Driver-phase-correlated fluctuations in the rotation of a strongly driven quantum bit

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The need to maximize the number of operations of a quantum bit within its decoherence time may require the ratio of Rabi frequency to transition frequency to be large enough to invalidate the rotating-wave approximation. The state of the quantum bit under any initial condition then depends explicitly on the phase of the driving field, resulting in driver-phase-correlated fluctuations and a violation of the rule that the degree of excitation depends only on the pulse area. This is due to the interference of the excitations caused by the corotating and counterrotating fields, and is a significant source of error, correctable only by controlling the driver phase. We present a scheme for observing this effect under currently realizable parameters.

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In order to minimize the decoherence rate of a two-state quantum bit (qubit) embodied in a massive particle, one often chooses to use low-energy transitions. In general, one is interested in performing these transitions as fast as possible [1–5] which demands a strong Rabi frequency. The ratio of Rabi frequency to qubit transition frequency is therefore not necessarily very small, thus invalidating the so-called rotating-wave approximation (RWA). A key effect due to violation of the RWA (VRWA) is the so-called Bloch-Siegert shift [6–9] which is negligible in optical transitions, but is manifested in nuclear magnetic resonance [10]. Here, we show that VRWA leads to another important effect, which can lead to controllable errors that are significant on the scale of precisions envisioned for a functioning quantum computer [11]. Specifically, we show that under VRWA the population difference between the two levels of the quantum bit, with any initial condition, depends explicitly on the phase of the driving field at the onset of an excitation pulse, which is a violation of the rule [6] that for a two-level system starting in the ground state, the population difference is a function of the integral of the field amplitude over the pulse duration and does not depend on the phase of the field. We provide a physical interpretation of this effect in terms of an interference of the excitations caused by the corotating and counterrotating fields, and present a scheme for observing this effect under currently realizable parameters.

To see the implication of this result, consider a scenario where one has a qubit, initialized to the ground state, and would like to prepare it to be in an equal superposition of the ground and excited states. To this end, one would apply a resonant pulse with an area of $\pi/2$ starting at a time $t=t_0$. Under the RWA, one does not have to know what the absolute phase of the field, ϕ_P, is at t_0, and the population difference for the qubit would be zero. Under VRWA, however, the desired excitation would only occur if $\phi_P=0$. Otherwise, the population difference would have a component varying as $\eta\sin(2\phi_P)$, where η is a parameter that is proportional to the ratio of Rabi frequency to transition frequency. Suppose one has to apply this pulse to many such qubits, with a potentially different ϕ_P for each (e.g., because the pulses are applied at different times or the qubits are spatially separated), but with identical pulse areas. The population difference for the qubits will then exhibit a fluctuation, correlated to their respective values of ϕ_P. For a quantum computer, this variation would represent a source of error. For some experiments (e.g., Ref. [5]), the value of η is already close to 0.01, so that the magnitude of this error is much larger than the ultimate accuracy (10^{-6}) desirable for a large-scale quantum computer [11] and must be controlled.

To illustrate this effect, we consider an ideal two-level system where a ground state $|0\rangle$ is coupled to a higher-energy state $|1\rangle$. We also assume that the $0\leftrightarrow1$ transition is magnetic dipolar, with a transition frequency ω, and the magnetic field is of the form $B=B_0\cos(\omega t+\phi)$. We now summarize briefly the two-level dynamics without the RWA. In the dipole approximation, the Hamiltonian can be written as

$$\hat{H}=\epsilon(\sigma_0-\sigma_z)/2+g(t)\sigma_x, \tag{1}$$

where $g(t)=-g_0[\exp(i\omega t+i\phi)+\text{c.c.}]/2$, σ_i are the Pauli matrices, and $\epsilon=\omega$ corresponds to resonant excitation. The state vector is written as

$$|\xi(t)\rangle=\begin{bmatrix}C_0(t)\\C_1(t)\end{bmatrix}. \tag{2}$$

We perform a rotating-wave transformation by operating on $|\xi(t)\rangle$ with the unitary operator $\hat{Q}$, where

$$\hat{Q}=(\sigma_0+\sigma_z)/2+\exp(i\omega t+i\phi)(\sigma_0-\sigma_z)/2. \tag{3}$$

The Schrödinger equation then takes the form (setting $\hbar=1$) $|\dot{\tilde{\xi}}\rangle=-iH(t)|\tilde{\xi}(t)\rangle$ where the effective Hamiltonian is given by

$$\tilde{H}=\alpha(t)\sigma_++\alpha^*(t)\sigma_-, \tag{4}$$

with $\alpha(t)=-(g_0/2)[\exp(-i2\omega t-i2\phi)+1]$, and in the rotating frame the state vector is

$$|\tilde{\xi}(t)\rangle \equiv \hat{Q}|\xi(t)\rangle = \begin{bmatrix} \tilde{C}_0(t) \\ \tilde{C}_1(t) \end{bmatrix}. \tag{5}$$

Now, one may choose to make the RWA, corresponding to dropping the fast-oscillating term in $\alpha(t)$. This corresponds to ignoring effects (such as the Bloch-Siegert shift) of the order of (g_0/ω), which can easily be observable in an experiment if g_0 is large [6–10]. On the other hand, by choosing g_0 to be small enough, one can make the RWA for any value of ω. We explore both regimes in this paper. As such, we find the general results without the RWA.

From Eqs. (4) and (5), one gets two coupled differential equations

$$\dot{\tilde{C}}_0(t) = -(g_0/2)[1+\exp(-i2\omega t - i2\phi)]\tilde{C}_1(t), \tag{6a}$$

$$\dot{\tilde{C}}_1(t) = -(g_0/2)[1+\exp(+i2\omega t + i2\phi)]\tilde{C}_0(t). \tag{6b}$$

We assume $|C_0(t)|^2=1$ to be the initial condition and proceed further to find an approximate analytical solution of Eq. (6). Given the periodic nature of the effective Hamiltonian, the general solution to Eq. (6) can be written in the form

$$|\tilde{\xi}(t)\rangle = \sum_{n=-\infty}^{\infty} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \exp[n(-i2\omega t - i2\phi)]. \tag{7}$$

Inserting Eq. (7) into Eq. (6) and equating coefficients with same the frequencies, one gets for all n,

$$\dot{a}_n = i2n\omega a_n + ig_0(b_n + b_{n-1})/2, \tag{8a}$$

$$\dot{b}_n = i2n\omega b_n + ig_0(a_n + a_{n+1})/2. \tag{8b}$$

Here, the coupling between a_0 and b_0 is the conventional one present when the RWA is made. The couplings to the nearest neighbors, $a_{\pm1}$ and $b_{\pm1}$, are detuned by an amount 2ω and so on. To the lowest order in (g_0/ω), we can ignore terms with $|n|>1$, thus yielding a truncated set of six equations

$$\dot{a}_0 = ig_0(b_0 + b_{-1})/2, \tag{9a}$$

$$\dot{b}_0 = ig_0(a_0 + a_1)/2, \tag{9b}$$

$$\dot{a}_1 = i2\omega a_1 + ig_0(b_1 + b_0)/2, \tag{9c}$$

$$\dot{b}_1 = i2\omega b_1 + ig_0 a_1/2, \tag{9d}$$

$$\dot{a}_{-1} = -i2\omega a_{-1} + ig_0 b_{-1}/2, \tag{9e}$$

$$\dot{b}_{-1} = -i2\omega b_{-1} + ig_0(a_{-1} + a_0)/2. \tag{9f}$$

We consider g_0 to have a time dependence of the form $g_0(t)=g_{0M}[1-\exp(-t/\tau_{sw})]$, where the switching time constant $\tau_{sw} \gg \omega^{-1}, g_{0M}^{-1}$. We can solve these equations by employing the method of adiabatic elimination, which is valid to first order in $\eta \equiv (g_0/4\omega)$. Note that η is also a function of time and can be expressed as $\eta(t)=\eta_0[1-\exp(-t/\tau_{sw})]$, where $\eta_0 \equiv (g_{0M}/4\omega)$. We consider first Eqs. (9e) and (9f). In order to simplify these two equations further, one needs to diagonalize the interaction between a_{-1} and b_{-1}. Define $\mu_- \equiv (a_{-1}-b_{-1})$ and $\mu_+ \equiv (a_{-1}+b_{-1})$, which now can be used to reexpress these two equations in a symmetric form as

$$\dot{\mu}_- = -i(2\omega + g_0/2)\mu_- - ig_0 a_0/2, \tag{10a}$$

$$\dot{\mu}_+ = -i(2\omega - g_0/2)\mu_+ + ig_0 a_0/2. \tag{10b}$$

Adiabatic following then yields (again, to lowest order in η) $\mu_- \approx -\eta a_0$ and $\mu_+ \approx \eta a_0$, which in turn yields $a_{-1}\approx 0$ and $b_{-1} \approx \eta a_0$. In the same manner, we can solve Eqs. (9c) and (9d), yielding $a_1 \approx -\eta b_0$ and $b_1 \approx 0$.

Note that the amplitudes of a_{-1} and b_1 are vanishing (each proportional to η^2) to lowest order in η and thereby justifying our truncation of the infinite set of relations in Eq. (9). It is easy to show now

$$\dot{a}_0 = ig_0 b_0/2 + i\Delta(t)a_0/2, \tag{11a}$$

$$\dot{b}_0 = ig_0 a_0/2 - i\Delta(t)b_0/2, \tag{11b}$$

where $\Delta(t)=g_0^2(t)/4\omega$ is essentially the Bloch-Siegert shift. Equation (11) can be thought of as a two-level system excited by a field detuned by Δ. For simplicity, we assume that this detuning is dynamically compensated for by adjusting the driving frequency ω. This assumption does not affect the essence of the results to follow, since the resulting correction to η is negligible. With the initial condition of all the population in $|0\rangle$ at $t=0$, the only nonvanishing (to lowest order in η) terms in the solution of Eq. (9) are

$$a_0(t) \approx \cos[g_0'(t)t/2], \quad b_0(t) \approx i\sin[g_0'(t)t/2],$$

$$a_1(t) \approx -i\eta\sin[g_0'(t)t/2], \quad b_{-1}(t) \approx \eta\cos[g_0'(t)t/2],$$

where

$$g_0'(t) = 1/t\int_0^t g_0(t')dt' = g_0[1-(t/\tau_{sw})^{-1}\exp(-t/\tau_{sw})].$$

We have verified this solution via numerical integration of Eq. (6) as shown later. Inserting this solution into Eq. (6) and reversing the rotating-wave transformation, we get the following expressions for the components of Eq. (2):

$$C_0(t) = \cos[g_0'(t)t/2] - 2\eta\Sigma\sin[g_0'(t)t/2], \tag{12a}$$

$$C_1(t) = ie^{-i(\omega t+\phi)}\{\sin[g_0'(t)t/2] + 2\eta\Sigma^*\cos[g_0'(t)t/2]\}, \tag{12b}$$

where we have defined $\Sigma \equiv (i/2)\exp[-i(2\omega t + 2\phi)]$. To lowest order in η, this solution is normalized at all times. Note that if one wants to carry this excitation on an ensemble of atoms using a $\pi/2$ pulse and measure the population of the state $|1\rangle$ after the excitation terminates [at $t=\tau$ when $g'(\tau)\tau/2=\pi/2$], the result would be an output signal given by

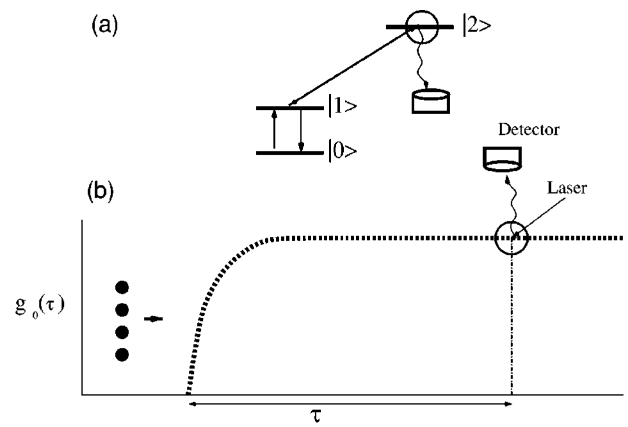

FIG. 1. Schematic illustration of an experimental arrangement for measuring the phase dependence of the population of the excited state $|1\rangle$: (a) The microwave field couples the ground state ($|0\rangle$) to the excited state ($|1\rangle$). A third level, $|2\rangle$, which can be coupled to $|1\rangle$ optically, is used to measure the population of $|1\rangle$ via fluorescence detection. (b) The microwave field is turned on adiabatically with a switching time constant τ_{sw}, and the fluorescence is monitored after a total interaction time of τ.

$$|C_1(g_0'(\tau),\phi)|^2 = \frac{1}{2}[1 + 2\eta \sin(2\phi_\tau)], \qquad (13)$$

where we have defined the phase of the field at $t=\tau$ to be $\phi_\tau \equiv \omega\tau + \phi$. This signal contains information of both the amplitude and phase of the driving field.

This result can be appreciated best by considering an experimental arrangement of the type illustrated in Fig. 1. Consider, for example, a collection of ^{87}Rb atoms, caught in a dipole force trap, where the states $|0\rangle \equiv 5^2S_{1/2}:|F=1,m=1\rangle$ and $|1\rangle \equiv 5^2S_{1/2}:|F=2,m=2\rangle$ form the two-level system. These states differ in frequencies by 6.683 47 GHz. When illuminated by resonant right-circularly polarized light at a frequency of 3.844×10^{14} Hz, state $|1\rangle$ couples only to the state $|2\rangle \equiv 5^2P_{3/2}:|F=3,m=3\rangle$, which in turn can decay only to state $|1\rangle$. This cycling transition can thus be used to pump the system into state $|1\rangle$. When a right-circularly polarized microwave field at 6.683 47 GHz is applied, state $|1\rangle$ couples only to state $|0\rangle$, even when the RWA approximation breaks down. The strong-coupling regime (e.g., η_0 of the order of 0.1) can be reached, for example, by using a superconducting, high-Q (10^{10}) microwave cavity [12]. The theoretical model developed above is then a valid description of the coupling between $|0\rangle$ and $|1\rangle$.

The strong microwave field is turned on adiabatically with a switching time constant τ_{sw}, starting at $t=0$. After an interaction time of τ, chosen so that $g_0'(\tau)\tau = \pi/2$, the population of state $|1\rangle$ can be determined by coupling this state to the state $|2\rangle$ with a short (faster than $1/\omega$ and $1/g_{0M}$) laser pulse and monitoring the resulting fluorescence [13].

We have simulated this process explicitly for the following parameters: $\omega = 2\pi\times 6.683\,47\times 10^9$ sec^{-1}, $g_{0M}=0.1$, and $\tau_{sw}=0.1$. These numbers are easily achievable experimentally. The laser pulse width τ_{LP} is chosen to be 10^{-12} sec in order to satisfy the constraint that $\tau_L \ll 1/\omega$ and $\tau_L \ll 1/g_{0M}$.

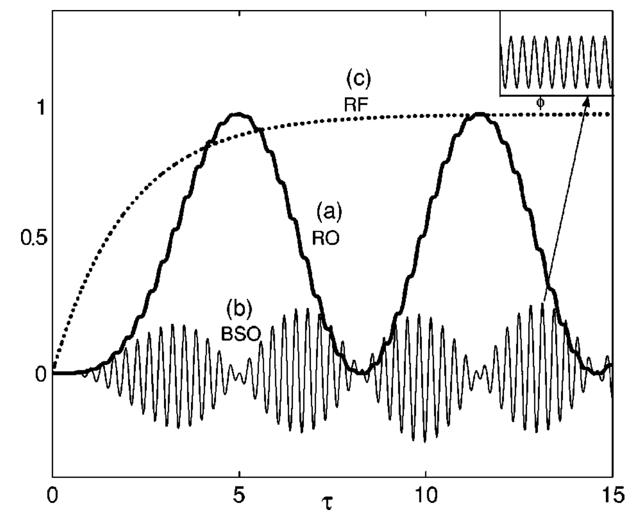

FIG. 2. Illustration of the Bloch-Siegert oscillation (BSO): (a) the population of state $|1\rangle$, as a function of the interaction time τ, showing the BSO superimposed on the conventional Rabi oscillation; (b) the BSO oscillation (amplified scale) by itself, produced by subtracting the Rabi oscillation from the plot in (a); and (c) the time dependence of the Rabi frequency. Inset: BSO as a function of the absolute phase of the field.

In order to optimize the signal, the laser Rabi frequency Ω_L is chosen to be such that $\Omega_L\tau_L = \pi$, so that all the populations of state $|1\rangle$ are excited to state $|2\rangle$ at the end of the pulse. For the cycling transition (1-2) and a pulse focused to an area of 25 μm^2, the power needed for achieving this Rabi frequency is 1.2 W, which is achievable experimentally. After the laser pulse is turned off, the fluorescence is collected for a duration longer than the spontaneous-decay lifetime (32 nsec) of state $|2\rangle$. Under this condition, our simulation verifies that the detector signal is essentially proportional to the population of state $|1\rangle$, as given by Eq. (13), with the proportionality constant determined by the efficiency of the detection system. If 10^6 atoms are used (a number easily achievable in a dipole trap), the signal-to-noise ratio can be more than 100 for the parameters considered here, assuming a detector solid angle of 0.1π and a quantum efficiency of 0.8. In Fig. 2(a), we have shown the evolution of the excited-state population as a function of the interaction time τ using the analytical expression of Eq. (12). Under the RWA, this curve would represent the conventional Rabi oscillation. However, we notice here some additional oscillations, which are magnified and shown separately in Fig. 2(b), produced by subtracting the conventional Rabi oscillation $(\sin^2[g_0'(\tau)\tau/2])$ from Fig. 2(a). That is, Fig. 2(b) corresponds to what we call the Bloch-Siegert oscillation (BSO), given by $\eta\sin[g_0'(\tau)\tau]\sin(2\phi_\tau)$. The dashed curve (c) shows the time dependence of the Rabi frequency. These analytical results agree closely with the results obtained via direct numerical integration of Eq. (7). Consider next a situation where the interaction time τ is fixed so that we are at the peak of the BSO envelope. The experiment is now repeated many times, with a different value of ϕ each time. The corresponding population of $|1\rangle$ is given by $\eta\sin(2\phi_\tau)$ and is plotted as a function of ϕ in the inset of Fig. 2. This dependence of the population of $|1\rangle$ on the initial phase ϕ (and, therefore, on the final phase ϕ_τ) makes it possible to measure these quantities.

Note, of course, that the speed of the detection system is limited fundamentally by the spontaneous decay rate γ^{-1} ($\sim$32 nsec in this case) of state $|2\rangle$. As such, it is impossible in this explicit scheme to monitor the phase of the microwave field on a time scale shorter than its period. If one were interested in monitoring the phase of a microwave field of a lower frequency (so that $\omega^{-1} \gg \gamma^{-1}$), it would be possible to track the phase on a time scale much shorter than its period. One possible set of atomic levels that can be used for this purpose is the Zeeman sublevels (e.g., those of the $5^2S_{1/2}:F=1$ hyperfine level of ^{87}Rb atoms), where the energy spacing between the sublevels can be tuned by a dc magnetic field to match the microwave field to be measured. However, the number of sublevels that get coupled is typically more than 2. A simple extension of our theoretical analysis shows that the signature of the phase of the microwave field still appears in the population of any of these levels and can be used to measure the phase. More generally, the phase signature is likely to appear in the population of the atomic levels, no matter how many levels are involved, as long as the Rabi frequency is strong enough for the RWA to break down.

A recent experiment by Martinis *et al.* [5] is an example where a qubit is driven very fast. In this experiment, a qubit is made using the two states of a current-biased Josephson junction, the resonance frequency is $\omega=6.9$ GHz, and the Rabi frequency is $g=80$ MHz. If this experiment is carried out without keeping track of the phase of the driving field, the degree of qubit excitation will fluctuate due to the BSO, leading to an error which is of the order of $g/\omega=0.01$—i.e., nearly 1%. This error is much larger than the permissible error rate of 10^{-6} for an error-correcting quantum computer that would consist of 10^6 qubits [11]. In order to eliminate the BSO-induced error, one can design the driving system such that the phase is measured, e.g., by using an auxiliary cluster of bits located close to the qubit of interest, at the onset of the qubit excitation, and the measured value of the phase is used to determine the duration of the excitation pulse, in order to ensure the desired degree of excitation of the qubit [14,15]. Finally, we point out that by making use of distant entanglement, the BSO process may enable teleportation of the phase of a field that is encoded in the atomic state amplitude, for potential applications to remote frequency locking [16–19].

In conclusion, we have shown that when a two-level atomic system is driven by a strong periodic field, the Rabi oscillation is accompanied by another oscillation at twice the transition frequency, and this oscillation carries information about the absolute phase of the driving field. One can detect this phase by simply measuring only the population of the excited state. This leads to a phase-correlated fluctuation in the excitation of a qubit and violation of the rule that the degree of excitation depends only on the pulse area. We have shown how the resulting error may be significant and must be controlled for low-energy fast qubit operations.

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