# Nonresonant multiphoton ionization in atomic hydrogen

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The infinite summations over the complete set of unperturbed atomic states appearing in the $N$th-order perturbation theory for multiphoton ionization are performed using the Dalgarno-Lewis method. The relevant transition matrix elements are written in a closed integral form exhibiting all the analytic properties of the amplitude as a function of incident photon energy. The cross sections for two- and three-photon ionization for atomic hydrogen are calculated for both linearly and circularly polarized light with a wide range of photon energy spectrum including the near resonance, and numerical comparison is made with the values obtained by different methods.

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## I. INTRODUCTION

Since the advent of very high power lasers the evaluation of multiphoton transition matrix element in atomic system has very important role to play in atomic physics. When the $N$th-order time-dependent perturbation is applied, a major difficulty in the evaluation of these matrix elements is the infinite summation over the complete set of intermediate states, which includes discrete as well as continuum states. Different procedures [1–12] are used to perform this intermediate sum, but in general none of them can be efficiently used for the entire range of wavelengths of the incident photons.

Here we have outlined a simple and efficient alternate analytical method for obtaining a closed-form expression for the two-photon and three-photon radiative transition matrix elements (bound-free) from the hydrogenic ground state and calculated the corresponding cross sections for the entire range of photon spectrum. We have performed the intermediate sum exactly using the Dalgarno-Lewis procedure [8] and obtained a set of inhomogeneous second-order coupled differential equations. The integral representation of solutions to these equations for two- and three-photon transitions is obtained and its analytical properties are studied. Finally the numerical results for two- and three-photon cross sections essentially covering the range of the available numerical results are presented.

## II. TWO- AND THREE-PHOTON SCATTERING CROSS SECTIONS

Following the perturbation theory and using the nonrelativistic dipole approximation, the differential cross section for $N$-photon [13,14] ionization in atomic units is given by

$$\frac{d\sigma^{(N)}}{d\Omega} = \frac{\alpha}{2\pi}\left(\frac{I}{2I_0}\right)^{N-1} |\mathcal{M}_{fg}^{(N)}|^2 a_0^2 \omega k, \tag{1}$$

where $\alpha$ is the fine-structure constant, $a_0 = 5.2917 \times 10^{-9}$ cm is the first Bohr radius, $I$ is the field strength intensity of the radiation field, $I_0 = 7.019 \times 10^{16}$ W/cm$^2$ is the atomic unit of field strength intensity, and $k$ is the momentum of the photoelectron ejected in the direction of the unit vector $\hat{k}$. For transitions from ground state of atomic hydrogen $k$ is given as

$$k = \sqrt{2N\omega - 1}, \tag{2}$$

and the $N$th-order transition amplitude $\mathcal{M}_{fg}^{(N)}$ corresponding to a transition from the initial ground state $|g\rangle$ to a final state $|f\rangle$ belonging to the continuum reads [1]

$$\mathcal{M}_{fg}^{(N)} = \sum_{i_{N-1}\cdots i_1} \frac{\langle f|\vec{\epsilon}\cdot\vec{r}|i_{N-1}\rangle \cdots \langle i_2|\vec{\epsilon}\cdot\vec{r}|i_1\rangle\langle i_1|\vec{\epsilon}\cdot\vec{r}|g\rangle}{(E_{i_{N-1}} - E_g - (N-1)\omega)\cdots(E_{i_1} - E_g - \omega)}, \tag{3}$$

where $\vec{\epsilon}$ is the unit polarization vector for the incident radiation field and $\vec{\epsilon}\cdot\vec{r}$ is in units of the Bohr radius $a_0$, and the energies $E_g$, $E_i$, and $\omega$ are in units of $(e^2/a_0)$. The $N-1$ intermediate state sum is over the complete set of states including the continuum.

The infinite summation over the intermediate state in Eq. (3) can be performed exactly by defining a set of $N$ operators $M_n$ with $n = 0, 1, \ldots, N-1$ such that

$$(\vec{\epsilon}\cdot\vec{r})M_{n-1}|g\rangle = (M_n H_0 - H_0 M_n + n\omega M_n)|g\rangle, \tag{4}$$

where we take $M_0$ as unit operator $\hat{I}$ and $H_0 = -\nabla^2/2 - 1/r$ the Hamiltonian for the hydrogen atom in atomic units. Using Eq. (4) in Eq. (3) and the closure relation $\Sigma_i |i\rangle\langle i| = \hat{I}$ the transition matrix element is reduced to

$$\mathcal{M}_{fg}^{(N)} = \langle f|(\vec{\epsilon}\cdot\vec{r})M_{N-1}|g\rangle. \tag{5}$$

Thus the problem of infinite summation is reduced to the determination of analytic form of the operators $M_n$ for $n = 0, 1, \ldots, N-1$. The angular separation of the functions $M_n(\mathbf{r})$ can be performed by expanding it in terms of spherical harmonics. For simplicity first we consider linearly polarized light so that $\vec{\epsilon}\cdot\vec{r} = z$. Then the angular separation of $M_n$'s needed for two- and three-photon transitions can be achieved by writing

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$$M_1(\mathbf{r}) = r\psi_1(r)P_1(\cos\theta), \tag{6}$$

$$M_2(\mathbf{r}) = \psi_0(r) + r^2\psi_2(r)P_2(\cos\theta), \tag{7}$$

where $P_l(\,)$ are Legendre polynomials of order $l$. Now substituting Eqs. (6) and (7) in Eq. (4) and by taking $|g\rangle = \exp(-r)/\sqrt{\pi}$ we get the following differential equations for the unknown radial functions $\psi_1$, $\psi_0$, and $\psi_2$:

$$r\frac{d^2}{dr^2}\psi_1(r) + (4-2r)\frac{d}{dr}\psi_1(r) + (2\omega r - 2)\psi_1(r) = 2r, \tag{8}$$

$$r\frac{d^2}{dr^2}\psi_2(r) + (6-2r)\frac{d}{dr}\psi_2(r) + (4\omega r - 4)\psi_2(r) = \frac{4}{3}r\psi_1(r), \tag{9}$$

$$r\frac{d^2}{dr^2}\psi_0(r) + (2-2r)\frac{d}{dr}\psi_0(r) + 4\omega r\psi_0(r) = \frac{2}{3}r^3\psi_1(r). \tag{10}$$

It is straightforward to see from Eq. (5) that the infinite summation over the intermediate states are reduced to the determination of the solutions for the above inhomogeneous differential equations. From Eqs. (9) and (10) it is clear that the evaluation of three-photon transition amplitude requires the knowledge of the two-photon transition amplitude. This is also clear from the Eq. (4) that the appearance of $(N-1)$th-order transition amplitude as inhomogeneous term for the determination of the $N$th-order transition amplitude is a general feature of this method.

By using the method of Laplace transform [15] for the solution of differential equation we can obtain the solutions to the above differential equations as

$$\psi_1(r) = \frac{1}{\omega} - \frac{1}{2\omega^3}\Phi(1,1,\lambda_1,1,r), \tag{11}$$

$$\begin{aligned}\psi_0(r) = {} & \frac{2}{3\omega^2}\left\{\frac{r^2}{4} + \frac{1}{4\omega^2}(1-2\lambda_1) - \frac{3}{8\omega}\right\} - \frac{1}{3\lambda_1\omega^4} \\ & \times\Phi(0,0,\lambda_2,1,r) - \frac{1}{6\omega^4}\left\{r^2 + \frac{\lambda_1}{\omega}r - \frac{1}{\omega}\left(1+\frac{1}{\lambda_1}\right)\right\} \\ & \times\Phi(1,1,\lambda_1,1,r) - \frac{1}{3\omega^4}\left(1+\frac{1}{\lambda_1}\right)\int_{\lambda_1}^{1} dt\frac{K(1,1,\lambda_1,t)}{K(1,1,\lambda_2,t)} \\ & \times\left\{\frac{1-2\omega}{\omega(1+\lambda_1)} - \frac{t}{\omega} - \frac{2}{t+\lambda_1} + \frac{2}{(t+\lambda_1)^2}\right\} \\ & \times\Phi(0,0,\lambda_2,t,r),\end{aligned} \tag{12}$$

$$\begin{aligned}\psi_2(r) = {} & \frac{1}{3\omega^2} - \frac{1}{3\omega^4}\Phi(1,1,\lambda_1,1,r) \\ & + \frac{2}{3\omega^4}\int_{\lambda_1}^{1} dt\frac{K(1,1,\lambda_1,t)t}{K(3,3,\lambda_2,t)}\Phi(2,2,\lambda_2,t,r),\end{aligned} \tag{13}$$

where

$$K(p,q,\lambda,s) = \left(\frac{1-\lambda}{1+\lambda}\right)^{1/\lambda}(s+\lambda)^{p+1/\lambda}(s-\lambda)^{q-1/\lambda}, \tag{14}$$

and

$$\Phi(p,q,\lambda,t,r) = \int_{\lambda}^{1} ds\; e^{-r(s-1)}K(p,q,\lambda,s), \tag{15}$$

where $p,q,$ and $\lambda$ in general can be complex numbers, and $\lambda_1 = \sqrt{1-2\omega}$ and $\lambda_0 = \lambda_2 = \sqrt{1-4\omega}$. If the limits of the integration are complex numbers, the contour of integration in the complex plane should not encounter any singularities of the integrand in Eq. (15). It is also useful to note that the above integrals are defined only for $\mathrm{Re}(q-1/\lambda) > -1$. But it is easy to analytically continue this definition for $\mathrm{Re}(q-1/\lambda) < -1$. This is done by writing $(s-\lambda)^{q-1/\lambda} = \partial/\partial s\{(s-\lambda)^{q+1-1/\lambda}\}/(q+1-1/\lambda)$ in Eq. (15) and performing a partial integration. Thus we obtain

$$\begin{aligned}\Phi(p,q,\lambda,t,r) = {} & \frac{1}{q+1-\frac{1}{\lambda}}\Bigg\{e^{-r(t-1)}K(p,q+1,\lambda,t) \\ & + r\Phi(p,q+1,\lambda,t,r) - \left(p+\frac{1}{\lambda}\right) \\ & \times\Phi(p-1,q+1,\lambda,t,r)\Bigg\}.\end{aligned} \tag{16}$$

This is an important recurrence relation which is used to study the analytic properties of the transition amplitude [16].

The amplitude for continuum transitions can be calculated by taking the final-state wave function to be

$$|f\rangle = 4\pi\sum_{lm} i^l R_{kl}(r)Y_{lm}(\hat{r})Y^*_{lm}(\hat{k}), \tag{17}$$

where $Y_{lm}(\,)$ are the well-known spherical harmonics. The radial part of hydrogen atom wave function for an attractive Coulomb potential can be taken as

$$\begin{aligned}R_{kl}(r) = {} & e^{\pi\gamma/2}\frac{\Gamma(l+1-i\gamma)}{\Gamma(2l+2)}(2rk)^l \\ & \times e^{ikr}F(l+1-i\gamma, 2l+2, -2ikr),\end{aligned} \tag{18}$$

with $F(\,)$ as the usual confluent hypergeometric functions.

Now substituting Eq. (17) in Eq. (5) and after performing the angular integration we obtain

$$\mathcal{M}^{(2)} = \mathcal{D}_0^{(2)}Y_{00}(\hat{k}) + \mathcal{D}_2^{(2)}Y_{20}(\hat{k}), \tag{19}$$

$$\mathcal{M}^{(3)} = \mathcal{D}_3^{(3)}Y_{30}(\hat{k}) + \mathcal{D}_1^{(3)}Y_{10}(\hat{k}), \tag{20}$$

where $\mathcal{M}^{(2)}$ and $\mathcal{M}^{(3)}$ are, respectively, the two- and three-photon transition amplitudes and

$$\mathcal{D}_0^{(2)} = \frac{8\pi}{3}\int dr e^{-r}R^*_{k0}(r)r^4\psi_1, \tag{21}$$

TABLE I. Three-photon scattering cross section per unit squared intensity $\sigma^{(3)}/I^2$ ($\mathrm{cm}^6/\mathrm{W}^2$) for both linear and circular polarization vs wavelength above one-photon ionization threshold $\omega > 1/2$.

| | $\sigma_L^{(3)}/I^2$ | | $\sigma_C^{(3)}/I^2$ | |
|---|---|---|---|---|
| λ (Å) | Klar-Maq[a] | Present | Klar-Maq[a] | Present |
| 20 | | 5.596(−66) | | 2.470(−66) |
| 100 | | 1.959(−59) | | 9.254(−60) |
| 200 | 1.16(−56) | 1.169(−56) | 6.38(−57) | 6.417(−57) |
| 300 | | 4.799(−55) | | 3.023(−55) |
| 400 | 6.60(−54) | 6.632(−54) | 4.72(−54) | 4.744(−54) |
| 500 | | 5.065(−53) | | 4.078(−53) |
| 600 | 2.65(−52) | 2.667(−52) | 2.39(−52) | 2.398(−52) |
| 700 | | 1.089(−51) | | 1.085(−51) |
| 800 | 3.68(−51) | 3.702(−51) | 4.04(−51) | 4.065(−51) |
| 900 | | 1.972(−50) | | 1.317(−50) |

[a]Klarsfeld and Maquet [23].

$$\mathcal{D}_2^{(2)} = \frac{16\pi}{3\sqrt{5}} \int dr R_{k2}^*(r) r^4 \psi_1 e^{-r}, \tag{22}$$

$$\mathcal{D}_3^{(3)} = i\frac{24\pi}{5\sqrt{7}} \int dr R_{k3}^*(r) r^5 \psi_2 e^{-r}, \tag{23}$$

$$\mathcal{D}_1^{(3)} = -i\frac{8\pi}{\sqrt{3}} \int dr R_{k1}^*(r)\left\{ r^3\psi_0 + \frac{2}{5} r^5 \psi_2 \right\} e^{-r}. \tag{24}$$

It is useful to note that the form of the amplitudes is consistent with the well-known selection rules for multiphoton transitions. Using Eqs. (11)–(13) the radial integrals in Eqs. (21)–(24) can be easily done using the standard integrals [17]

$$\int_0^\infty dr e^{-sr} r^{2l+1} F\left( l+1+\frac{i}{k}, 2l+2, 2ikr \right) = \Gamma(2l+2)\frac{s^{(i/k)-l-1}}{(s-2ik)^{(i/k)+l+1}}. \tag{25}$$

These integrals can be put in a neat form using the following notations:

$$I_{kl}(s,k) = \int_0^\infty dr R_{kl}(r) e^{-rs} r^k, \tag{26}$$

$$\mathcal{I}_{kl}(p,q,\lambda,t,k) = \int_0^\infty dr\ R_{kl}(r) e^{-r} r^k \Phi(p,q,\lambda,t,r) = \int_\lambda^t ds\ K(p,q,\lambda,s) I_{kl}(s,k). \tag{27}$$

With this we get the integrals in Eqs. (21)–(24) as

TABLE II. Three-photon scattering cross section per unit squared intensity $\sigma^{(3)}/I^2$ ($\mathrm{cm}^6/\mathrm{W}^2$) for both linear and circular polarization when two-photon ionization is energetically allowed ($1/4 < \omega < 1/2$).

| | $\sigma_L^{(3)}/I^2$ | | | $\sigma_C^{(3)}/I^2$ | |
|---|---|---|---|---|---|
| λ (Å) | Karule[a] | Klar[b] | Present | Klar[b] | Present |
| 1000 | | | 1.484(−48) | | |
| 1100 | | | 3.430(−50) | | 5.472(−50) |
| 1200 | 5.55(−48) | 5.55(−48) | 6.121(−48) | 4.67(−48) | 5.234(−48) |
| 1300 | | | 1.168(−48) | | 2.379(−48) |
| 1400 | 1.01(−48) | 1.00(−48) | 1.007(−48) | 2.39(−48) | 2.402(−48) |
| 1500 | 1.35(−48) | | 1.355(−48) | | 3.370(−48) |
| 1600 | 2.02(−48) | 2.03(−48) | 2.024(−48) | 5.07(−48) | 5.056(−48) |
| 1700 | 3.09(−48) | | 3.104(−48) | | 7.663(−48) |
| 1800 | 4.74(−48) | 4.67(−48) | 4.761(−48) | 1.13(−47) | 1.151(−47) |

[a]Karule [19].
[b]Klarsfeld and Maquet [23].

$$\int_0^\infty dr\ R_{kl}(r) e^{-r} r^5 \psi_2(r) = \frac{1}{3\omega^2} I_{kl}(1,5) - \frac{1}{3\omega^4}\mathcal{I}_{kl}(1,1,\lambda_1,1,5) + \frac{2}{3\omega^4}\int_{\lambda_1}^1 dt \frac{K(1,1,\lambda_1,t)t}{K(3,3,\lambda_2,t)}\mathcal{I}_{kl}(2,2,\lambda_2,t,5), \tag{28}$$

$$\begin{aligned}\int_0^\infty dr R_{kl}(r) e^{-r} r^3 \psi_0(r) &= \frac{1}{6\omega^2} I_{kl}(1,5) + \frac{2}{3\omega^2}\left\{ \frac{1}{4\omega^2}(1-2\lambda_1) - \frac{3}{8\omega} \right\} I_{kl}(1,3) \\ &\quad - \frac{1}{3\lambda_1\omega^4}\mathcal{I}_{kl}(0,0,\lambda_2,1,3) - \frac{1}{6\omega^4}\left\{ \mathcal{I}_{kl}(1,1,\lambda_1,1,5) \right. \\ &\quad \left. + \frac{\lambda_1}{\omega}\mathcal{I}_{kl}(1,1,\lambda_1,1,4) - \frac{1}{\omega}\left(1+\frac{1}{\lambda_1}\right)\mathcal{I}_{kl}(1,1,\lambda_1,1,3) \right\} \\ &\quad - \frac{1}{3\omega^4}\left(1+\frac{1}{\lambda_1}\right)\int_{\lambda_1}^1 dt \frac{K(1,1,\lambda_1,t)}{K(1,1,\lambda_2,t)}\left\{ \frac{1-2\omega}{\omega(1+\lambda_1)} - \frac{t}{\omega} \right. \\ &\quad \left. - \frac{2}{t+\lambda_1} + \frac{2}{(t+\lambda_1)^2} \right\} \mathcal{I}_{kl}(0,0,\lambda_2,t,3). \end{aligned} \tag{29}$$

Substituting Eqs. (19) and (20) in Eq. (1) and integrating over the emitted electron direction we obtain

$$\frac{\sigma_L^{(2)}}{I} = \frac{\alpha}{4\pi I_0} a_0^2 \{ |\mathcal{D}_0^{(2)}|^2 + |\mathcal{D}_2^{(2)}|^2 \} \omega k, \tag{30}$$

$$\frac{\sigma_L^{(3)}}{I^2} = \frac{\alpha}{8\pi}\left(\frac{a_0}{I_0}\right)^2 \{ |\mathcal{D}_3^{(3)}|^2 + |\mathcal{D}_1^{(3)}|^2 \} \omega k, \tag{31}$$

where $\sigma_L^{(2)}$ and $\sigma_L^{(3)}$ are, respectively, the two- and three-photon cross sections for linear polarization. Using the

TABLE III. Three-photon scattering cross section per unit squared intensity $\sigma^{(3)}/I^2$ ($cm^6/W^2$) for both linear and circular polarization when three-photon ionization is energetically possible.

| | $\sigma_L^{(3)}/I^2$ | | | | $\sigma_C^{(3)}/I^2$ | | | |
|---|---|---|---|---|---|---|---|---|
| λ(Å) | Karule[a] | Lapla[b] | Gao[c] | Present | Karule[a] | Lapla[b] | Gao[c] | Present |
| 1900 | | 1.186(−46) | 1.172(−46) | 1.17(−46) | 1.65(−45) | 2.479(−46) | 2.459(−46) | 2.454(−46) |
| 2000 | 5.58(−48) | 5.581(−48) | 5.429(−48) | 5.423(−48) | 1.37(−47) | 1.365(−47) | 1.326(−47) | 1.324(−47) |
| 2100 | 2.58(−47) | 2.542(−47) | 2.541(−47) | 2.539(−47) | 5.85(−47) | 5.771(−47) | 5.776(−47) | 5.772(−47) |
| 2200 | 1.59(−47) | 1.593(−47) | 1.589(−47) | 1.588(−47) | 3.95(−47) | 3.957(−47) | 3.945(−47) | 3.943(−47) |
| 2300 | 2.60(−47) | 2.650(−47) | 2.641(−47) | 2.640(−47) | 3.84(−47) | 3.853(−47) | 3.840(−47) | 3.839(−47) |
| 2400 | 6.42(−46) | 7.125(−46) | 7.057(−46) | 7.056(−46) | 3.92(−47) | 3.935(−47) | 3.917(−47) | 3.917(−47) |
| 2500 | 3.03(−46) | 2.980(−46) | 2.948(−46) | 2.948(−46) | 3.97(−47) | 3.998(−47) | 3.972(−47) | 3.972(−47) |
| 2600 | 1.01(−46) | 1.008(−46) | 1.002(−46) | 1.002(−46) | 3.91(−47) | 3.947(−47) | 3.906(−47) | 3.906(−47) |
| 2700 | 6.74(−47) | | | 6.705(−47) | 3.65(−47) | | | 3.641(−47) |

[a]Karule [21].
[b]Laplanche *et al.* [22].
[c]Gao and Starace [7].

Clebsh-Gordan algebra we can similarly write the cross sections for circular polarization as

$$\frac{\sigma_C^{(2)}}{I} = \frac{\alpha}{4\pi I_0} a_0^2 \frac{3}{2} |\mathcal{D}_2^{(2)}|^2 \omega k, \tag{32}$$

$$\frac{\sigma_C^{(3)}}{I^2} = \frac{\alpha}{8\pi} \left(\frac{a_0}{I_0}\right)^2 \frac{5}{2} |\mathcal{D}_3^{(3)}|^2 \omega k. \tag{33}$$

The closed-form expressions in Eqs. (30)–(33) for scattering cross section are more elegant than the expressions obtained by other methods.

### III. NUMERICAL RESULTS AND DISCUSSION

We have calculated the two- and three-photon cross sections for values of photon energy both below and above threshold ionization. This is achieved by performing the integrals in Eqs. (27)–(29) numerically and substituting it in Eqs. (21)–(24) and finally using Eqs. (30)–(33). In the case of three-photon ionization many authors dealt with only the problem of above threshold ionization, and there is no single formulation for a wide energy range. But we are able to perform a detailed numerical calculation for a wide range of incident wavelengths of physical interest using this method. The scattering cross section for three-photon transitions including the near-resonance contribution for both linear and circular polarization are given in Tables I–III and our results are in excellent agreement with those previously obtained by other authors through different methods.

The closed integral form of the solution in Eqs. (11)–(13) is very convenient for numerical computation. This form is also very useful to separate out the resonance singularities present in the original perturbation-theory result given by Eq. (3). By repeatedly using the recurrence relation given by Eq.

TABLE IV. Three-photon scattering cross section per unit intensity $\sigma^{(2)}/I$ for both linear and circular polarization when two-photon ionization is energetically possible.

| | $\sigma_L^{(2)}/I$ | | | | $\sigma_C^{(2)}/I$ | |
|---|---|---|---|---|---|---|
| λ(Å) | Karule[a] | Gao[b] | Chan-Tan[c] | Present | Laplache[d] | Present |
| 1100 | 4.00(−33) | 4.024(−34) | 4.013(−34) | 4.001(−34) | 4.284(−34) | 4.241(−34) |
| 1200 | 6.42(−32) | 6.441(−32) | 6.303(−32) | 6.422(−32) | 7.725(−32) | 7.731(−32) |
| 1300 | 1.27(−32) | 1.276(−32) | 1.276(−32) | 1.274(−32) | 1.879(−32) | 1.874(−32) |
| 1400 | 8.45(−33) | 8.451(−33) | 8.450(−33) | 8.446(−33) | 1.270(−32) | 1.266(−32) |
| 1500 | | | | 8.337(−33) | | 1.252(−32) |
| 1600 | 9.15(−33) | 9.153(−33) | 9.154(−33) | 9.151(−33) | 1.358(−32) | 1.353(−32) |
| 1700 | 1.03(−32) | 1.025(−32) | 1.025(−32) | 1.024(−32) | 1.498(−32) | 1.493(−32) |
| 1800 | | | | 1.143(−32) | | 1.636(−32) |

[a]Karule [20].
[b]Gao and Starace [7].
[c]Chan and Tang [24].
[d]Laplanche *et al.* [22].

(16) all the singularities will now appear as simple poles in the expression for the transition amplitude [16]. It is very important to note that we always take the analytic expressions in Eqs. (11)–(13) with proper use of Eq. (16) as solution to Eqs. (8)–(10), irrespective of the value of the incident photon energy $\omega$. Thus for the case of above one-photon ionization threshold $(\omega > 1/2)$ both $\lambda_1$ and $\lambda_2$ become purely imaginary and above two-photon threshold $(1/2 > \omega > 1/4)$ only $\lambda_2$ is purely imaginary. If the integration limit becomes imaginary, we can make appropriate linear transformation of the integration variable to transform the limit of integration to be real. Thus the amplitude for transition to continuum states is analytic continuation of the solutions obtained for the bound to bound transition [18].

It may be seen that our data agree quite well with the available results of Klarsfeld and Maquet [23], where they have used Pade-Sturmian method for the determination of the matrix element, but near the vicinity of threshold their method fails. For the case of the above two-photon ionization threshold numerical values at selected wavelengths are compared and are given in Table II. Here a comparison is also made with the available values of Karule [19] for linear polarization, where analytical continuation of the Sturmian expansion of the Coulomb Green's function is used for the above threshold ionization calculation. Results compared here show good agreement with these values. Most of the available data is for the range of wavelengths from 1824.472 Å to 2736.708 Å, which corresponds to below threshold ionization. In this case also our results are in good agreement with the results obtained by various methods [7,11,21,22] for both linear and circular polarization.

As a verification of the method of analytic continuation of the amplitude we have used the auxiliary operator $M_1$ in Eq. (6) to evaluate the two-photon transition amplitude. With the same analytic solution to the amplitude with proper value of $\lambda_1$ we have obtained cross sections for both below and above transitions. The additional results along with the results already published [13] are given in Table IV. At $\lambda = 1200$ Å we get the same result $(6.42 \times 10^{-32})$ published by Chang and Poe [25], where they have used the same implicit summation technique.

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