

Entanglement dynamics and quantum-state transport in spin chains

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We study the dynamics of a Heisenberg XY spin chain with an unknown state coded into one qubit or a pair of entangled qubits, with the rest of the spins being in a polarized state. The time evolution involves magnon excitations, and through them the entanglement is transported across the channel. For a large number of qubits, explicit formulas for the concurrences, measures for two-qubit entanglements, and the fidelity for recovering the state some distance away are calculated as functions of time. Initial states with an entangled pair of qubits show better fidelity, which takes its first maximum value at earlier times, compared to initial states with no entangled pair. In particular initial states with a pair of qubits in an unknown state $\alpha|\uparrow\uparrow + \beta|\downarrow\downarrow\rangle$ are best suited for quantum-state transport.

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Quantum entanglement has been recognized as an important resource for quantum information and computation [1], for transmission of quantum states through a channel [2]. There have been many proposals of physical systems to serve as channels for quantum communication [3], and in particular the spin chains [4]. The essential idea is to encode one particular qubit (spin-1/2 degree of freedom), and let it be transported across the chain to recover the code from another qubit some distance away. The quantum spin chains are well suited for state transport, as one could use the Schrödinger dynamics to propagate entanglement, and thus achieve the desired quantum communication.

The dynamics of entanglement of spin systems, viz., the study of time evolution of an initial state, can be classified into three categories: The dynamics of an S^z definite state or an S^z nondefinite state under S^z conserving time evolution (for example, the Heisenberg XY model), and the time evolution of an initial state under an S^z nonconserving dynamics (for example, the transverse Ising model). The structure of entanglement sharing and the time scales for entanglement transport are very different between these categories. In this paper, we will study the Heisenberg XY model, where S^z is conserved through the time evolution. The main features of the dynamics of entanglement that will be addressed are the appearance of pairwise entanglements between distant spins, viz., concurrences, starting from an initial state with no entangled pairs or exactly one entangled pair of spins, and the time scales for the transport of entanglement. The maximally entangled initial states $|\uparrow\downarrow \pm \downarrow\uparrow\rangle$ (the Bell state $B1$) and $|\uparrow\uparrow \pm \downarrow\downarrow\rangle$ (the Bell state $B2$) evolve quite differently [5]. The dynamics of entanglement in these states involve one-magnon and two-magnon excitations. The entanglement sharing in one-magnon and two-magnon eigenstates of the Heisenberg XY model has been studied [6,7]. The time evolution of $B1$ states involves the one-magnon excitations, which are not affected by s^z - s^z interactions, whereas for the

$B2$ states two-magnon excitations are involved, which introduce many complications. Here, not only two-magnon scattering states (equivalent to two one-magnon excitations) that have a weak signature of the interactions, but two-magnon bound states with a strong interaction effects [7] have a significant contribution for entanglement dynamics.

Let us consider an anisotropic Heisenberg XY model for a linear chain of spins ($s=1/2$), with a Hamiltonian

$$H = K_z \sum s_i^z s_{i+1}^z - \frac{K}{2} \sum s_i^+ s_{i+1}^- + \text{H.c.} - B \sum s_i^z - E_0, \quad (1)$$

where K_z, K are the interaction strengths for the z components and the xy components, respectively, and B is the strength of the magnetic field along the z direction. The constant $E_0 = -NK_z/4 - NB/2$ is the energy of the ferromagnetic state, with all the spins polarized along the z direction. The above Hamiltonian generates unitary evolution from the Schrödinger equation which conserves the total S^z in the state. The energy eigenstates and eigenvalues are known exactly through the Bethe ansatz [9]. The ferromagnetic state $|F\rangle = |\uparrow \cdots \uparrow\rangle$ has no dynamics, being an eigenstate with zero eigenvalue, and has no entanglement between any pair of spins. Let us first consider an unentangled state with one spin, at the site l , in an unknown state, at time $t=0$ given by

$$|\psi_u(0)\rangle = |\uparrow \cdots \uparrow\rangle |\alpha\uparrow + \beta\downarrow\rangle_l \equiv (\alpha + \beta s_l^-)|F\rangle. \quad (2)$$

This state has no entanglement, being a direct product state of different site states. Let us denote $|n\rangle = s_n^-|F\rangle$, a state with one down spin at site n . At a later time t , the state can be written as

$$|\psi_u(t)\rangle = \alpha|F\rangle + \beta \sum \phi_n(t)|n\rangle, \quad (3)$$

where $\phi_n = (1/N) \sum_q \exp i[q(n-l) - E_q t/\hbar]$, and the one-magnon energy $E_q = -K \cos q + B$. In the limit of large number of spins, $N \rightarrow \infty$, the wave function can be expressed in terms of the Bessel function J_{n-l} as

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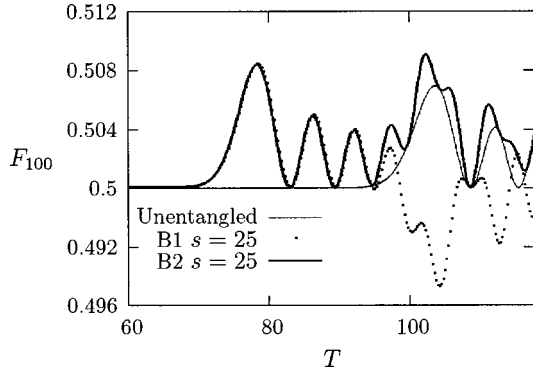


FIG. 1. The average fidelity F_r , as a function of T , for $r=100$, for the three types of states discussed. For the entangled states, $s=l-m=25$ has been used. The first maximum for the unentangled is at $T=r=100$, and for the B1 and B2 states, at $T=r-s$.

$$\phi_n(t) = e^{-i(Bt/\hbar)} e^{i(\pi/2)(n-1)} J_{n-1}(Kt/\hbar). \quad (4)$$

The time scale for the structure in the wave function is $\tau = \hbar/K$, and from now on we will write the time as a multiple of τ , as $T=t/\tau$. We can estimate the time scale, using $K \sim 0.01$ eV, as $\tau \sim 10^{-13}$ sec. The magnetic field adds on a constant phase to the wave function, and thus can be dropped. At time t the mixed state of a given site i can be denoted by the reduced density matrix $\rho_i = \text{tr}'_{1, \dots, N, i} |\psi_u\rangle\langle\psi_u|$, where the prime indicates a partial trace over all states except at site i . It is straightforward to write down the reduced density matrix as

$$\rho_i = [1 - |\beta|^2 |\phi_i(t)|^2] |\uparrow\rangle\langle\uparrow| + |\beta|^2 |\phi_i(t)|^2 |\downarrow\rangle\langle\downarrow|. \quad (5)$$

Now the initial unknown state encoded in the l th qubit can be extracted from the i th qubit with a fidelity $F_i = \text{Tr} \rho_u \rho_i$ where $\rho_u = |\alpha\uparrow + \beta\downarrow\rangle\langle\alpha\uparrow + \beta\downarrow|$, which works out to be (with $i=l+r$, a distance r away from the initial site) $F_r(T) = |\alpha|^2 + |\beta|^2 (|\beta|^2 - |\alpha|^2) J_r^2(T)$. By averaging over all possible initial states, i.e., the Bloch sphere, the average fidelity is

$$F_r(T) = \frac{1}{2} + \frac{1}{6} J_r^2. \quad (6)$$

A similar formula has been derived in Ref. [4], except that our calculation is simplified due to the limit of large N . Now, we are interested in propagating the initial code to a distance r , and recover it. The fidelity F_r has a maximum value for $T \approx r$, which means the quantum state is transported at a rate $v_i = 1/\tau$. After waiting for a time $t = v_i r$ we have the best recovery of the quantum state at site, a distance r away from the initial site l . In Fig. 1, the fidelity has been plotted as a function of time for $r=100$; the first maximum is for $T_c \approx r$, along with the result for entangled initial states we shall discuss below.

Though the initial state has no pairwise entanglement, for $T \neq 0$, the state develops entanglement. We use the concurrence measure [8] for the pairwise entanglement, which can be calculated from the two-site reduced density matrix ρ_{ij} (which is obtained by tracing over all spins except those at sites i and j). The time-reversed density matrix is denoted by $\tilde{\rho}_{ij}$, and the eigenvalues of $\rho_{ij} \tilde{\rho}_{ij}$ by $\lambda_1, \dots, \lambda_4$ in the descend-

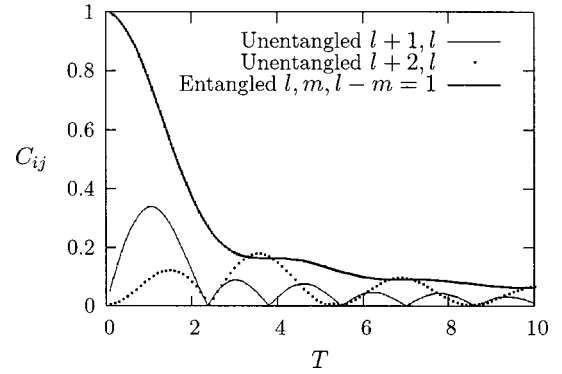


FIG. 2. The concurrence C_{ij} as a function of T , using $|\beta|^2 = 1/2$. It can be seen how concurrence builds up between the initial sites l , $l+1$, and $l+r$ for the unentangled initial state, and how the concurrence decreases between the initial sites l and m for B1 and B2 Bell states.

ing order. Then the concurrence between the two sites is [8] $C_{ij} = \max(\lambda_1^{1/2} - \lambda_2^{1/2} - \lambda_3^{1/2} - \lambda_4^{1/2}, 0)$. Here, the two-site density matrix has the form

$$\rho_{ij} = \begin{pmatrix} 1 - |\beta|^2 (|\phi_i|^2 + |\phi_j|^2) & & & & \\ & |\beta|^2 |\phi_j|^2 & |\beta|^2 \phi_i \phi_j^* & & \\ & |\beta|^2 \phi_i^* \phi_j & |\beta|^2 |\phi_i|^2 & & \\ & & & & 0 \end{pmatrix}.$$

Now the concurrence is given by [6] $C_{ij} = 2|\beta|^2 |J_{i-i}(T) J_{j-l}(T)|$. A plot of concurrences vs T is shown in Fig. 2 for $i=l+1, j=l$ and $i=l+2, j=l$. For small T , the concurrences grow as $C_{ij} \approx (2|\beta|^2/r!) (T/2)^r$. For large T , we have $C_{ij} \approx 2|\beta|^2/\pi T$ with oscillations.

Now we will turn to initial states with entangled pairs of spins. Entangled states are expected to be better than the unentangled states considered above. The dynamics will transport and further generate entanglement, as we shall see below. Let us first consider an initial state with a pair of qubits at sites l and m in an entangled state (B1 state) $|\alpha|\uparrow\uparrow\rangle + |\beta|\downarrow\downarrow\rangle$, all other spins polarized, which is represented as

$$|\psi_1(T=0)\rangle = (\alpha s_l^- + \beta s_m^-) |F\rangle = \sum \phi_n(0) |n\rangle. \quad (7)$$

For $T=0$ all concurrences are zero except $C_{lm}=1$. For $T \neq 0$, this entanglement spreads, and is transported to other pairs. Again using the one-magnon excited states, we can write down the wave function as a function of T ,

$$\phi_n(T) = \beta e^{i(\pi/2)(n-l)} J_{i-l}(T) + \alpha e^{i(\pi/2)(n-m)} J_{i-m}(T). \quad (8)$$

Since the above state is a one-magnon state (though it may not be an eigenstate) the concurrence is given by [6] $C_{ij} = 2|\phi_i^* \phi_j|$. In particular, the concurrence between the sites l and m for later times, for the maximally entangled initial state ($\alpha = \beta = 1/\sqrt{2}$), for $l-m$ even or odd, respectively, is given as

$$C_{lm} = (J_0 + (-1)^{(l-m)/2} J_{l-m})^2 \quad \text{or} \quad (J_0^2 + J_{l-m}^2). \quad (9)$$

In Fig. 2, C_{lm} has been plotted as a function of T for $l-m=1$, which drops from the initial value of unity as T^{-2} .

Now, the fidelity of recovering a quantum state $|\alpha\uparrow + \beta\downarrow\rangle$ at a site i can be calculated straightforwardly, analogous to the unentangled state we calculated before, as (for $i=m+r$, a distance r away from the site m) $F_r(T) = |\alpha|^2 + (|\beta|^2 - |\alpha|^2)|\phi_{r+m}(T)|^2$. For $|\alpha| \approx |\beta|$, the fidelity is close to $1/4$, as before for the unentangled case, which means the channel is noisy for a good recovery. In this case the unknown state can be recovered from two sites, as we shall discuss below. The average fidelity, after averaging over the Bloch sphere, is given by

$$F_r(T) = \frac{1}{2} + \frac{1}{6}(J_{r-(l-m)}^2 - J_r^2), \quad (10)$$

which should be compared with the formula we obtained [Eq. (6)] for the unentangled initial state. The first maximum of the fidelity now depends on $J_{r-(l-m)}$, which will occur for an earlier time, and the maximum value will be more than for J_r . The rate at which the state propagates is still $v_i = 1/\tau$, but we need to wait for a shorter time interval $t = v_i[r - (l-m)]$ to recover the state a distance r away from the m th site. The distance between the two initial sites can be chosen conveniently for a given distance over which the transport is desired. The fidelity for this case has been shown in Fig. 1 for a few values of r as a function of time. Due to the presence of the competing term (the last term in the above), the fidelity could fall below the value for the unentangled state, after the first maximum.

The initial entangled state at sites l and m , $\rho_{B1} = |\alpha\uparrow\downarrow + \beta\downarrow\uparrow\rangle\langle\alpha\uparrow\downarrow + \beta\downarrow\uparrow|$, can be extracted from sites i and j at a later time with a fidelity $G_{ij} = \text{Tr}\rho_{B1}\rho_{ij}$, which can be calculated after some manipulations simply as $G_{ij} = |\alpha\phi_j + \beta\phi_i|^2$. This has a maxima structure even for $|\alpha| \approx |\beta|$, unlike the fidelity function F_i discussed earlier. For $i=l+r, j=m+r$, i.e., for a pair of sites translated by r from the initial pair l, m , the fidelity takes the form (using $s=l-m$) $G_r = |2\alpha\beta J_r + \alpha^2 \exp(i\pi s/2) J_{r+s} + \beta^2 \exp(-i\pi s/2) J_{r-s}|^2$. After averaging over the Bloch sphere, it has a simple form

$$G_r = \frac{1}{3}(J_{r-s}^2 + J_{r+s}^2) + \frac{2}{3}J_r^2. \quad (11)$$

In Fig. 3, we show the fidelity G_r as a function of T for $r=50, 100$.

Let us now turn to the most difficult case of an initial state with a B2 state at sites l and m , and the rest of the spins polarized, given as $(\alpha + \beta s_l^- s_m^-)|F\rangle \equiv \alpha|F\rangle + \beta|\Phi(T=0)\rangle$. The state can be written as

$$|\psi_2(T=0)\rangle = \alpha|F\rangle + \beta \sum |ij\rangle, \quad (12)$$

where $|ij\rangle$ stands for a two-magnon basis state with two down spins at sites i and j . Initially for $i=l, j=m$ the wave function is unity, and zero for all other values of i and j . The time evolution of the second term above can be worked out in terms of the two-magnon excitations, using the Bethe ansatz solution for two down spins. The concurrences in the two-magnon eigenstates have been worked out [7] both for

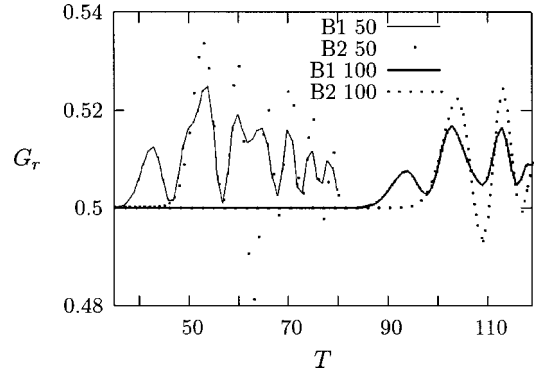


FIG. 3. The average fidelity G_r as a function of T for $r=50, 100$ using $s=l-m=10$. A constant 0.5 has been added on for B1 state to show both on the same graph.

the scattering and the bound states. The wave function is extremely complicated due to the magnon interactions arising for a nonzero K_z , which can be vastly simplified by taking the limit $K \gg K_z$, i.e., dropping the interaction K_z terms in the Hamiltonian. In the XY limit, the wave function as a function of time takes the form (after taking $N \rightarrow \infty$ limit)

$$\phi_{ij}(T) = e^{i(\pi/2)(i+j-l-m)} [J_{i-l}(T)J_{j-m} - J_{i-m}J_{j-l}], \quad (13)$$

which is antisymmetric in the two indices, reflecting the underlying fermionic nature of the moving down spins. Now, following through the steps as before, the two-site reduced density matrix has the form

$$\rho_{ij} = \begin{pmatrix} |\alpha|^2 + \beta^2 u_{ij} & & \alpha\beta^* \phi_{ij}^* \\ & |\beta|^2 w_{1ij} & |\beta|^2 z_{ij}^* \\ & |\beta|^2 z_{ij} & |\beta|^2 w_{2ij} \\ \alpha^* \beta \phi_{ij} & & & |\beta|^2 v_{ij} \end{pmatrix}.$$

In the above, the various matrix elements stand for $u_{ij} = \langle (1/2 + s_i^z)(1/2 + s_j^z) \rangle$, $v_{ij} = \langle (1/2 - s_i^z)(1/2 - s_j^z) \rangle$, $w_{1ij} = \langle (1/2 - s_i^z)(1/2 + s_j^z) \rangle$, $w_{2ij} = \langle (1/2 + s_i^z)(1/2 - s_j^z) \rangle$, $z_{ij} = \langle s_i^+ s_j^- \rangle$, where the expectation value is taken in the two-magnon state $|\Phi(T)\rangle$ only. Now following through the further steps of Wootters [8], the eigenvalues of $\rho_{ij}\tilde{\rho}_{ij}$ for the above density matrix are $(\sqrt{(|\alpha|^2 + u|\beta|^2)}|\beta|^2 v \pm |\alpha^* \beta \phi_{ij}|)^2$, $|\beta|^4 (\sqrt{w_1 w_2} \pm |z|)^2$. This gives two regimes for the concurrence as

$$C_{ij} = 2|\beta|^2 |z_{ij}| - 2|\beta| \sqrt{v_{ij}} \sqrt{|\alpha|^2 + |\beta|^2 u_{ij}}$$

or

$$= 2|\alpha^* \beta \phi_{ij}| - 2|\beta|^2 \sqrt{w_{1ij} w_{2ij}} \quad (14)$$

whichever term is positive, and otherwise zero. The off-diagonal matrix element can be calculated as (taking $i > j$)

$$z_{ij} = \sum \phi_{in}^* \phi_{jn} - 2 \sum_{j+1}^{i-1} \phi_{in}^* \phi_{jn} \equiv \eta_{ij} - 2\zeta_{ij}, \quad (15)$$

where the sum in η_{ij} is over all values of n , which can be calculated using the addition rule $J_n(x+y) = \sum_k J_k(x) J_{n-k}(y)$

and setting the site m at the middle of the chain for convenience,

$$\eta_{ij} = e^{i(\pi/2)(j-i)}(J_{i-l}J_{j-l} + J_{i-m}J_{j-m}). \quad (16)$$

ζ_{ij} , which is just the finite sum, is quite complicated to calculate in general. The diagonal matrix elements are $v_{ij} = |\phi_{ij}|^2$, $u_{ij} \approx 1$, $w_{1ij} = \eta_{ii} - |\phi_{ij}|^2$, $w_{2ij} = \eta_{jj} - |\phi_{ij}|^2$. This simplifies the expression for the concurrence between two sites i and j as

$$C_{ij} = |\beta| \max(0, |\beta||z| - |\phi|, 2(|\alpha\phi| - |\beta|\sqrt{w_1 w_2})). \quad (17)$$

For $l-m$ odd, the off-diagonal matrix element $z_{lm} = 0$, and the expression for the maximally entangled initial state is simple, $C_{lm} = J_0^2 + J_{l-m}^2$, which is what we got for $B1$ Bell states. For $l-m$ even, the expression is still complicated. For $l-m=2$, $C_{lm} = \max(0, |z|/2 - |\phi|/\sqrt{2}, |\phi| - w_1)$, where $\phi = (J_0^2 - J_2^2)$, $w_1 = J_0^2 + J_2^2 - |\phi|^2$, $z = 2J_0J_2 + J_1^2(J_0 + J_2)^2$. The concurrence between the sites l and m is plotted as a function of T for $l-m=1$ in Fig. 2, along with the result for the $B1$ states and the unentangled state.

The fidelity of recovering the state $|\alpha\uparrow + \beta\downarrow\rangle$ at a site i is again straightforward, $F_i = |\alpha|^2 + |\beta|^2(|\beta|^2 - |\alpha|^2)\eta_{ii}$. The average fidelity, for $i=m+r$,

$$F_r = \frac{1}{2} + \frac{1}{6}(J_{r-(l-m)}^2 + J_r^2). \quad (18)$$

As compared to the expression for the $B1$ states, in the above there is no competition between J_r and $J_{r-(l-m)}$. The fidelity here is greater than that of the unentangled state for all times.

The first maximum value is determined by the first term, as in the case of $B1$ states, which occurs for $t = v_t(r-l+m)$. A comparison of the fidelity as a function of T for all the three cases, from Fig. 1 for $r=100$, shows that the $B2$ states have better fidelity. Now, the fidelity of recovering $|\alpha\uparrow + \beta\downarrow\rangle$ from sites i and j , analogous to the $B1$ states we discussed before, is given as $G_{ij} = |\alpha|^2 + |\beta|^4|\phi_{ij}|^2 + |\alpha\beta|^2(\phi_{ij}^* + \phi_{ij})$. This function also exhibits a maxima structure for $|\alpha| \approx |\beta|$. The average fidelity is (for $i=l+r, j=m+r$)

$$G_r = \frac{1}{2} + \frac{1}{3}(J_r^2 - J_{r-s}J_{r+s})(J_r^2 - J_{r-s}J_{r+s} + e^{i\pi r}). \quad (19)$$

The fidelity as a function of T is shown in Fig. 3, for $r=50, 100, s=10$, along with the result for $B1$ states. The $B2$ states exhibit better fidelity here also, as is the case in Fig. 1.

In conclusion, we have investigated the quantum-state transport across a channel of qubits, the spin chain, using the Heisenberg XY dynamics. The presence of entanglement and its dynamics is crucial for communication over the channel. Initial states with a pair of qubits in a state $\alpha\uparrow\uparrow + \beta\downarrow\downarrow$ show better fidelity. Here, it will be interesting to investigate the effect of a nonzero K_z ; significant changes in the wave functions, the concurrences, and the fidelity are expected. Finally, states with many entangled pairs in an optimized network may demonstrate an almost-ideal quantum communication, that is, a teleportation protocol with the sender and the receiver at a fixed distance and the code transported with negligible interference from the network channel.

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