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Improving single-photon sources via linear optics and photodetection

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Triggered single-photon sources produce the vacuum state with nonzero probability, but produce a much smaller multiphoton component. It is therefore reasonable to approximate the output of these photon sources as a mixture of the vacuum and single-photon states with probabilities $1-p$ and p, respectively. Here we are concerned with increasing the efficiency p by directing multiple copies of the single-photon-vacuum mixture into a linear optical device and applying photodetection on some outputs. We prove that it is impossible, under certain conditions, to increase p via linear optics and conditional preparation based on photodetection. We also establish a class of photodetection events for which p can be improved, although with an added multiphoton component. In addition we prove that it is not possible to obtain perfect single-photon states via this method from imperfect $(p<1)$ inputs.

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Single-photon sources are important, for applications such as secure quantum key distribution [1] and linear optical quantum computation [2], yet generating single photons remains challenging. The traditional method involves photodetection on one output mode from a nondegenerate parametric down-conversion process to postselect a single photon in the correlated mode [3,4]. More recently alternative single-photon sources have been employed, including molecules [5], quantum wells [6], color centers [7], ions [8], and quantum dots [9]. Although these sources do not have as high a fidelity as can be achieved using parametric down-conversion [4], they have the advantage that they are triggered. For a triggered single-photon source, the probability of more than one photon being produced is much lower than that for a Poissonian process [10], but the vacuum contribution can be quite high. That is, the coefficients q_n of the single-mode output field density matrix $\Sigma_n q_n|n\rangle\langle n|$ are negligible for $n\geqslant 2$, but the vacuum contribution q_0 is substantial. For this study we consider an ideal single-mode single-photon source with limited efficiency p, which may be represented by the density operator

$$\hat{\rho}_p = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|. \tag{1}$$

Increasing the efficiency p is important because of requirements for quantum optics experiments, especially those concerned with quantum information processing. Much of this effort is directed to improving sources, but here we pose the question as to whether it is possible to perform postprocessing to obtain higher efficiency, while maintaining a negligible multiphoton contribution. A promising method of postprocessing is via linear optics and photodetection. It has been shown that linear optics and photodetection can be used to perform quantum computation [2], and optical controlled-NOT gates have recently been demonstrated [11]; however, there are also no-go theorems for linear optics [12].

Below we show that it is impossible to increase the single-photon efficiency p, provided we consider detection results where all but one of the photons are detected. This eliminates the most straightforward possibility for ensuring that the multiphoton contribution is negligible. If we allow other detection results, we show it is possible for low-efficiency (small p) single-photon states to yield, via linear optics and conditional preparation based on photodetection, an output with a larger probability for a single photon. However, these schemes also yield multiphoton contributions comparable to the Poisson distribution.

In the general case we start with a supply of N mixed states of the form (1). For additional generality we allow the different inputs to have different probabilities for a single photon, p_i, and we denote the maximum of these probabilities by $p_{\max}$. The initial input state may be described by

$$\begin{aligned}\hat{\rho}_{\text{in}}^{(N)} &= \bigotimes_{i=1}^{N} [(1-p_i)|0\rangle\langle 0| + p_i|1\rangle\langle 1|] \\ &= \sum_{\boldsymbol{s}} P_{\boldsymbol{s}}\left(\prod_i (\hat{a}_i^\dagger)^{s_i}|0\rangle \otimes \text{H.c.}\right),\end{aligned} \tag{2}$$

where $P_{\boldsymbol{s}}=\Pi_i p_i^{s_i}(1-p_i)^{1-s_i}$, and the vector $\boldsymbol{s}=(s_1,\ldots,s_N)^T$ $(s_i=0,1)$ gives the photon numbers in the inputs (see Fig. 1).

This input is then passed through a passive interferometer. A passive interferometer is comprised of beam splitters, mirrors, and phase shifters. Mathematically, a passive interferometer transforms the amplitude operators of the incoming fields $\hat{\boldsymbol{a}}$ via the matrix transformation $\hat{\boldsymbol{a}}^\dagger \mapsto \boldsymbol{\Lambda}^T\hat{\boldsymbol{a}}^\dagger$ with $\Lambda \in \mathrm{U}(N)$ [13], yielding

$$\hat{\rho}_{\text{trans}}^{(N)} = \sum_{\boldsymbol{s}} P_{\boldsymbol{s}}\left[\prod_i \left(\sum_k \Lambda_{ki}\hat{a}_k^\dagger\right)^{s_i}|0\rangle \otimes \text{H.c.}\right]. \tag{3}$$

Without loss of generality, we take mode 1 to be the mode in which we wish to improve the photon statistics. We perform

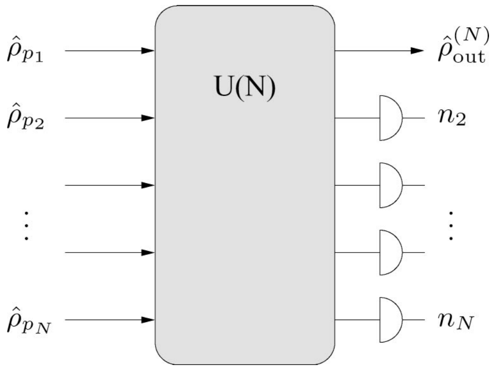

FIG. 1. Schematic setup of the network. We assume N incoming modes prepared in the state (2) with different p_i.

photodetections on the other $N-1$ modes, and examine the final state in mode 1 conditioned on the results of these photodetections. It is easy to see that no better result can be obtained by performing photodetections on fewer than $N-1$ modes; this would be equivalent to averaging over the photocounts for some of the modes. We assume ideal photodetection in this analysis in order to determine the best results possible using linear optics and photodetection.

Before determining the conditional output state, we first introduce some notation. The total number of photons detected is D, and the maximum possible number of photons in the N outputs is M. As some of the p_i may be equal to zero, M may be less than N; M is equal to the number of nonzero values of p_i. For $j>1$, n_j is the number of photons detected in mode j, and n_1 is the photon number in mode 1 (the output mode). We use the notation $\Sigma_{\boldsymbol{n}}=\Sigma_i n_i$ (so $\Sigma_{\boldsymbol{n}}=D+n_1$) and $\Sigma_{\boldsymbol{s}}=\Sigma_i s_i$. In addition, we define the set $\Phi_{\boldsymbol{s}}=\{i|s_i=1\}$, and let $Y_{\boldsymbol{s}}$ be the set that consists of all vectors comprised of the elements of $\Phi_{\boldsymbol{s}}$.

The conditional state in mode 1 after photodetection in modes 2 to N is

$$\hat{\rho}_{\text{out}}^{(N)} = \sum_{n_1=0}^{N} c_{n_1}|n_1\rangle\langle n_1|. \tag{4}$$

Each coefficient c_{n_1} is given by

$$c_{n_1} = K\langle \boldsymbol{n}|\hat{\rho}_{\text{trans}}^{(N)}|\boldsymbol{n}\rangle, \tag{5}$$

where $|\boldsymbol{n}\rangle$ is a tensor product of number states in each of the output modes and K is a normalization constant. Evaluating c_{n_1} gives

$$c_{n_1} = \frac{K'}{n_1!}\sum_{\boldsymbol{s};\Sigma_{\boldsymbol{s}}=\Sigma_{\boldsymbol{n}}} P_{\boldsymbol{s}}|S_{\boldsymbol{s},\boldsymbol{n}}|^2, \tag{6}$$

where $K'=K/\Pi_{j=2}^{N} n_j!$, and

$$S_{\boldsymbol{s},\boldsymbol{n}} = \sum_{\boldsymbol{\sigma}\in Y_{\boldsymbol{s}}} (\Lambda_{1,\sigma_1}\cdots\Lambda_{1,\sigma_{n_1}})\cdots(\Lambda_{N,\sigma_{\Sigma_{\boldsymbol{s}}-n_N+1}}\cdots\Lambda_{N,\sigma_{\Sigma_{\boldsymbol{s}}}}). \tag{7}$$

Two figures of merit for the output states are

$$R_{\text{out}} = \frac{c_1}{c_0}, \quad G_{\text{out}} = \frac{c_2/c_1}{c_1/c_0}. \tag{8}$$

The quantity R_{out} is the final ratio between the probabilities for one and zero photons. Similarly, we define the maximum initial ratio $R_{\text{in}}=p_{\max}/(1-p_{\max})$. The figure of merit G_{out} characterizes the two-photon contribution, and is equal to $1/2$ for Poisson photon statistics. For the input $G_{\text{in}}=0$, as there is no two-photon component.

If the multiphoton component in the output is zero, then comparing R_{in} and R_{out} immediately tells us if there is an improvement in the probability for a single photon. Even if the multiphoton component is nonzero, using R_{out} has the following advantages.

(1) The common constant K' cancels, so it is possible to evaluate this quantity analytically.

(2) If $R_{\text{out}}\leqslant R_{\text{in}}$, then it is clear that $c_1\leqslant p_{\max}$. Thus we can determine those cases where there is *no* improvement.

(3) For $p_{\max}\ll 1$, $c_0\approx 1$ and $R_{\text{in}}\approx p_{\max}$. Therefore the improvement in R_{out} over R_{in} is approximately the same as the improvement in the probability of a single photon over $p_{\max}$.

Ideally, we would determine the interferometer and detection pattern such that R_{out} is maximized, but this does not appear to be possible analytically. However, we can place an upper limit on R_{out} in the following way. Let us express the summation for c_0 as

$$c_0 = \frac{K'}{N-D}\sum_{\boldsymbol{s};\Sigma_{\boldsymbol{s}}=D+1}\sum_{k;s_k=1} P_{\boldsymbol{s}^k}|S_{\boldsymbol{s}^k,\boldsymbol{n}}|^2, \tag{9}$$

where $s_i^k=s_i$ except for $s_k^k=0$, n_j is the combination of detections for $j>1$, and $n_1=0$. The quotient of $N-D$ takes account of a redundancy in the sum. Each alternative input $\boldsymbol{s}^k$ has $N-D$ zeros, so there are $N-D$ possible alternative values of $\boldsymbol{s}$ that give the same $\boldsymbol{s}^k$. We may reduce this quotient slightly if we take account of the fact that some of the inputs may be vacuum states. The maximum total number of photons is M, so there are $N-M$ inputs with $p_i=0$. Therefore, if we limit the first sum in Eq. (9) to $\boldsymbol{s}$ such that $P_{\boldsymbol{s}}\neq 0$, then the redundancy is $M-D$. Therefore we obtain

$$\begin{aligned} c_0 &= \frac{K'}{M-D}\sum_{\substack{\boldsymbol{s};P_{\boldsymbol{s}}\neq 0\\ \Sigma_{\boldsymbol{s}}=D+1}}\sum_{k;s_k=1} P_{\boldsymbol{s}^k}|S_k|^2 \\ &= \frac{K'}{M-D}\sum_{\substack{\boldsymbol{s};P_{\boldsymbol{s}}\neq 0\\ \Sigma_{\boldsymbol{s}}=D+1}} P_{\boldsymbol{s}}\sum_{k;s_k=1}\frac{1-p_k}{p_k}|S_k|^2, \end{aligned} \tag{10}$$

where $S_k=S_{\boldsymbol{s}^k,\boldsymbol{n}}$. Since we have limited the sum to terms where $P_{\boldsymbol{s}}\neq 0$, p_k is nonzero, and thus the ratio $(1-p_k)/p_k$ does not diverge. Since p_k does not exceed $p_{\max}$, we have the inequality

$$c_0 \geqslant \frac{K'/R_{\text{in}}}{M-D}\sum_{\boldsymbol{s};\Sigma_{\boldsymbol{s}}=D+1} P_{\boldsymbol{s}}\sum_{k;s_k=1}|S_k|^2. \tag{11}$$

Here we are able to omit the condition $P_{\boldsymbol{s}}\neq 0$ because terms with $P_{\boldsymbol{s}}=0$ are zero anyway. We may reexpress the equation for c_1 as

$$c_1 = K' \sum_{\mathbf{s};\Sigma_{\mathbf{s}}=D+1} P_{\mathbf{s}} \left| \sum_{k;s_k=1} \Lambda_{1k} S_k \right|^2. \tag{12}$$

We therefore obtain

$$c_1 \leq K' \sum_{\mathbf{s};\Sigma_{\mathbf{s}}=D+1} P_{\mathbf{s}} \sum_{k;s_k=1} |S_k|^2. \tag{13}$$

Combining Eqs. (11) and (13) gives

$$R_{\rm out} = \frac{c_1}{c_0} \leq R_{\rm in}(M-D). \tag{14}$$

This yields an upper limit on the ratio between the one- and zero-photon probabilities. One application of this result is that it is impossible to obtain one photon with unit probability, as it would be necessary for this ratio to be infinite. Another consequence of Eq. (14) is that, for $D=M-1$ (i.e., the number of photons detected one less than the maximum input number), an improvement can never be achieved. This case is important because it is the most straightforward way of eliminating the possibility of two or more photons in the output mode.

Next we investigate situations in which the single-photon contribution can be enhanced. As $M \leq N$ and $D \geq 0$, the upper limit on the improvement in $R_{\rm out}$ is simply N. This is also the upper limit on how far c_1 can be increased above $p_{\rm max}$. We now consider a scheme that gives a linear improvement in $R_{\rm out}$, though not as high as N. In order to obtain a large value for the ratio $R_{\rm out}$, we want the inequality in Eq. (13) to be as close to equality as possible. In turn, this means that we want the vectors (Λ_{1k}) and (S_k) to be as close to parallel as possible. For this, we consider the interferometer given by

$$\Lambda_{12} = -\epsilon, \quad \Lambda_{22} = \sqrt{1-\epsilon^2},$$

$$\Lambda_{1i} = \sqrt{(1-\epsilon^2)/(N-1)}, \quad \Lambda_{2i} = \epsilon/\sqrt{N-1}, \tag{15}$$

for $i \neq 2$ (the values of Λ_{ji} for $j > 2$ do not enter into the analysis). Here ϵ is a small number, and we will ignore terms of order ϵ or higher. Now let $p_i = p_{\rm max}$, and consider the measurement record where zero photons are detected in modes 3 to N, so the number of photons detected in mode 2 is D. To determine c_{n_1}, note first that $\Lambda_{22} \gg \Lambda_{2i}$ for $i \neq 2$, so we may ignore those terms in the sum for $S_{\mathbf{s},\mathbf{n}}$ where Λ_{22} does not appear. Each term has magnitude $\Lambda_{11}^{n_1}\Lambda_{22}\Lambda_{21}^{D-1}$ [14], and there are $D(D+n_1-1)!$ such terms. Therefore, provided $s_2=1$,

$$S_{\mathbf{s},\mathbf{n}} \approx D(D+n_1-1)!\,\Lambda_{11}^{n_1}\Lambda_{22}\Lambda_{21}^{D-1}. \tag{16}$$

In the summation for c_{n_1}, we have $\binom{N-1}{D+n_1-1}$ different combinations of inputs such that $\Sigma_{\mathbf{s}} = D+n_1$ and $s_2=1$. Combining these results, we have

$$\begin{aligned} c_{n_1} &\approx \frac{K'}{n_1!} p_{\rm max}^{D+n_1}(1-p_{\rm max})^{N-D-n_1} \\ &\quad \times \frac{(N-1)!\,D^2(D+n_1-1)!}{(N-D-n_1)!} \Lambda_{11}^{2n_1}\Lambda_{22}^2\Lambda_{21}^{2D-2} \\ &\approx K''\left(\frac{R_{\rm in}}{N-1}\right)^{n_1} \frac{(D+n_1-1)!}{n_1!\,(N-D-n_1)!}. \end{aligned} \tag{17}$$

We have combined those factors that do not depend on n_1 into a new constant K'', and used $\Lambda_{11} \approx 1/\sqrt{N-1}$.

Using Eq. (17) we find that

$$R_{\rm out} \approx R_{\rm in}\frac{D(N-D)}{N-1}. \tag{18}$$

The maximum improvement in $R_{\rm out}$ is obtained for $D = \lceil N/2 \rceil$, where $R_{\rm out} \approx R_{\rm in} \lfloor N^2/4 \rfloor/(N-1)$. Here $\lceil\ \rceil$ denotes the ceiling function and $\lfloor\ \rfloor$ denotes the floor function. The multiplicative factor $\lfloor N^2/4 \rfloor/(N-1)$ is larger than 1 for all $N \geq 4$. Thus we find that, provided there are at least four modes, we may obtain an improvement in $R_{\rm out}$. For $p_{\rm max} \ll 1$, $c_1 \approx p_{\rm max}\lfloor N^2/4 \rfloor/(N-1)$. For large N, the probability of a single photon increases approximately as $N/4$. This is linear with N, but does not achieve the upper bound of N.

Although we find an improvement in the measure $R_{\rm out}$, the two-photon contribution is not negligible. Using the measure $G_{\rm out}$, we find

$$G_{\rm out} = \frac{c_2/c_1}{c_1/c_0} \approx \frac{(D+1)(N-D-1)}{2D(N-D)}. \tag{19}$$

For $D = \lceil N/2 \rceil$, this measure is close to $1/2$, so the two-photon component is similar to that for a Poisson distribution. By taking $D = N-2$, it is possible to obtain an improvement in $R_{\rm out}$ of about a factor of 2, with a value of $G_{\rm out}$ about half that for a Poisson distribution. However, this two-photon contribution is still much greater than for good single-photon sources [10].

The multiphoton contributions are especially important for larger $p_{\rm max}$. Although the improvement in $R_{\rm out}$ is independent of $p_{\rm max}$, the multiphoton component means that improvements in c_1 are obtained only for values of $p_{\rm max}$ below $1/2$. That is, this method can only be used to obtain improvements in the probability of a single photon up to $1/2$, but not to make the probability of a single photon arbitrarily close to 1.

The above method only gives $c_1 > p_{\rm max}$ for four or more modes. We will now show that it is impossible to obtain an improvement in the probability of a single photon with fewer than four modes, and for various combinations of detections with larger numbers of modes. We first examine the case $D=0$. Then there is only one term in the sum for c_0, and $c_0 = K' P_{\mathbf{0}}$. The expression for c_1 becomes

$$\begin{aligned} c_1 &= K' \sum_{k=1}^{N} \frac{p_k}{1-p_k} P_{\mathbf{0}} |\Lambda_{1k}|^2 \leq K' R_{\rm in} \sum_{k=1}^{N} P_{\mathbf{0}} |\Lambda_{1k}|^2 \\ &= K' R_{\rm in} P_{\mathbf{0}} = c_0 R_{\rm in}. \end{aligned} \tag{20}$$

Thus we have shown that $R_{\rm out} \leq R_{\rm in}$, so $c_1 \leq p_{\rm max}$. Hence there can be no improvement in the photon statistics if zero photons are detected.

We can also obtain a similar result for the case $D=1$, provided all the input p_i are equal. In that case, we have

$$c_0 = K' \sum_k \frac{p_{\rm max}}{1-p_{\rm max}} P_{\mathbf{0}} |\Lambda_{2k}|^2 = K' R_{\rm in} P_{\mathbf{0}}. \tag{21}$$

The value of c_1 is given by

$$c_1 = \frac{1}{2}K'\sum_k \sum_{l;l\neq k} R_{\rm in}^2 P_{\mathbf{0}} |\Lambda_{1l}\Lambda_{2k} + \Lambda_{1k}\Lambda_{2l}|^2 \leq \frac{1}{2}K' R_{\rm in}^2 P_{\mathbf{0}} \sum_{k,l} |\Lambda_{1l}\Lambda_{2k} + \Lambda_{1k}\Lambda_{2l}|^2 = K' R_{\rm in}^2 P_{\mathbf{0}}. \quad (22)$$

Thus we again find $R_{\rm out} \leq R_{\rm in}$, so $c_1 \leq p_{\rm max}$.

These results clearly eliminate the possibility of improving the probability of finding one photon with a two-mode interferometer. We have shown that detecting zero photons does not give an improvement, and if one photon is detected, then we must have $M-D=1$ or 0, so there again can be no improvement. Along the same lines we can also eliminate the three-mode interferometer.

We have shown that it is impossible to improve the efficiency of a single-photon source by channelling more than one low-efficiency single-photon state into a linear optical interferometer and detecting all but one of the photons. This eliminates the most straightforward scheme for obtaining an output state with no more than one photon. It is possible to obtain an improvement for more general detection results, but at the expense of nonzero probabilities for two or more photons. We have not proven that it is impossible to obtain an improvement in the probability of a single photon while restricting to zero probability for two or more photons; however, numerical searches indicate that it is unlikely.

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[14] Here we are using Λ_{21} and Λ_{11} to indicate the values of Λ_{2i} and Λ_{1i} for $i \neq 2$.