063\alpha+0.466\alpha^2$. As expected, the fitted equation shows that the linear term in α carries less weight and the intercept is a good approximation to $2(1+\ln \, \pi)$. The ground state saturates the entropic sum only for $\alpha=0$.

The role of α, primarily, is to change the shape of the potential which leads to other qualitative changes in the classical dynamics of the system. As α increases, the potential develops "channels" (see Fig. 1) but the ground state does not occupy the channels. The potential restricts the ground state to remain within the central region and not expand into the channels of the potential. Hence the spatial entropy decreases with increase in α indicating stronger localization of the "particle" near the origin of the potential. Correspondingly, the momentum entropy increases. But they do not can-

cel one another exactly, and the increase in entropic sum is of the order $O(\alpha^2)$. Thus, it is a rather slow but definite increase, as evident from Fig. 3.

V. LARGE PARAMETER

We note that the limit $\alpha \to \infty$ is a singular limit for the Hamiltonian in Eq. (6). It does not seem feasible to extend the present method to this limit. Hence we look at $\alpha \gg 1$, but not $\alpha \to \infty$.

A. Position eigenstate

For $\alpha \gg 1$, we cannot Taylor expand as done in Eq. (9). Hence, now the adiabatic Hamiltonian will be

$$H_{ad} = \frac{p_x^2}{2} + \frac{k_1^2 x^2}{2} + \Omega_x J_y. \tag{30}$$

Presently, exact solutions are not known for this problem. To go further, we quantize the y degree of freedom exactly and treat the x degree of freedom using the variational method. This leads to

$$H_{ad} = \frac{p_x^2}{2} + \frac{k_1^2 x^2}{2} + \frac{1}{2}\sqrt{k_2^2 + 2\alpha x^2}, \tag{31}$$

where Ω_x is substituted from Eq. (8). Now, we will use a variational method to obtain the ground state. The trial wave function of the form

$$\psi_{tr}(x) = \left(\frac{b}{\pi}\right)^{1/4} \exp(-bx^2/2) \tag{32}$$

is assumed. Note that the trial wave function is similar to the adiabatic ground state in Eq. (14), except for ω being replaced by the variational parameter b, to be determined. Since we are applying the standard variational technique as expounded in many quantum-physics books, we refer the reader to, say Ref. [17], for details of the variational method. We first determine the energy functional,

$$E_v = \langle \psi_{tr}(x)|H_{ad}|\psi_{tr}(x)\rangle \tag{33}$$

$$= \frac{b}{4} + \frac{k_1^2}{4b} + \sqrt{\frac{\alpha}{2b}} \ U(-1/2, 0, bk_2^2/2\alpha), \tag{34}$$

where $U(,,)$ is the confluent Hypergeometric function. Now, we optimize E_v to obtain b. Hence,

$$\frac{dE_v}{db} = \frac{1}{4} - \frac{k_1^2}{4b^2} + \frac{e^{bk_2^2/4\alpha}k_2^2 K_0(bk_2^2/4\alpha)}{4\sqrt{2b\pi\alpha}}$$

$$- \frac{\sqrt{\alpha}}{2\sqrt{2}b^{3/2}}U(-1/2, 0, bk_2^2/2\alpha), \tag{35}$$

where $K_0(\cdot)$ is the modified Bessel function of second kind.

Now, we solve for b by setting $dE_v/db = 0$. Once again, solving for b in a general case does not seem feasible. However, in the limit $\alpha \gg 1$, we use the asymptotic forms as $z = (bk_2^2/2\alpha) \to 0$ [18],

$$U\left(\frac{-1}{2}, 0, z\right) \sim \frac{1}{\sqrt{\pi}}\left\{1 + \frac{z}{2}\left[\ln z + \Psi\left(\frac{1}{2}\right) + 2\gamma - 1\right]\right\},$$

$$K_0(z/2) \sim -\ln(z/2), \tag{36}$$

where $\gamma = 0.5772...$ is the Euler's constant and $\Psi(\cdot)$ is the digamma function. We use these asymptotic forms and ignore terms that contain α in the denominator. Then, we obtain,

$$\frac{dE_v}{db} \simeq \frac{1}{4} - \frac{k_1^2}{4b^2} - \frac{\sqrt{\alpha}}{2\sqrt{2\pi}b^{3/2}} = 0. \tag{37}$$

If the terms are rearranged, this turns out to be a quartic algebraic equation in b.

$$b^4 - 2k_1^2 b^2 - \frac{2\alpha b}{\pi} + k_1^4 = 0. \tag{38}$$

This can be exactly solved for b but leads to rather complicated terms. As an aside, we also point out that using asymptotic forms for $U(\cdot)$ and $K_0(\cdot)$ as $z \to \infty$, i.e., $\alpha \to 0$, we can recover the position eigenstate obtained in Eq. (15).

From this point, we specialize to our case $k_1 = k_2 = 1$. Then, for $\alpha \gg 1$, the second term in Eq. (38) can be neglected since b increases monotonically with increasing α. Now, setting $dE_v/db = 0$, we obtain for $\alpha \gg 1$,

$$b \approx \left(\frac{2\alpha}{\pi}\right)^{1/3}. \tag{39}$$

In general, b is dependent on k_1 and k_2 as evident from the exact solution to Eq. (38) but we have taken $k_1 = k_2 = 1$. Plugging in this b in Eq. (32), we obtain the adiabatic ground state valid for $\alpha \gg 1$ as

$$\Psi(x,y) = \frac{(b\Omega_x)^{1/4}}{\sqrt{\pi}}e^{-\Omega_x y^2/2}e^{-bx^2/2}, \tag{40}$$

where $\Omega_x = \sqrt{1 + 2\alpha x^2}$ since we have taken $k_1 = k_2 = 1$ to make the analysis tractable. It is correctly normalized but does not give the correct limit for $\alpha = 0$. It does not possess the symmetry $x \leftrightarrow y$.

1. Position entropy for $\alpha \gg 1$

The probability density in position representation is

$$\rho(x,y) = |\Psi(x,y)|^2 = \frac{\sqrt{b\Omega_x}}{\pi}e^{-\Omega_x y^2}e^{-bx^2}. \tag{41}$$

From this, the information entropy with $k_1 = k_2 = 1$ can be evaluated [see Eq. (23)] as

$$S_q(\alpha) = 1 + \ln \pi - \ln \sqrt{b} - \frac{I_2}{\sqrt{\pi}}, \tag{42}$$
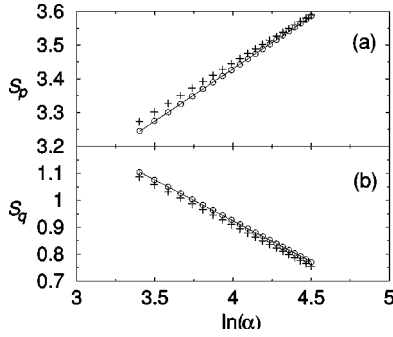
where I_2 is given by

FIG. 4. (a) Entropy in momentum representation and in (b) spatial representation for the Hamiltonian in Eq. (6) with $k_1=k_2=1$ as a function of α ranging from 30 to 90. Circles are the numerical entropies and solid line is the best fit to the numerical data. Crosses represent the theoretical result based on the adiabatic approach.

$$I_2 = \int_{-\infty}^{\infty} e^{-x^2} \ln\left(1 + \frac{2\alpha}{b}x^2\right) dx$$

$$= \sqrt{\pi}\left[\gamma + \pi\, \text{erfi}\left(\sqrt{\frac{b}{2\alpha}}\right) + \ln\left(\frac{\alpha}{2b}\right) - \frac{b}{\alpha}\right.$$

$$\left. \times {}_2F_2\left(1,1;\frac{3}{2},2;\frac{b}{2\alpha}\right)\right] \qquad (43)$$

and $\text{erfi}(\cdot)$ is the imaginary error function and ${}_2F_2(\cdot)$ is the generalized hypergeometric function. In this form, the information entropy in Eq. (42) is not particularly illuminating. For $\alpha \gg 1$, the contributions from second and fourth terms in Eq. (43) are negligible. Thus, we obtain a simplified form for the spatial entropy,

$$S_q(\alpha) = 1 + \ln\,\pi - \ln\,\sqrt{b} + \frac{\gamma}{4} - \frac{1}{4}\ln\left(\frac{\alpha}{2b}\right). \qquad (44)$$

Now, substituting for b from Eq. (38), we obtain

$$S_q(\alpha) = 1 + \ln\,\pi + \frac{\gamma}{4} - \frac{1}{3}\ln\left(\frac{\pi}{2}\right) - \frac{1}{3}\ln\,\alpha. \qquad (45)$$

Thus, for large α, the ground-state entropy falls logarithmically with a slope $1/3$. This is verified numerically through exact calculations. In Fig. 4(b), we show the position entropy plotted as a function of $\ln(\alpha)$. A linear regression gives the best fit line as $S_q = 2.1368 - 0.3033\ln\,\alpha$, verifying the approximate theoretical slope in Eq. (45). The intercept is approximately $1 + \ln\,\pi$, the entropy of the unperturbed oscillator, but we stress that we cannot recover $\alpha=0$ result from Eq. (45). We also point out a systematic difference seen in Figs. 4(a) and 4(b) between the theoretical estimate of entropy (crosses) given by Eq. (45) and the numerical result (circles). This is due to the adiabatic approximation becoming less accurate near the origin of the potential and this is showing up in the results.

B. Momentum eigenstate and entropy

We obtain adiabatic ground state in the momentum representation for $\alpha \gg 1$ by Fourier transforming the position
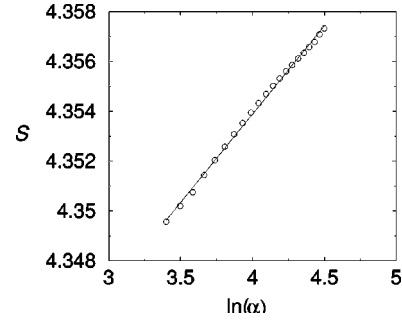

FIG. 5. (a) Entropy sum for the ground state of the Hamiltonian in Eq. (6) with $k_1=k_2=1$ as a function of α. Circles are numerical values and solid line is the best fit to the numerical data as given by Eq. (48).

eigenstate in Eq. (40). We need to perform the integral in Eq. (16) using the ground state in Eq. (40). Once again, by transforming variables, we can reduce it to a one-dimensional integral,

$$\widetilde{\Psi}(p_x,p_y) = \frac{b^{1/4}}{\sqrt{2}\pi}\int_{-\infty}^{\infty} dx \frac{e^{-bx^2/2}e^{-p_y^2/2\Omega_x}e^{ip_x x}}{\Omega_x^{1/4}}, \qquad (46)$$

where Ω_x is given by Eq. (8) and b is the variational parameter determined in Eq. (38). This integral could only be performed numerically. Hence, we first determine the adiabatic momentum eigenstate by numerically integrating Eq. (46) from which the entropies are computed. The entropies obtained numerically by the adiabatic approach, shown as crosses in Fig. 4(a), and those obtained numerically without any approximation [circles in Fig. 4(a)] are in fair agreement with each other for the range of α considered in this work. A linear regression gives the line of best fit,

$$S_p(\alpha) = 2.1754 + 0.3134\ln\,\alpha. \qquad (47)$$

Momentum entropy increases logarithmically with α. We notice from our numerical calculations, exact as well as the one based on adiabatic approach, that the slope in the equation of best fit for the momentum entropy is consistently higher than the one for the position entropy. This difference accounts for the behavior of the entropic sum presented in the following section.

C. Entropic sum for large α

Finally, we put together the results to look at the entropic sum for large α. Figure 5 shows the entropic sum for the ground state for a range of parameters $\alpha=30,\ldots,90$. The best-fit line (solid line) is given by

$$S = 4.3249 + 0.007\,08\ln\,\alpha. \qquad (48)$$

First, it satisfies the entropic uncertainty inequality in Eq. (4). The positive slope in this equation shows that the entropic sum increases with the parameter but rather slowly. It must also be pointed out that this slope in Eq. (48) corresponds closely to the difference in slopes corresponding to the best-fit equations for position and momentum entropy.
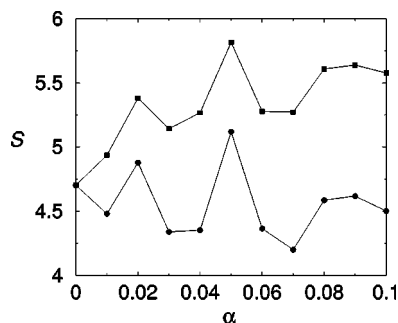
FIG. 6. Spatial (circle) and momentum (square) entropy for a highly excited state characterized by approximate quantum number pair (110,0) with $k_1=k_2=1$ as a function of α.

The functional form of Eq. (48) is quite analogous to the entropic sum in Eq. (5) conjectured for atoms, clusters, and nuclei [10,11] on the basis of theoretical arguments and numerical results. Such logarithmic increase in entropic sum is noted earlier for 1D power-law potentials in the semiclassical limit [9]. Thus, this result provides an approximate (semi)thereotical approach to see the emergence of the functional form in Eq. (48) in a simple coupled system.

VI. HIGHLY EXCITED STATES

In this section, we look at the gray areas that are not yet clearly understood. We briefly discuss the EUR for the highly excited states. Most eigenstates of chaotic systems in the semiclassical limit are irregular or random looking states that could be modeled by random-matrix theory [19]. Such random-matrix averages represent a particular limit at which the system's entropy becomes independent of the nonlinearity parameter. Hence we focus our attention on the highly excited "regular" states which are characterized by N quanta of excitation in one degree of freedom and 0 in the other, often called the Born-Oppenheimer type of states [15,16]. Thus they could be approximately labeled by the quantum number pair $(N,0;\alpha=\alpha_0)$, where $N \gg 1$ [20]. For these states, we will numerically explore the variation of entropies as a function of α. In a sense, the entropies of highly excited states have already been reported before in a different context [21]. First, we choose a particular eigenstate characterized by $(N,0;\alpha_0)$, in our case (110,0;0.0), and then follow the eigenstate as a function of α. The calculations are also somewhat cumbersome and hence we sample only a small parametric window.

The results, obtained from exact numerical basis-set diagonalization and then a subsequent entropy calculation, shown in Fig. 6 reveal that the spatial and momentum entropies display similar trend as a function of α. This is qualitatively different from that of the ground state. From the figure, it is also clear that the variation with α is not monotonic. The entropic sum evidently obeys the inequality in Eq. (4). It is known that for Hamiltonian in Eq. (6) there are certain special periodic orbits which are responsible for supporting regular states in the system [22]. The probability density structures are related to the classical phase-space structures

[23] and hence the information entropy is also intimately connected to the properties of the local classical phase-space structures that support such regular states. Thus the entropy is modulated by the qualitative nature of local classical dynamics as a function of α [20].

VII. CONCLUSIONS AND DISCUSSIONS

This work can be broadly divided into two parts. Focussing on the ground state of a coupled system, first, we show, within the framework of adiabatic theory, that spatial and momentum eigenstates can be constructed. Second, using the adiabatic eigenstates we obtain approximate results for the entropic uncertainty relations. For the ground state, we show that the spatial entropy decreases as a function of the coupling parameter α in the system while the momentum entropy increases. However, the entropic sum increases with the coupling parameter α and gets saturated only for $\alpha=0$. This is reminiscent of the results reported numerically for the entropic sums of atoms, molecules, and clusters [11]. Thus, this work provides an approach to obtain analytical results for the information entropies and their sums, for *coupled* nonlinear systems as a function of coupling parameter.

However, for highly excited states, the spatial and momentum entropies qualitatively look similar and seem to be related to the qualitative nature of local classical dynamics. In general, the relation between the classical phase-space structures and the quantum eigenstates is not completely understood yet. In this context, one might mention the semiclassical approaches proposed by Berry and Bogomolny [23] based on the Gutzwiller's trace formula. Thus, we only point out the intricacies involved in the entropic sums for the highly excited states. In due course, it might become possible to relate information entropy to classical orbits through these approaches. There are already empirical results pointing to this connection [20].

The adiabatic theory based approach presented here is not without demerits. Strictly speaking, the adiabatic effects take over when there is clear separation in the time scales of motion in the two modes that constitute the system. Thus, in the case of Hamiltonian in Eq. (6), the potential develops channels (see Fig. 1) for large values of α and thus facilitates adiabatic effects. The adiabatic theory is particularly suitable if the probability density develops structures within these channels, as it happens for some of the highly excited states. Since the ground state does not enter the channel for large α, it is likely to become less accurate for very large values of α. Further, this work leaves an interesting question unanswered. What happens in the limit $\alpha=\infty$? The present method of treatment does not seem adequate to answer this question. We hope the results here would stimulate work on more rigorous approaches to entropic sums in a wide variety of coupled systems.

[1] H. P. Robertson, Phys. Rev. **34**, 163 (1929).

[2] J. S. Dehesa *et al.*, J. Comput. Appl. Math. **133**, 23 (2001); V. Majernik and L. Richterek, Eur. J. Phys. **18**, 79 (1997).

[3] M. J. W. Hall, Phys. Rev. A **59**, 2602 (1999).

[4] I. Bialynicki-Birula and J. Mycielski, Commun. Math. Phys. **44**, 129 (1975).

[5] V. Majernik and L. Richterek, J. Phys. A **30**, L49 (1997).

[6] R. J. Yanez, W. Van Assche, and J. S. Dehesa, Phys. Rev. A **50**, 3065 (1994); W. Van Assche, R. J. Yanez, and J. S. Dehesa, J. Math. Phys. **36**, 4106 (1995); M. W. Coffey, J. Phys. A **36**, 7441 (2003).

[7] S. Abe *et al.*, Phys. Lett. A **295**, 74 (2002); S. Abe and A. K. Rajagopal, J. Phys. A **34**, 8727 (2001).

[8] J. S. Dehesa *et al.*, Int. J. Bifurcation Chaos Appl. Sci. Eng. **12**, 2387 (2002).

[9] J. S. Dehesa *et al.*, Phys. Rev. A **66**, 062109 (2002).

[10] S. R. Gadre, Phys. Rev. A **30**, 620 (1984).

[11] S. E. Massen and C. P. Panos, Phys. Lett. A **246**, 530 (1998); J. C. Angulo, J. Phys. A **26**, 6493 (1993).

[12] P. R. Certain and N. Moiseyev, J. Chem. Phys. **86**, 2146 (1987).

[13] K. Ganesan and M. Lakshmanan, Phys. Rev. Lett. **62**, 232 (1989).

[14] Ira N. Levine, *Molecular Spectroscopy* (Wiley, New York, 1975).

[15] C. C. Martens *et al.*, J. Chem. Phys. **90**, 2328 (1989); J. Zakrzewski and R. Marcinek, Phys. Rev. A **42**, 7172 (1990); B. Eckhardt *et al.*, *ibid.* **39**, 3776 (1989).

[16] R. E. Prange *et al.*, Phys. Scr., T **T90**, 134 (2001).

[17] C. Cohen-Tannoudji, B. Diu, and F. Laloe, *Quantum Mechanics* (Wiley, New York, 1997).

[18] Frank W. J. Olver, *Asymptotics and Special Functions* (A.K. Peters Ltd., Natick, MA, 1997).

[19] K. R. W. Jones, J. Phys. A **23**, L1247 (1990); K. Muller *et al.*, Phys. Rev. Lett. **78**, 215 (1997).

[20] M. S. Santhanam *et al.*, Phys. Rev. E **57**, 345 (1998).

[21] M. S. Santhanam *et al.*, Mol. Phys. **88**, 325 (1996).

[22] J. L. Anchell, J. Chem. Phys. **92**, 4342 (1990).

[23] M. V. Berry, Proc. R. Soc. London, Ser. A **423**, 219 (1989); E. B. Bogomolny, Physica D **31**, 169 (1988).