Frequency and damping of hydrodynamic modes in a trapped Bose-condensed gas

Tetsuro Nikuni
Department of Physics, Faculty of Science, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo, Japan 162-9601

Allan Griffin
Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7

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Recently it was shown that the Landau-Khalatnikov two-fluid hydrodynamics describes the collision-dominated region of a trapped Bose condensate interacting with a thermal cloud. We use these equations to discuss the low frequency hydrodynamic collective modes in a trapped Bose gas at finite temperatures. We derive variational expressions based on these equations for both the frequency and damping of collective modes. A new feature is our use of frequency-dependent transport coefficients, which produce a natural cutoff by eliminating the collisionless low-density tail of the thermal cloud. Above the superfluid transition, our expression for the damping in trapped inhomogeneous gases is analogous to the result first obtained by Landau and Lifshitz for uniform classical fluids. We also use the moment method to discuss the crossover from the collisionless to the hydrodynamic region. Recent data for the monopole-quadrupole mode in the hydrodynamic region of a trapped gas of metastable ^{4}He is discussed. We also present calculations for the damping of the analogous $m=0$ monopole-quadrupole condensate mode in the superfluid phase.

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I. INTRODUCTION

In recent papers, Zaremba and the authors have derived a closed set of the two-fluid hydrodynamic equations of a trapped Bose-condensed gas starting from a simplified microscopic model describing the coupled dynamics of the condensate and noncondensate atoms [1–5]. These equations can be written in the Landau-Khalatnikov (LK) form, well known in the study of superfluid ^{4}He [6,7]. These simplified hydrodynamic equations include dissipative terms associated with the shear viscosity, the thermal conductivity, and the four second-viscosity coefficients. Explicit formulas for these transport coefficients were obtained in Ref. [5] and used to define three characteristic transport relaxation times [8]. These define the crossover between the collisionless and hydrodynamic regions. Detailed calculations of these transport relaxation times in a trapped Bose gas [8] show that the collisions between the condensate and noncondensate enhance the transport relaxation rates significantly in the MIT data [9], so that one is in the hydrodynamic region below the Bose-Einstein condensation temperature T_{BEC}. We also note that the recent Bose condensate observed in metastable He* [10–12] appears to be well within the collision-dominated hydrodynamic region, even above T_{BEC}. This is because of the relatively large density of He* atoms and their large *s*-wave scattering length.

The present paper calculates the damping of collective modes in trapped Bose gases in the collisional hydrodynamic limit described by local equilibrium. In this limit, as noted above, the generalized GP equation for the condensate plus the kinetic equation for the thermal cloud (derived in Ref. [3]) have been proven [5] to be equivalent to the two-fluid hydrodynamic equations (including damping due to transport processes) of Landau and Khalatnikov [6,7]. This equivalence is important since it shows that the simplified theory of nonequilibrium behavior developed in Ref. [3] properly describes the local equilibrium hydrodynamic region and the approach to equilibrium. There are several other alternative theories of the nonequilibrium dynamics of trapped Bose gases [13–15]. However, up to the present, these formulations have not been used to derive the analogue of the LK two-fluid hydrodynamic equations.

In the present paper, we derive a general expression for the frequency and damping of hydrodynamic collective modes in a trapped Bose-condensed gas at finite temperatures, starting from the two-fluid hydrodynamic equations derived in Ref. [5]. These two-fluid equations are briefly reviewed in Sec. II, and reformulated as a closed set of equations for the condensate and noncondensate velocity fields. In Sec. III, we derive a variational expression for undamped normal-mode frequencies in the Landau limit ($\omega\tau\ll 1$), extending an approach first developed in Ref. [3]. In Sec. IV, we obtain a general expression for the damping, which only depends on knowing the undamped normal-mode solutions. This kind of expression is very convenient in working out the damping of hydrodynamic modes in trapped Bose gases, as first pointed out by Kavoulakis *et al.* [16]. As an illustration, in Sec. VI we give a detailed discussion of the the $m=0$ monopole-quadrupole collective mode above T_{BEC} studied in the recent experiments [12]. In Sec. VII, we also calculate the damping of the $m=0$ hydrodynamic mode in the superfluid phase. In Sec. VIII, we review some puzzling aspects of the recent data analysis given in Refs. [10–12] in the light of the present calculations.

Appendix A gives some details of the damping calculations based on the use of frequency-dependent transport coefficients. The moment method for a degenerate normal Bose gas is reviewed in Appendix B.

II. TWO-FLUID HYDRODYNAMICS OF A TRAPPED BOSE GAS: A REVIEW

In this paper, we consider a Bose-condensed gas confined in an external anisotropic harmonic trap potential

$$U_{\mathrm{ext}}(\mathbf{r})=\frac{m}{2}(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2). \tag{1}$$

Our starting point is the two-fluid hydrodynamics derived from the finite-temperature kinetic theory by Zaremba, Nikuni, and Griffin (ZNG) [3]. In the ZNG theory, the coupled dynamics of the condensate and noncondensate [3] is described by the generalized Gross-Pitaevskii (GP) equation for the condensate wave function $\Phi(\mathbf{r},t)$

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t}=\left[-\frac{\hbar^2\nabla^2}{2m}+U_{\mathrm{ext}}(\mathbf{r})+gn_c(\mathbf{r},t)+2g\tilde{n}(\mathbf{r},t)-iR(\mathbf{r},t)\right]\Phi(\mathbf{r},t), \tag{2}$$

and the semiclassical kinetic equation for the noncondensate distribution function $f(\mathbf{r},\mathbf{p},t)$,

$$\frac{\partial f(\mathbf{r},\mathbf{p},t)}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}f(\mathbf{r},\mathbf{p},t)-\nabla U(\mathbf{r},t)\cdot\nabla_{\mathbf{p}}f(\mathbf{r},\mathbf{p},t)=C_{12}[f,\Phi]+C_{22}[f]. \tag{3}$$

Here $n_c(\mathbf{r},t)=|\Phi(\mathbf{r},t)|^2$ is the condensate density, and $\tilde{n}(\mathbf{r},t)$ is the noncondensate density,

$$\tilde{n}(\mathbf{r},t)=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f(\mathbf{r},\mathbf{p},t), \tag{4}$$

and $U(\mathbf{r},t)=U_{\mathrm{ext}}(\mathbf{r})+2g[n_c(\mathbf{r},t)+\tilde{n}(\mathbf{r},t)]$ is the time-dependent effective potential acting on the noncondensate, including the Hartree-Fock (HF) mean field. As usual, we approximate the interaction in the s-wave scattering approximation $g=4\pi\hbar^2a/m$. The dissipative term $R(\mathbf{r},t)$ in the generalized GP equation (2) is due to the collisional exchange of atoms in the condensate and noncondensate. This is related to the C_{12} collision integral in Eq. (3), namely,

$$R(\mathbf{r},t)=\frac{\hbar\Gamma_{12}(\mathbf{r},t)}{2n_c(\mathbf{r},t)},$$

$$\Gamma_{12}(\mathbf{r},t)=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}C_{12}[f(\mathbf{r},\mathbf{p},t),\Phi(\mathbf{r},t)]. \tag{5}$$

Explicit expressions for the two collision integrals (C_{22} and C_{12}) in the kinetic equation (3) can be found in Ref. [3].

The GP equation (5) can be written in the hydrodynamic form in terms of the amplitude and phase of $\Phi(\mathbf{r},t)=\sqrt{n_c(\mathbf{r},t)}e^{i\theta(\mathbf{r},t)}$, which leads to

$$\frac{\partial n_c}{\partial t}+\nabla\cdot(n_c\mathbf{v}_c)=-\Gamma_{12}[f,\Phi], \tag{6a}$$

$$m\left(\frac{\partial}{\partial t}+\mathbf{v}_c\cdot\nabla\right)\mathbf{v}_c=-\nabla\mu_c, \tag{6b}$$

where the superfluid velocity is $\mathbf{v}_c\equiv\hbar\nabla\theta(\mathbf{r},t)/m$ and the condensate chemical potential is given by

$$\mu_c(\mathbf{r},t)=-\frac{\hbar^2\nabla^2\sqrt{n_c(\mathbf{r},t)}}{2m\sqrt{n_c(\mathbf{r},t)}}+U_{\mathrm{ext}}(\mathbf{r})+gn_c(\mathbf{r},t)+2g\tilde{n}(\mathbf{r},t). \tag{7}$$

One sees that Γ_{12} in Eq. (6a) plays the role of a "source function" in the continuity equation for the condensate, arising from the fact that C_{12} collisions do not conserve the number of condensate atoms [3].

Hydrodynamic equations for the noncondensate can be derived by following the standard procedure first developed in the kinetic theory of classical gases. We take moments of the kinetic equation (3) with respect to 1,$\mathbf{p}$ and p^2 to derive the most general form of "hydrodynamic-type" equations for the noncondensate. These moment equations take the form (μ and ν are Cartesian components)

$$\frac{\partial\tilde{n}}{\partial t}+\nabla\cdot(\tilde{n}\mathbf{v}_n)=\Gamma_{12}[f,\Phi], \tag{8a}$$

$$m\tilde{n}\left(\frac{\partial}{\partial t}+\mathbf{v}_n\cdot\nabla\right)v_{n\mu}=-\frac{\partial P_{\mu\nu}}{\partial x_\nu}-\tilde{n}\frac{\partial U}{\partial x_\mu}-m(v_{n\mu}-v_{c\mu})\Gamma_{12}[f,\Phi], \tag{8b}$$

$$\frac{\partial\tilde{\epsilon}}{\partial t}+\nabla\cdot(\tilde{\epsilon}\mathbf{v}_n)=-\nabla\cdot\mathbf{Q}-D_{\mu\nu}P_{\mu\nu}+\left[\frac{1}{2}m(\mathbf{v}_n-\mathbf{v}_c)^2+\mu_c-U\right]\Gamma_{12}[f,\Phi]. \tag{8c}$$

Here and elsewhere, repeated Greek subscripts are summed. The noncondensate density was defined earlier in Eq. (4), while the noncondensate local velocity $\mathbf{v}_n(\mathbf{r},t)$ is defined by

$$\tilde{n}(\mathbf{r},t)\mathbf{v}_n(\mathbf{r},t)\equiv\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{\mathbf{p}}{m}f(\mathbf{r},\mathbf{p},t). \tag{9}$$

In addition, we have introduced the following quantities:

$$P_{\mu\nu}(\mathbf{r},t)\equiv m\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left(\frac{p_\mu}{m}-v_{n\mu}\right)\left(\frac{p_\nu}{m}-v_{n\nu}\right)f(\mathbf{r},\mathbf{p},t), \tag{10a}$$

$$\mathbf{Q}(\mathbf{r},t)\equiv\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2\left(\frac{\mathbf{p}}{m}-\mathbf{v}_n\right)f(\mathbf{r},\mathbf{p},t), \tag{10b}$$

$$\tilde{\epsilon}(\mathbf{r},t)\equiv\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2f(\mathbf{r},\mathbf{p},t). \tag{10c}$$

Finally, the symmetric rate-of-strain tensor appearing in Eq. (8c) is defined as

$$D_{\mu\nu}(\mathbf{r},t)\equiv\frac{1}{2}\left(\frac{\partial v_{n\mu}}{\partial x_\nu}+\frac{\partial v_{n\nu}}{\partial x_\mu}\right). \tag{11}$$

The "hydrodynamic" equations (8)–(11) are exact, but not closed as they stand.

We next apply the Chapman-Enskog procedure to obtain a closed set of hydrodynamic equations. This procedure yields [5]

$$P_{\mu\nu}=\delta_{\mu\nu}\tilde{P}-2\eta\left(D_{\mu\nu}-\frac{1}{3}\mathrm{Tr}\,D\,\delta_{\mu\nu}\right), \tag{12}$$

$$\mathbf{Q}=-\kappa\boldsymbol{\nabla}T, \tag{13}$$

where $\tilde{P}$ is the kinetic pressure and T is the temperature. The above formulas involve the position-dependent shear viscosity η and the thermal conductivity κ. These local position-dependent transport coefficients will be discussed in more detail below.

In order to study small amplitude oscillations, we linearize the hydrodynamic equations around static equilibrium as $n_c=n_{c0}+\delta n_c$, $\mathbf{v}_c=\delta\mathbf{v}_c$, $\tilde{n}=\tilde{n}_0+\delta\tilde{n}$, $\mathbf{v}_n=\delta\mathbf{v}_n$, $\tilde{P}=\tilde{P}_0+\delta P$, where the subscript 0 denotes static equilibrium. The equilibrium condensate density profile is determined by (within the Thomas-Fermi approximation, which neglects the quantum pressure or kinetic energy term in the GP equation)

$$n_{c0}(\mathbf{r})=\frac{1}{g}[\mu_{c0}-U_{\mathrm{ext}}(\mathbf{r})]-2\tilde{n}_0(\mathbf{r}), \tag{14}$$

while the equilibrium noncondensate distribution function f_0 is given by

$$f_0(\mathbf{r},\mathbf{p})=\frac{1}{e^{\beta_0[p^2/2m+U_0(\mathbf{r})-\mu_{c0}]}-1}. \tag{15}$$

The local density $\tilde{n}_0(\mathbf{r})$ and the local kinetic pressure $\tilde{P}_0(\mathbf{r})$ of the noncondensate atoms are given from Eq. (15) as

$$\tilde{n}_0(\mathbf{r})=\frac{1}{\Lambda^3}g_{3/2}(z_0), \tag{16}$$

$$\tilde{P}_0(\mathbf{r})=\frac{k_{\mathrm{B}}T_0}{\Lambda^3}g_{5/2}(z_0), \tag{17}$$

where $z_0(\mathbf{r})=e^{[\mu_{c0}-U_0(\mathbf{r})]/k_{\mathrm{B}}T_0}$ is the local equilibrium fugacity, $\Lambda=(2\pi\hbar^2/mk_{\mathrm{B}}T_0)^{1/2}$ is the thermal de Broglie wavelength, and $g_n(z)=\sum_{l=1}^{\infty}z^l/l^n$.

The linearized hydrodynamic equations for the condensate are given by

$$\frac{\partial\delta n_c}{\partial t}=-\boldsymbol{\nabla}\cdot(n_{c0}\delta\mathbf{v}_c)-\delta\Gamma_{12}, \tag{18a}$$

$$m\frac{\partial\delta\mathbf{v}_c}{\partial t}=-g\boldsymbol{\nabla}(\delta n_c+2\delta\tilde{n}), \tag{18b}$$

while the linearized hydrodynamic equations for the noncondensate atoms are given by

$$\frac{\partial\delta\tilde{n}}{\partial t}+\boldsymbol{\nabla}\cdot(\tilde{n}_0\delta\mathbf{v}_n)=\delta\Gamma_{12}, \tag{19a}$$

$$\begin{aligned}m\tilde{n}_0\frac{\partial\delta v_{n\mu}}{\partial t}=&-\frac{\partial\delta\tilde{P}}{\partial x_\mu}-\delta\tilde{n}\frac{\partial U_0}{\partial x_\mu}-2g\tilde{n}_0\frac{\partial\delta n}{\partial x_\mu}\\&+\frac{\partial}{\partial x_\nu}\left\{2\eta\left[D_{\mu\nu}-\frac{1}{3}(\mathrm{Tr}D)\delta_{\mu\nu}\right]\right\},\end{aligned} \tag{19b}$$

$$\begin{aligned}\frac{\partial\delta\tilde{P}}{\partial t}=&-\frac{5}{3}\boldsymbol{\nabla}\cdot(\tilde{P}_0\delta\mathbf{v}_n)+\frac{2}{3}\delta\mathbf{v}_n\cdot\boldsymbol{\nabla}\tilde{P}_0+(\mu_{c0}-U_0)\delta\Gamma_{12}\\&+\frac{2}{3}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}\delta T).\end{aligned} \tag{19c}$$

The above equations involve the fluctuations of the source function $\delta\Gamma_{12}$ and the temperature δT. These can also be written in terms of the condensate and noncondensate velocity fields [5]. One finds $\delta\Gamma_{12}=\delta\Gamma_{12}^{(1)}+\delta\Gamma_{12}^{(2)}$, where

$$\delta\Gamma_{12}^{(1)}=\sigma_H\left\{\boldsymbol{\nabla}\cdot[n_{c0}(\delta\mathbf{v}_c-\delta\mathbf{v}_n)]+\frac{1}{3}n_{c0}\boldsymbol{\nabla}\cdot\delta\mathbf{v}_n\right\}, \tag{20}$$

$$\delta\Gamma_{12}^{(2)}=-\tau_\mu\frac{\partial}{\partial t}\delta\Gamma_{12}^{(1)}-\frac{2\sigma_H\sigma_1}{3g\tilde{n}_0}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}\delta T), \tag{21}$$

$$\frac{\partial\delta T}{\partial t}=-\frac{2}{3}T_0\boldsymbol{\nabla}\cdot\delta\mathbf{v}_n+\frac{2T_0}{3\tilde{n}_0}\sigma_1\delta\Gamma_{12}^{(1)}. \tag{22}$$

Here σ_H and σ_1 only involve the static local thermodynamic functions of the gas, and are defined in Eqs. (25) and (51) of Ref. [5]. In Eq. (20), τ_μ is a new relaxation time describing how fast the condensate and noncondensate atoms reach diffusive local equilibrium with each other. More explicitly, it is given by

$$\frac{1}{\tau_\mu}=\left(\frac{gn_{c0}}{k_{\mathrm{B}}T}\right)\frac{1}{\tau_{12}\sigma_H}, \tag{23}$$

where τ_{12} is a collision time of the condensate atoms with the noncondensate atoms as defined in Ref. [5].

The above two-fluid hydrodynamic equations have dissipative terms involving the shear viscosity η and the thermal conductivity κ. These transport coefficients are given by the following expressions [5,8]:

$$\kappa=\frac{5}{2}\tau_\kappa\frac{\tilde{n}_0k_{\mathrm{B}}^2T_0}{m}\left\{\frac{7g_{7/2}(z_0)}{2g_{3/2}(z_0)}-\frac{5}{2}\left[\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]^2\right\}, \tag{24}$$

$$\eta=\tau_\eta\tilde{n}_0k_{\mathrm{B}}T_0\left[\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right], \tag{25}$$

where z_0 is the local fugacity defined earlier. These expressions involve the characteristic transport relaxation times τ_κ and τ_η, which are defined in Refs. [5,8]. The three relaxation times τ_κ, τ_η and τ_μ characterize how fast the two-component system reaches local equilibrium, and thus they define the crossover frequency between the collision-dominated hydrodynamic region and the so-called collisionless region dominated by mean fields. We note that the two-fluid equations (18) and (19) can also be rewritten in the LK form [5,6], which involve the four second viscosity coefficients $\zeta_1, \zeta_2, \zeta_3$, and ζ_4. As shown in Ref. [5], they are all related to the relaxation time τ_μ as follows:

$$\zeta_1 = \zeta_4 = \frac{g n_{c0}}{3m}\sigma_H \tau_\mu ,$$

$$\zeta_2 = \frac{g n_{c0}^2}{9}\sigma_H \tau_\mu , \quad \zeta_3 = \frac{g}{m^2}\sigma_H \tau_\mu . \tag{26}$$

To derive a closed set of equations for the velocity fields $\mathbf{v}_c$ and $\mathbf{v}_n$, we take time derivatives of Eqs. (18b) and (19b). We obtain

$$m\frac{\partial^2 \delta \mathbf{v}_c}{\partial t^2} = g\boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot(n_{c0}\delta\mathbf{v}_c) + 2\boldsymbol{\nabla}\cdot(\tilde{n}_0 \delta\mathbf{v}_n)] - g\boldsymbol{\nabla}\delta\Gamma_{12} \tag{27}$$

and

$$\begin{aligned} m\tilde{n}_0 \frac{\partial^2 \delta v_{n\mu}}{\partial t^2} = {} & \frac{5}{3}\frac{\partial}{\partial x_\mu}[\boldsymbol{\nabla}\cdot(\tilde{P}_0 \mathbf{v}_n)] - \frac{2}{3}\frac{\partial}{\partial x_\mu}(\delta\mathbf{v}_n\cdot\boldsymbol{\nabla}\tilde{P}_0) \\ & + \boldsymbol{\nabla}\cdot(\tilde{n}_0\delta\mathbf{v}_n)\frac{\partial U_0}{\partial x_\mu} + 2g\tilde{n}_0\frac{\partial}{\partial x_\mu}[\boldsymbol{\nabla}\cdot(n_{c0}\delta\mathbf{v}_c) \\ & + \boldsymbol{\nabla}\cdot(\tilde{n}_0\delta\mathbf{v}_n)] - \frac{1}{3}\frac{\partial U_0}{\partial x_\mu}\delta\Gamma_{12} + \frac{2}{3}g n_{c0}\frac{\partial \delta\Gamma_{12}}{\partial x_\mu} \\ & + \frac{\partial}{\partial x_\nu}\left\{2\eta\left[\frac{\partial}{\partial t}D_{\mu\nu} - \frac{1}{3}\left(\mathrm{Tr}\frac{\partial D}{\partial t}\right)\delta_{\mu\nu}\right]\right\} \\ & - \frac{2}{3}\frac{\partial}{\partial x_\mu}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}\delta T). \end{aligned} \tag{28}$$

We then look for normal-mode solutions of Eqs. (27) and (28) of the form

$$\mathbf{v}_n(\mathbf{r},t) = \mathbf{u}_n(\mathbf{r})e^{-i\omega t}, \quad \mathbf{v}_c(\mathbf{r},t) = \mathbf{u}_c(\mathbf{r})e^{-i\omega t}. \tag{29}$$

In this case, the coupled equations (27) and (28) reduce to

$$\begin{aligned} m\omega^2\mathbf{u}_c = {} & -g\boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot(n_{c0}\mathbf{u}_c)] - 2g\boldsymbol{\nabla}[\boldsymbol{\nabla}\cdot(\tilde{n}_0\mathbf{u}_n)] \\ & + g\boldsymbol{\nabla}\delta\Gamma_{12,\omega}[\mathbf{u}_n,\mathbf{u}_c], \end{aligned} \tag{30}$$

$$\begin{aligned} m\tilde{n}_0\omega^2 u_{n\mu} = {} & -\frac{5}{3}\frac{\partial}{\partial x_\mu}[\boldsymbol{\nabla}\cdot(\tilde{P}_0\mathbf{u}_n)] + \frac{2}{3}\frac{\partial}{\partial x_\mu}(\mathbf{u}_n\cdot\boldsymbol{\nabla}\tilde{P}_0) \\ & - \boldsymbol{\nabla}\cdot(\tilde{n}_0\mathbf{u}_n)\frac{\partial U_0}{\partial x_\mu} - 2g\tilde{n}_0\frac{\partial}{\partial x_\mu}[\boldsymbol{\nabla}\cdot(n_{c0}\mathbf{u}_c) \\ & + \boldsymbol{\nabla}\cdot(\tilde{n}_0\mathbf{u}_n)] + \frac{1}{3}\frac{\partial U_0}{\partial x_\mu}\delta\Gamma_{12,\omega}[\mathbf{u}_n,\mathbf{u}_c] \\ & - \frac{2}{3}g n_{c0}\frac{\partial\delta\Gamma_{12,\omega}[\mathbf{u}_n,\mathbf{u}_c]}{\partial x_\mu} \\ & + i\omega\frac{\partial}{\partial x_\nu}\left\{2\eta\left[u_{\mu\nu} - \frac{1}{3}(\boldsymbol{\nabla}\cdot\mathbf{u}_n)\delta_{\mu\nu}\right]\right\} \\ & + \frac{2}{3}\frac{\partial}{\partial x_\mu}\boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}\delta T_\omega[\mathbf{u}_n,\mathbf{u}_c]). \end{aligned} \tag{31}$$

Here the symmetric tensor $u_{\mu\nu}$ is defined by

$$u_{\mu\nu} \equiv \frac{1}{2}\left(\frac{\partial u_{n\nu}}{\partial x_\mu} + \frac{\partial u_{n\mu}}{\partial x_\nu}\right). \tag{32}$$

The source function $\delta\Gamma_{12}$ appearing in the above equations can be expressed in terms of of $\mathbf{u}_n$ and $\mathbf{u}_c$ as $\delta\Gamma_{12} = \delta\Gamma_{12,\omega}[\mathbf{u}_n,\mathbf{u}_c]e^{-i\omega t}$, where

$$\delta\Gamma_{12,\omega}[\mathbf{u}_n,\mathbf{u}_c] = \delta\Gamma_{12}^{(1)}[\mathbf{u}_n,\mathbf{u}_c] + \delta\Gamma_{12,\omega}^{(2)}[\mathbf{u}_n,\mathbf{u}_c], \tag{33}$$

with

$$\delta\Gamma_{12}^{(1)}[\mathbf{u}_n,\mathbf{u}_c] = \sigma_H\left\{\boldsymbol{\nabla}\cdot[n_{c0}(\mathbf{u}_c - \mathbf{u}_n)] + \frac{1}{3}n_{c0}\boldsymbol{\nabla}\cdot\mathbf{u}_n\right\}, \tag{34}$$

$$\delta\Gamma_{12,\omega}^{(2)} = i\omega\tau_\mu\delta\Gamma_{12}^{(1)}[\mathbf{u}_n,\mathbf{u}_c] - \frac{2\sigma_H\sigma_1}{3g\tilde{n}_0}\boldsymbol{\nabla}\cdot(\kappa\delta T_\omega[\mathbf{u}_n,\mathbf{u}_c]). \tag{35}$$

Similarly, the temperature fluctuation is given by $\delta T = i\delta T_\omega[\mathbf{u}_n,\mathbf{u}_c]$, where

$$\delta T_\omega[\mathbf{u}_n,\mathbf{u}_c] = \frac{1}{\omega}\left[-\frac{2}{3}T_0(\boldsymbol{\nabla}\cdot\mathbf{u}_n) + \frac{2T_0}{3\tilde{n}_0}\sigma_1\delta\Gamma_{12}^{(1)}[\mathbf{u}_n,\mathbf{u}_c]\right]. \tag{36}$$

Using these results in Eqs. (30) and (31), we see that we have obtained a *closed* set of equations for both local velocity components $\mathbf{u}_n$ and $\mathbf{u}_c$.

III. UNDAMPED NORMAL MODE FREQUENCY

We first consider the undamped normal-mode solutions of our hydrodynamic equations, neglecting all hydrodynamic dissipation. Formally this means that we take $\eta, \kappa, \tau_\mu \to 0$ in the two-fluid hydrodynamic equations. As discussed in Refs. [3,5], this limit corresponds to the Landau two-fluid hydrodynamics without dissipation. In this limit, the coupled equations for $\mathbf{u}_n$ and $\mathbf{u}_c$ simplify to

$$m\omega^2\mathbf{u}_c = -g\nabla[\nabla\cdot(n_{c0}\mathbf{u}_c)] - 2g\nabla[\nabla\cdot(\tilde{n}_0\mathbf{u}_n)] + g\nabla\,\delta\Gamma^{(1)}_{12,\omega}[\mathbf{u}_n,\mathbf{u}_c], \tag{37}$$

$$m\omega^2\mathbf{u}_n = -\frac{5}{3}\nabla[\nabla\cdot(\tilde{P}_0\mathbf{u}_n)] + \frac{2}{3}\nabla[\mathbf{u}_n\cdot\nabla\tilde{P}_0] - \frac{2}{3}\nabla(gn_{c0}\delta\Gamma^{(1)}_{12,\omega}[\mathbf{u}_n,\mathbf{u}_c]) - \delta\Gamma^{(1)}_{12}[\mathbf{u}_n,\mathbf{u}_c]\nabla U_0 - \nabla\cdot(\tilde{n}_0\mathbf{u}_n)\nabla U_0 - 2g\tilde{n}_0\nabla[\nabla\cdot(\tilde{n}_0\mathbf{u}_n) + \nabla\cdot(n_{c0}\mathbf{u}_c)]. \tag{38}$$

In general, solutions of these coupled hydrodynamic equations are very complicated. However, one can reformulate this problem so that the solutions are given in terms of a variational functional. The present discussion closely follows the variational analysis developed in Ref. [3] for two-fluid hydrodynamic equations, introduced in Ref. [1], which omitted the contribution of the source term from C_{12} collisions, i.e., $\delta\Gamma_{12}=0$. However, as Ref. [3] showed, $\delta\Gamma_{12}$ plays a crucial role in obtaining the correct Landau two-fluid hydrodynamic limit. The same formalism was also used in Ref. [17] to calculated the hydrodynamic mode frequencies for a trapped Bose gas above T_{BEC}.

Introducing the six-component local velocity vector

$$\mathbf{u} = \begin{pmatrix}\mathbf{u}_n\\ \mathbf{u}_c\end{pmatrix}, \tag{39}$$

we combine the coupled equations for $\mathbf{u}_n$ and $\mathbf{u}_c$ in Eqs. (38) and (37) into a matrix equation

$$\boldsymbol{L}\mathbf{u} = \omega^2\boldsymbol{D}\mathbf{u}. \tag{40}$$

The 6×6 matrix $\boldsymbol{L}$ has the block structure,

$$\boldsymbol{L} = \begin{pmatrix}\hat{L}_{11} & \hat{L}_{12}\\ \hat{L}_{21} & \hat{L}_{22}\end{pmatrix}, \tag{41}$$

with the 3×3 matrix elements being defined as

$$(\hat{L}_{11}\mathbf{u}_n)_\mu = -\frac{5}{3}\frac{\partial}{\partial x_\mu}[\nabla\cdot(\tilde{P}_0\mathbf{u}_n)] + \frac{2}{3}\frac{\partial}{\partial x_\mu}(\mathbf{u}_n\cdot\nabla\tilde{P}_0) - \nabla\cdot(\tilde{n}_0\mathbf{u})\frac{\partial U_0}{\partial x_\mu} - 2g\tilde{n}_0\frac{\partial}{\partial x_\mu}[\nabla\cdot(\tilde{n}_0\mathbf{u}_n)] - \frac{\partial}{\partial x_\mu}\left(\frac{2}{3}gn_{c0}\delta\Gamma^{(1)}_{12}[\mathbf{u}_n,0]\right) + \delta\Gamma^{(1)}_{12}[\mathbf{u}_n,0]\frac{\partial U_0}{\partial x_\mu}, \tag{42a}$$

$$(\hat{L}_{12}\mathbf{u}_c)_\mu = -2g\tilde{n}_0\frac{\partial}{\partial x_\mu}\nabla\cdot(n_{c0}\mathbf{u}_c) - \frac{\partial}{\partial x_\mu}\left(\frac{2}{3}gn_{c0}\delta\Gamma^{(1)}_{12}[0,\mathbf{u}_c]\right) + \delta\Gamma^{(1)}_{12}[0,\mathbf{u}_c]\frac{\partial U_0}{\partial x_\mu}, \tag{42b}$$

$$(\hat{L}_{21}\mathbf{u}_n)_\mu = -2gn_{c0}\frac{\partial}{\partial x_\mu}\nabla\cdot(\tilde{n}_0\mathbf{u}_n) + gn_{c0}\frac{\partial}{\partial x_\mu}\delta\Gamma^{(1)}_{12}[\mathbf{u}_n,0], \tag{42c}$$

$$(\hat{L}_{22}\mathbf{u}_c)_\mu = -gn_{c0}\frac{\partial}{\partial x_\mu}\nabla\cdot(n_{c0}\mathbf{u}_c) + gn_{c0}\frac{\partial}{\partial x_\mu}\delta\Gamma^{(1)}_{12}[0,\mathbf{u}_c]. \tag{42d}$$

Similarly, the matrix $\boldsymbol{D}$ is block-diagonal ($\hat{D}_{12}=\hat{D}_{21}=0$), with elements $\hat{D}_{11}=m\tilde{n}_0\hat{1}, \hat{D}_{22}=mn_{c0}\hat{1}$. Here, $\hat{1}$ is a 3×3 unit matrix. We note that the matrix $\boldsymbol{L}$ has the Hermitian property

$$\int d\mathbf{r}\,\mathbf{u}'\cdot(\boldsymbol{L}\mathbf{u}) = \int d\mathbf{r}\,\mathbf{u}\cdot(\boldsymbol{L}\mathbf{u}'). \tag{43}$$

The coupled equations (37) and (38) can be rewritten in terms of the variational functional

$$J[\mathbf{u}_n,\mathbf{u}_c] = \frac{U[\mathbf{u}_n,\mathbf{u}_c]}{K[\mathbf{u}_n,\mathbf{u}_c]}, \tag{44}$$

where

$$U[\mathbf{u}_n,\mathbf{u}_c] \equiv \frac{1}{2}\int d\mathbf{r}\,\mathbf{u}\cdot(\hat{L}\mathbf{u}), \quad K[\mathbf{u}_n,\mathbf{u}_c] \equiv \frac{1}{2}\int d\mathbf{r}\,\mathbf{u}\cdot(\hat{D}\mathbf{u}). \tag{45}$$

Using the Hermitian property of $\boldsymbol{L}$ in Eq. (43), one can prove that the requirement that the functional J be stationary leads to the required equations in Eqs. (37) and (38) and ω^2 is identified with the stationary value of the functional J. One can therefore evaluate the collective mode frequency using a variational ansatz for $\mathbf{u}_n$ and $\mathbf{u}_c$ in the variational functional $J[\mathbf{u}_n,\mathbf{u}_c]$.

IV. GENERAL EXPRESSION FOR DAMPING OF HYDRODYNAMIC MODES

In this section, we derive a general expression of hydrodynamic damping of a collective mode due to transport coefficients. The general expression for hydrodynamic damping of collective modes in a trapped Bose gas was first derived in Ref. [18] above T_{BEC}.

We now include hydrodynamic dissipation involving κ,η and τ_μ in the two-fluid equations. Similarly to Eq. (40), one can write the coupled equations for $\mathbf{u}_n$ and $\mathbf{u}_c$, which are given in Eqs. (30) and (31), in a matrix form

$$\boldsymbol{L}\mathbf{u} + \boldsymbol{F}\mathbf{u} = \omega^2\boldsymbol{D}\mathbf{u}. \tag{46}$$

Here $\boldsymbol{F}$ represents the dissipative terms in the two-fluid equations and has the block structure

$$\boldsymbol{F} = \begin{pmatrix}\hat{F}_{11} & \hat{F}_{12}\\ \hat{F}_{21} & \hat{F}_{22}\end{pmatrix}, \tag{47}$$

with the matrix elements

$$
\begin{aligned}
(\hat{F}_{11}\mathbf{u}_n)_\mu = & -\frac{\partial}{\partial x_\mu}\Bigg\{\frac{2}{3}gn_{c0}(i\omega\tau_\mu)\delta\Gamma_{12}^{(1)}[\mathbf{u}_n,0] \\
& -\left(\frac{2}{3}+\frac{4n_{c0}\sigma_H\sigma_1}{9\tilde{n}_0}\right)\nabla\cdot(\kappa\nabla\delta T_\omega[\mathbf{u}_n,0])\Bigg\} \\
& +\Bigg\{i\omega\tau_\mu\delta\Gamma_{12}^{(1)}[\mathbf{u}_n,0] \\
& -\frac{2\sigma_H\sigma_1}{3g\tilde{n}_0}\nabla\cdot(\kappa\nabla\delta T_\omega[\mathbf{u}_n,0])\Bigg\}\frac{\partial U_0}{\partial x_\mu} \\
& +i\omega\frac{\partial}{\partial x_\nu}2\eta\left(u_{\mu\nu}-\frac{1}{3}\nabla\cdot\mathbf{u}_n\delta_{\mu\nu}\right),
\end{aligned} \tag{48a}
$$

$$
\begin{aligned}
(\hat{F}_{12}\mathbf{u}_c)_\mu = & -\frac{\partial}{\partial x_\mu}\Bigg\{\frac{2}{3}gn_{c0}(i\omega\tau_\mu)\delta\Gamma_{12}^{(1)}[0,\mathbf{u}_c] \\
& -\left(\frac{2}{3}+\frac{4n_{c0}\sigma_H\sigma_1}{9\tilde{n}_0}\right)\nabla\cdot(\kappa\nabla\delta T_\omega[0,\mathbf{u}_c])\Bigg\} \\
& +\Bigg\{i\omega\tau_\mu\delta\Gamma_{12}^{(1)}[0,\mathbf{u}_c] \\
& -\frac{2\sigma_H\sigma_1}{3g\tilde{n}_0}\nabla\cdot(\kappa\nabla\delta T_\omega[0,\mathbf{u}_c])\Bigg\}\frac{\partial U_0}{\partial x_\mu},
\end{aligned} \tag{48b}
$$

$$
\begin{aligned}
(\hat{F}_{21}\mathbf{u}_n) = gn_{c0}\frac{\partial}{\partial x_\mu}\Bigg[& i\omega\tau_\mu\delta\Gamma_{12}^{(1)}[\mathbf{u}_n,0] \\
& -\frac{2\sigma_H\sigma_1}{3g\tilde{n}_0}\nabla\cdot(\kappa\nabla\delta T_\omega[\mathbf{u}_n,0])\Bigg],
\end{aligned} \tag{48c}
$$

$$
\begin{aligned}
(\hat{F}_{22}\mathbf{u}_c) = gn_{c0}\frac{\partial}{\partial x_\mu}\Bigg[& i\omega\tau_\mu\delta\Gamma_{12}^{(1)}[0,\mathbf{u}_c] \\
& -\frac{2\sigma_H\sigma_1}{3g\tilde{n}_0}\nabla\cdot(\kappa\nabla\delta T_\omega[0,\mathbf{u}_c])\Bigg].
\end{aligned} \tag{48d}
$$

In our subsequent analysis, we treat $\boldsymbol{F}$ as a small perturbation to the undamped equation in Eq. (40). To find a solution for $\mathbf{u}_n, \mathbf{u}_c$ including damping, we expand the vector $\mathbf{u}$ in terms of undamped normal-mode solutions $\mathbf{u}_\alpha$ (α is the mode index):

$$
\mathbf{u}=\sum_\alpha C_\alpha\mathbf{u}_\alpha, \quad \boldsymbol{L}\mathbf{u}_\alpha=\omega_\alpha^2\boldsymbol{D}\mathbf{u}_\alpha. \tag{49}
$$

From the Hermitian property of the operator $\hat{L}$, one can show that these normal-mode solutions satisfy the orthonormality relation [3]

$$
\int d\mathbf{r}\ \mathbf{u}_\alpha\cdot(\boldsymbol{D}\mathbf{u}_\beta)=\delta_{\alpha\beta}. \tag{50}
$$

We note that in the above relation, we assume that the normal-mode solutions are normalized. Making use of Eq. (50), we obtain a linear equation for the coefficient C_α:

$$
(\omega^2-\omega_\alpha^2)C_\alpha=\int d\mathbf{r}\mathbf{u}_\alpha\cdot\sum_{\alpha'} C_{\alpha'}\hat{F}\mathbf{u}_{\alpha'}\equiv\sum_{\alpha'} V_{\alpha\alpha'}C_{\alpha'}, \tag{51}
$$

where the matrix element $V_{\alpha\alpha'}$ is defined by

$$
V_{\alpha\alpha'}\equiv\int d\mathbf{r}\mathbf{u}_\alpha\cdot\hat{F}\mathbf{u}_{\alpha'}. \tag{52}
$$

We note that $V_{\alpha\alpha'}$ also depend on the frequency ω.

Expanding the mode frequency to first order in the perturbation $\boldsymbol{F}$ as $\omega=\omega_\alpha+\Delta\omega_\alpha$, we find

$$
\Delta\omega_\alpha=\frac{1}{2\omega_\alpha}V_{\alpha\alpha}|_{\omega=\omega_\alpha}, \tag{53}
$$

where

$$
\begin{aligned}
V_{\alpha\alpha}|_{\omega=\omega_\alpha} = -i\omega_\alpha\int d\mathbf{r}\Bigg\{ & \frac{g\tau_\mu}{\sigma_H}(\delta\Gamma_{12}^{(1)}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}])^2 \\
& +\frac{\kappa}{T_0}|\nabla\delta T_{\omega_\alpha}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}]|^2+\frac{\eta}{2}\left(\frac{\partial u_{n\alpha\nu}}{\partial x_\mu}+\frac{\partial u_{n\alpha\mu}}{\partial x_\nu}\right. \\
& \left.-\frac{2}{3}\delta_{\mu\nu}\nabla\cdot\mathbf{u}_{n\alpha}\right)^2\Bigg\}.
\end{aligned} \tag{54}
$$

We thus find that $\Delta\omega_\alpha=-i\Gamma_\alpha$, where the damping rate Γ_α of the mode α is given by

$$
\begin{aligned}
\Gamma_\alpha=\int d\mathbf{r}\Bigg\{ & \frac{g\tau_\mu}{\sigma_H}(\delta\Gamma_{12}^{(1)}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}])^2 \\
& +\frac{\kappa}{T_0}|\nabla\delta T_{\omega_\alpha}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}]|^2+\frac{\eta}{2}\left(\frac{\partial u_{n\alpha\nu}}{\partial x_\mu}+\frac{\partial u_{n\alpha\mu}}{\partial x_\nu}\right. \\
& \left.-\frac{2}{3}\delta_{\mu\nu}\nabla\cdot\mathbf{u}_{n\alpha}\right)^2\Bigg\}\left[2\int d\mathbf{r}m(n_{c0}u_{c\alpha}^2+\tilde{n}_0u_{n\alpha}^2)\right]^{-1}.
\end{aligned} \tag{55}
$$

Here we explicitly display the normalization factor in Eq. (55). This expression for the damping rate can also be written in terms of the second viscosity coefficients:

$$
\begin{aligned}
\Gamma_\alpha=\int d\mathbf{r}\Bigg\{ & \zeta_2(\nabla\cdot\mathbf{u}_{n\alpha})^2+2\zeta_1(\nabla\cdot\mathbf{u}_{n\alpha})\nabla\cdot[mn_{c0}(\mathbf{u}_{c\alpha} \\
& -\mathbf{u}_{n\alpha})]+\zeta_3\{\nabla\cdot[mn_{c0}(\mathbf{u}_{c\alpha}-\mathbf{u}_{n\alpha})]\}^2 \\
& +\frac{\kappa}{T_0}|\nabla\delta T_{\omega_\alpha}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}]|^2+\frac{\eta}{2}\left(\frac{\partial u_{n\alpha\nu}}{\partial x_\mu}+\frac{\partial u_{n\alpha\mu}}{\partial x_\nu}\right. \\
& \left.-\frac{2}{3}\delta_{\mu\nu}\nabla\cdot\mathbf{u}_{n\alpha}\right)^2\Bigg\}\left[2\int d\mathbf{r}m(n_{c0}u_{c\alpha}^2+\tilde{n}_0u_{n\alpha}^2)\right]^{-1}.
\end{aligned} \tag{56}
$$

The formula in Eq. (56) for the damping rate can be understood in terms of the entropy production [19]. The local entropy production rate $R_s(\mathbf{r},t)$ in the two-fluid equations [6] is given in Eq. (87) of Ref. [5]. Assuming a normal-mode oscillation of the form

$$\mathbf{v}_n(\mathbf{r},t)=\mathbf{u}_{n\alpha}(\mathbf{r})\cos\omega_\alpha t,\quad \mathbf{v}_c(\mathbf{r},t)=\mathbf{u}_{c\alpha}(\mathbf{r})\cos\omega_\alpha t, \tag{57}$$

we find that the time average of the total entropy production rate is given by

$$\begin{aligned}\langle R_s\rangle &\equiv \frac{\omega_\alpha}{2\pi}\int_0^{2\pi/\omega_\alpha}dt\int d\mathbf{r}R_s(\mathbf{r},t)\\ &=\frac{1}{2}\int d\mathbf{r}\Big\{\zeta_2(\nabla\cdot\mathbf{u}_{n\alpha})^2\\ &\quad+2\zeta_1(\nabla\cdot\mathbf{u}_{n\alpha})\nabla\cdot[mn_{c0}(\mathbf{u}_{c\alpha}-\mathbf{u}_{n\alpha})]\\ &\quad+\zeta_3\{\nabla\cdot[mn_{c0}(\mathbf{u}_{c\alpha}-\mathbf{u}_{n\alpha})]\}^2\\ &\quad+\frac{\kappa}{T_0}|\nabla\delta T_{\omega_\alpha}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}]|^2\\ &\quad+\frac{\eta}{2}\left(\frac{\partial u_{n\alpha\nu}}{\partial x_\mu}+\frac{\partial u_{n\alpha\mu}}{\partial x_\nu}-\frac{2}{3}\delta_{\mu\nu}\nabla\cdot\mathbf{u}_{n\alpha}\right)^2\Big\}.\end{aligned} \tag{58}$$

On the other hand, the total mechanical energy is

$$\langle E_{\mathrm{mech}}\rangle=\frac{1}{2}\int d\mathbf{r}(m\tilde{n}_0u_{n\alpha}^2+mn_{c0}u_{c\alpha}^2). \tag{59}$$

One can then write the damping rate Γ_α as

$$\Gamma_\alpha=\frac{\langle R_s\rangle}{2\langle E_{\mathrm{mech}}\rangle}. \tag{60}$$

This general expression for damping was first given in the classic work in Landau and Lifshitz (LL) [19] for classical fluids. It was later used by Kavoulakis *et al.* [16] to study damping in trapped Bose gases above T_{BEC}. This kind of LL damping formula is discussed in the case of superfluid ^{4}He by Wilks [7].

So far we have not dealt with the problem arising from the fact that in a trapped Bose gas, the decreasing density in the tail of the thermal cloud always leads to the breakdown of the hydrodynamic description. As pointed out by Kavoulakis *et al.* [16], this causes trouble in using Eq. (55) or (56) to evaluate the damping of modes in a trapped Bose gas. In Refs. [18,16], this problem was handled in a physically motivated but *ad hoc* manner, by introducing a spatial cutoff in the integral. In this paper, we propose a new, more microscopic, procedure to deal with this problem. As we discussed in Ref. [5], the fact that the condensate and noncondensate atoms are not in complete local equilibrium can be taken into account by introducing the frequency-dependent second viscosity coefficients

$$\zeta_i(\omega)=\frac{\zeta_i}{1-i\omega\tau_\mu}. \tag{61}$$

Similarly, one can also introduce the frequency dependence in the shear viscosity and the thermal conductivity as

$$\kappa(\omega)=\frac{\kappa}{1-i\omega\tau_\kappa},\quad \eta(\omega)=\frac{\eta}{1-i\omega\tau_\eta}. \tag{62}$$

In Appendix A, we give a more detailed discussion and derivation of these frequency-dependent transport coefficients starting from the kinetic equation. Replacing the transport coefficients in Eq. (56) with $\kappa(\omega_\alpha),\eta(\omega_\alpha)$ and $\zeta_i(\omega_\alpha)$, and taking real part, we find the damping rate of a collective mode in a trapped Bose gas

$$\begin{aligned}\Gamma_\alpha=&\int d\mathbf{r}\Big\{\frac{1}{1+(\omega_\alpha\tau_\mu)^2}\frac{g\tau_\mu}{\sigma_H}(\delta\Gamma_{12}^{(1)}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}])^2\\ &+\frac{1}{1+(\omega_\alpha\tau_\kappa)^2}\frac{\kappa}{T_0}|\nabla\delta T_{\omega_\alpha}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}]|^2\\ &+\frac{1}{1+(\omega_\alpha\tau_\eta)^2}\frac{\eta}{2}\left(\frac{\partial u_{n\alpha\nu}}{\partial x_\mu}+\frac{\partial u_{n\alpha\mu}}{\partial x_\nu}-\frac{2}{3}\nabla\cdot\mathbf{u}_{n\alpha}\delta_{\mu\nu}\right)^2\Big\}\\ &\times\left[2\int d\mathbf{r}m(n_{c0}u_{c\alpha}^2+\tilde{n}_0u_{n\alpha}^2)\right]^{-1}.\end{aligned} \tag{63}$$

We recall that $\delta T_{\omega_\alpha}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}]$ and $\delta\Gamma_{12}^{(1)}[\mathbf{u}_{n\alpha},\mathbf{u}_{c\alpha}]$ are defined in Eqs. (34) and (36) in terms of the velocities $\mathbf{u}_{n\alpha}$ and $\mathbf{u}_{c\alpha}$. This result in Eq. (63) allows us to calculate the hydrodynamic damping due to various transport processes in a trapped Bose gas (both above and below T_{BEC}) and it is the major new result of this paper. The frequency-dependent transport coefficients in Eq. (63) automatically yield the factors $1/[1+(\omega_\alpha\tau_i)]^2$ $(i=\mu,\kappa,\eta)$, which effectively introduce a spatial cutoff for $\omega_\alpha\tau_i>1$, i.e., when hydrodynamics breaks down and we enter the collisionless regime.

V. A UNIFORM BOSE GAS

As an illustration of the physics implied by our results in the previous sections, we study first and second sound in a uniform Bose-condensed gas using our variational expressions for the frequency and damping. In a dilute gas, first sound mainly involves the noncondensate oscillation ($\mathbf{u}_n \gg \mathbf{u}_c$), while second sound mainly involves the condensate oscillation ($\mathbf{u}_c\gg\mathbf{u}_n$) (see, for example, Ref. [20]). To a first approximation, we can simply use $\mathbf{u}_c=0$ for first sound and $\mathbf{u}_n=0$ for second sound. Using the plane-wave solution $\mathbf{u}_n,\mathbf{u}_c\propto\hat{k}\cos\mathbf{k}\cdot\mathbf{r}$ in the variational formulas, we find

$$\omega_i=u_ik-i\Gamma_i\quad (i=1,2), \tag{64}$$

where the two sound velocities are given by

$$u_1^2=\frac{5\tilde{P}_0}{3m\tilde{n}_0}+\frac{2g\tilde{n}_0}{m}-\frac{4gn_{c0}^2\sigma_H}{9m\tilde{n}_0}, \tag{65}$$

$$u_2^2=\frac{gn_{c0}}{m}(1-\sigma_H), \tag{66}$$

and the damping rates are given by

$$\Gamma_1=\frac{k^2}{2m\tilde{n}_0}\left[\frac{4}{3}\eta+\zeta_2-mn_{c0}(\zeta_1+\zeta_4)+(mn_{c0})^2\zeta_3+\frac{4\kappa T_0}{9u_1^2}\left(1+\frac{2\sigma_1\sigma_H n_{c0}}{3\tilde{n}_0}\right)^2\right], \tag{67}$$

$$\Gamma_2=\frac{k^2}{2}\left[mn_{c0}\zeta_3+\frac{4\kappa T_0 n_{c0}}{9u_2^2 m\tilde{n}_0^2}(\sigma_1\sigma_H)^2\right]. \tag{68}$$

These results agree with Eqs. (C11)–(C14) of Appendix C in Ref. [5], which were directly derived from the Landau-Khalatnikov two-fluid equations for a dilute Bose gas. We note that the two sound velocities u_1 and u_2 are slightly different from those given in Ref. [20], because Ref. [20] worked in the limit $\omega\tau_\mu \gg 1$, and neglected the source term $\delta\Gamma_{12}$. As shown in Ref. [3], however, these differences are quantitatively very small in the case of a weakly interacting Bose gas.

VI. MONOPOLE-QUADRUPOLE MODE IN A DEGENERATE NORMAL BOSE GAS

Let us now consider collective modes in a trapped non-condensed Bose gas above $T_{\rm BEC}$. Here we consider the $m=0$ monopole-quadrupole collective mode in an axisymmetric trap ($\omega_x=\omega_y=\omega_\perp \neq \omega_z$). This type of collective mode above $T_{\rm BEC}$ was first observed in the pioneering MIT experiment [9], but the density was not considered large enough to probe the hydrodynamic regime (see, however, Ref. [8]). More recently, however, ENS experiments with metastable He* atoms studied the $m=0$ mode in a high-density thermal cloud [12].

Above $T_{\rm BEC}$, the Hartree-Fock mean field is negligible and thus the equilibrium density is simply given by Eq. (16) with $z_0(\mathbf{r})=e^{\beta[\mu_0-U_{\rm ext}(\mathbf{r})]}$. The chemical potential μ_0 is determined as a function of the temperature through $N=(k_{\rm B}T/\hbar\bar{\omega})^3 g_3(z_0)$, where $\bar{\omega}\equiv(\omega_x\omega_y\omega_z)^{1/3}$. The hydrodynamic modes in a trapped Bose gas with the $m=0$ symmetry were first discussed by Griffin, Wu, and Stringari [21]. The two normal-mode frequencies are temperature-independent, and are given by [21]

$$\Omega_\pm^2=\frac{1}{3}\left[5\omega_\perp^2+4\omega_z^2\pm\sqrt{25\omega_\perp^4+16\omega_z^4-32\omega_z^2\omega_\perp^2}\right]. \tag{69}$$

The corresponding velocity field is given by

$$\mathbf{u}_n=(ax,ay,bz), \tag{70}$$

where the coefficients a and b satisfy the following relations:

$$\frac{a_\pm}{b_\pm}=\left(\frac{3\Omega_\pm^2}{4\omega_z^2}-2\right),\quad \text{or}\quad \frac{b_\pm}{a_\pm}=\left(\frac{3\Omega_\pm^2}{2\omega_\perp^2}-5\right). \tag{71}$$

Kavoulakis *et al.* [16] discussed the hydrodynamic damping of this $m=0$ mode using the LL formula, with a spatial cutoff to deal with the crossover from the hydrodynamic to collisionless regime in the tail of the thermal cloud. In contrast, we calculate the damping rate using Eq. (63), which eliminates the collisionless regime in the tail of the thermal cloud through the use of the frequency-dependent transport coefficients. Above $T_{\rm BEC}$, there is no contribution from the second viscosity transport coefficients. Moreover, in the $m=0$ mode above $T_{\rm BEC}$, one can show $\boldsymbol{\nabla}T=0$ [16,21] and thus the thermal conductivity makes no contribution to the damping of the $m=0$ mode. Thus, only the shear viscosity in Eq. (63) contributes to the damping of this low-frequency collective mode. Using Eq. (70), we find (see also Ref. [16])

$$\Gamma_-=\frac{\frac{2}{3}(a_--b_-)^2\int d\mathbf{r}\,\frac{\eta(\mathbf{r})}{1+[\Omega_-\tau_\eta(\mathbf{r})]^2}}{\int d\mathbf{r}[a_-^2(x^2+y^2)+b_-^2z^2]m\tilde{n}_0(\mathbf{r})}. \tag{72}$$

Making use of the fact that the equilibrium density profile given by Eq. (16) is a function of $U_{\rm ext}(\mathbf{r})$, one can rewrite Eq. (72) as

$$\Gamma_-=\frac{\frac{2}{3}(a_--b_-)^2\int\frac{\eta(\mathbf{r})}{1+[\Omega_-\tau_\eta(\mathbf{r})]^2}}{\left(\frac{2a_-^2}{\omega_\perp^2}+\frac{b_-^2}{\omega_z^2}\right)\frac{m}{3}\int[\omega_\perp^2(x^2+y^2)+\omega_z^2z^2]\tilde{n}_0(\mathbf{r})}. \tag{73}$$

In Eq. (73), the factor involving the coefficients a_- and b_- can be written in a simple form in terms of the undamped frequencies Ω_- and Ω_+ as

$$\begin{aligned}\frac{\frac{2}{3}(a_-^2-b_-^2)}{\left(\frac{2a_-^2}{\omega_\perp^2}+\frac{b_-^2}{\omega_z^2}\right)}&=\frac{\frac{2}{3}\left(\frac{a_-}{b_-}-1\right)\left(1-\frac{b_-}{a_-}\right)}{\left(\frac{2a_-}{\omega_\perp^2b_-}+\frac{b_-}{\omega_z^2a_-}\right)}\\&=\frac{(\Omega_-^2-4\omega_z^2)(\Omega_-^2-4\omega_\perp^2)}{4(5\omega_\perp^2+4\omega_z^2-3\Omega_-^2)}\\&=\frac{(\Omega_-^2-4\omega_z^2)(\Omega_-^2-4\omega_\perp^2)}{2(\Omega_+^2-\Omega_-^2)}.\end{aligned} \tag{74}$$

Here we have used Eq. (71) and Eq. (69). We thus obtain the following simple expression for the damping rate:

$$\Gamma=\frac{\tilde{\tau}}{2(\Omega_+^2-\Omega_-^2)}(\Omega_-^2-4\omega_z^2)(\Omega_-^2-4\omega_\perp^2), \tag{75}$$

where we have introduced a new relaxation time [see Eq. (25)]:

$$\tilde{\tau} \equiv \frac{\displaystyle\int d\mathbf{r}\, \frac{\tilde{P}_0(\mathbf{r})\tau_\eta(\mathbf{r})}{1+(\Omega_-\tau_\eta)^2}}{\displaystyle\frac{m}{3}\int d\mathbf{r}[\omega_\perp^2(x^2+y^2)+\omega_z^2 z^2]\tilde{n}_0(\mathbf{r})}. \tag{76}$$

Here we have used the relation [see Eqs. (16), (17) and (25)] $\eta(\mathbf{r}) = \tilde{P}_0(\mathbf{r})\tau_\eta(\mathbf{r})$. Using the equilibrium relation $\nabla\tilde{P}_0 + \tilde{n}_0\nabla U_{\text{ext}} = 0$, which is valid when neglecting the HF mean field, one can reduce Eq. (76) to

$$\tilde{\tau} = \frac{\displaystyle\int d\mathbf{r}\, \frac{\tilde{P}_0(\mathbf{r})\tau_\eta}{1+(\Omega_-\tau_\eta)^2}}{\displaystyle\int d\mathbf{r}\tilde{P}_0(\mathbf{r})}. \tag{77}$$

Before presenting results given by an explicit evaluation of the relaxation time $\tilde{\tau}$ in Eq. (76) using the parameters for the ENS trap [12], it is useful to comment on the relation between the present calculation of hydrodynamic damping and the moment method developed by Guéry-Odelin *et al.* [22]. These authors applied the moment method to the classical-gas Boltzmann equation. It is straightforward to generalize their method to the kinetic equation for a Bose-degenerate gas (see, for example, Ref. [23]). This moment method is briefly reviewed in Appendix B, and here we simply give the final results. In the moment method for the monopole-quadrupole mode, collisions are characterized by a single parameter, the quadrupole relaxation time τ defined in Eq. (B7). In the hydrodynamic limit $\omega_z\tau \ll 1$, the moment method reproduces the hydrodynamic frequency in Eq. (69), but the damping is now given by Eq. (B12). The moment result for the damping has the same form as Eq. (75), except that τ replaces $\tilde{\tau}$. Both $\tilde{\tau}$ and τ are related to the same position-dependent viscous relaxation time $\tau_\eta(\mathbf{r})$, but involve different spatial averages [see Eq. (76) and Eq. (B7)].

Our evaluation of the damping Γ in Eq. (75) is based on the ENS trap parameters [11,12]: trap frequencies $\omega_\perp/2\pi = 988$, $\omega_z/2\pi = 115$, total number of atoms $N = 8.2\times 10^6$, and s-wave scattering length $a = 16$ nm. The ideal Bose gas transition temperature is given by $T_c = (\hbar\bar{\omega}/k_B)(N/1.202)^{1/3} = 4.39\ \mu$K. In the temperature region $T_c < T < 3T_c$, our calculations show that $\omega_z\tau_\eta(\mathbf{r}=0) \ll 1$. Thus the dominant contribution in the integral of Eq. (76) arises from the low density tail of the cloud where $\omega_z\tau_\eta(\mathbf{r}) \sim 1$. In Fig. 1, we plot the temperature dependence of the damping rate. For comparison, we also plot the damping calculated using the moment method. We find that the two methods give results of the same order of magnitude, but there are significant differences.

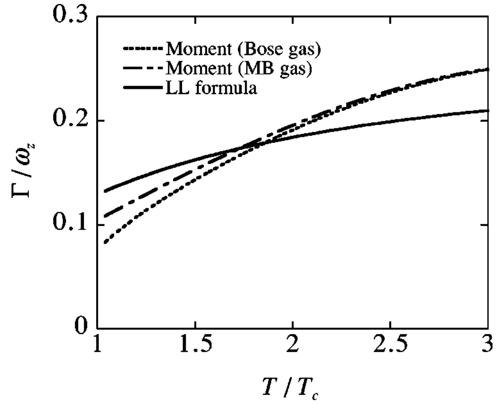

FIG. 1. Temperature dependence of the hydrodynamic damping rate of the $m=0$ mode in a degenerate Bose gas above T_{BEC}, calculated from Eq. (75). We also show the result obtained from the moment method by the broken line [see Fig. 8(a)]. For comparison, we also plot the moment result for the Maxwell-Boltzmann (MB) gas.

VII. MONOPOLE-QUADRUPOLE MODE IN A BOSE-CONDENSED GAS

In this section, we consider the $m=0$ collective mode in the superfluid phase below T_{BEC}. For this purpose, we first need to calculate various equilibrium quantities, solving Eq. (14) and Eq. (16) self-consistently. Figure 2 shows the temperature dependence of the condensate fraction. The effective Bose condensation temperature is lower than the free gas result T_c, due to the mean field. Figure 3 shows the associated density profile of the condensate and noncondensate components at $T = 3\ \mu\text{K} \simeq 0.68T_c$.

Below T_{BEC}, the gas in general exhibits coupled oscillations of the condensate and noncondensate components. At finite temperatures slightly below T_{BEC}, one has a "condensate mode," in which the condensate component mainly oscillates, and a "noncondensate mode," in which the noncondensate component mainly oscillates [3]. As one might expect, the frequencies of the modes are close to those of a pure condensate mode at $T=0$ and a pure noncondensate mode above T_{BEC}, respectively. These frequencies are slightly shifted due to coupling between the two components. In the calculations in this section, we focus entirely on the damping of the modes, and neglect these relatively small frequency shifts. Thus, we neglect the condensate oscillation in the noncondensate mode and noncondensate oscillation in the condensate mode. We calculate the hydrodynamic damping of these two modes. In contrast to the moment method results discussed in Appendix B, our present results are only valid in the hydrodynamic regime.

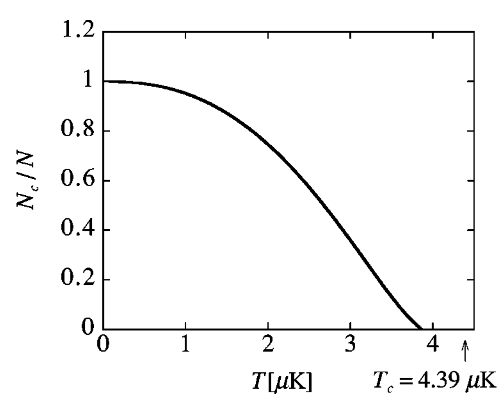

FIG. 2. Condensate fraction versus temperature.

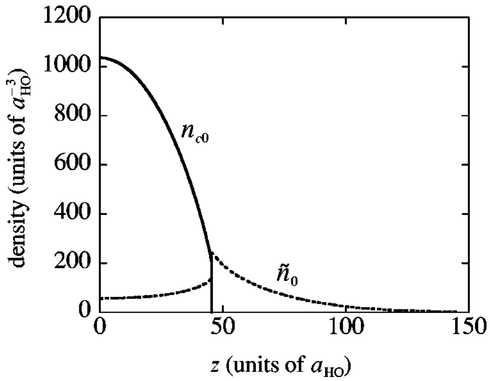

FIG. 3. Density profile of the condensate and noncondensate along the z axis at $T=3\ \mu$K ($\simeq 0.68T_c$). The length unit is the average harmonic oscillator length $a_{\rm HO}\equiv \hbar/m\overline{\omega}$, and the density unit is $a_{\rm HO}^{-3}$. The discontinuous change in n_{c0} and $\tilde{n}_0$ at the condensate boundary is a well-known artifact arising from using the Thomas-Fermi approximation.

A. Noncondensate mode

We first consider damping of the noncondensate mode of monopole-quadrupole symmetry. In Eq. (55), we approximate $\mathbf{u}_n$ using Eq. (70) and Eq. (71), and set $\mathbf{u}_c=0$. For simplicity, we assume that $\boldsymbol{\nabla} T=0$ also holds below $T_{\rm BEC}$. We then find that the damping consists of two contributions, $\Gamma=\Gamma_1+\Gamma_2$, where Γ_1 is the contribution from the shear viscosity, again given by Eq. (75), while Γ_2 is due to the second viscosity, which is given by

$$\Gamma_2=\frac{\displaystyle\int d\mathbf{r}\,\frac{1}{1+(\Omega_-\tau_\mu)^2}\frac{g\tau_\mu}{\sigma_H}(\delta\Gamma_{12}^{(1)}[\mathbf{u}_n,0])^2}{\displaystyle 2m\int d\mathbf{r}\tilde{n}_0 u_n^2}. \tag{78}$$

As noted above, the frequency Ω_- of this mode is well approximated by Ω_- as given in Eq. (69) for $T>T_c$. In Fig. 4, we plot the temperature dependence of the damping of the noncondensate mode below $T_{\rm BEC}$. The contribution from the shear viscosity is dominant for $T\gtrsim 0.3T_c$, with the contribution from the second viscosity coefficient only taking over at very low temperatures.

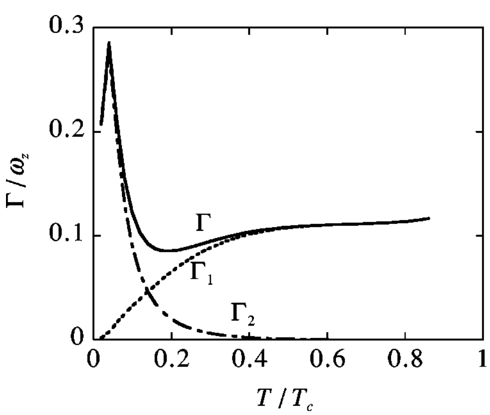

FIG. 4. Damping rate of the noncondensate mode below the BEC transition temperature. The broken lines gives the separate contributions from the shear viscosity (Γ_1) and the second viscosity (Γ_2). Compare with results above T_c shown in Fig. 1.

B. Condensate mode

We next consider the condensate mode below the superfluid transition temperature. The pure condensate mode frequencies at $T=0$ were first given by Stringari [24]:

$$\Omega_\pm^2=2\omega_\perp^2+\frac{3}{2}\omega_z^2\pm\frac{1}{2}\sqrt{16\omega_\perp^4+9\omega_z^4-16\omega_\perp^2\omega_z^2}. \tag{79}$$

The associated condensate velocity field is given by

$$\mathbf{u}_c=(ax,ay,bz), \tag{80}$$

where one finds

$$\frac{a_\pm}{b_\pm}=\left(\frac{\Omega_\pm^2}{2\omega_z^2}-\frac{3}{2}\right),\quad \text{or}\quad \frac{b_\pm}{a_\pm}=\left(\frac{\Omega_\pm^2}{\omega_\perp^2}-4\right)a_\pm. \tag{81}$$

Using this pure condensate mode solution for $\mathbf{u}_c$, and setting the noncondensate velocity $\mathbf{u}_n=0$ and ignoring any temperature fluctuation δT, we obtain a simple formula from Eqs. (63) and (34) for the damping of the low-frequency condensate mode,

$$\Gamma_-=\frac{\displaystyle\int d\mathbf{r}\,\frac{g\tau_\mu\sigma_H}{1+(\Omega_-\tau_\mu)^2}[\boldsymbol{\nabla}\cdot(n_{c0}\mathbf{u}_c)]^2}{\displaystyle 2m\int d\mathbf{r}n_{c0}u_c^2}. \tag{82}$$

Here Ω_- is the frequency given by Eq. (79) and is approximately $\Omega_-\approx\sqrt{\frac{5}{2}}\omega_z$, while $\mathbf{u}_c$ is given by Eq. (80), both of these describing the undamped hydrodynamic mode.

The expression for the damping rate in Eq. (82) which has been found here in the hydrodynamic regime is similar to Eq. (79) of Ref. [23], which gives the collisional damping of condensate collective modes in the collisionless regime. In fact, one can show that in the limit $\Omega_-\tau_\mu\gg 1$, Eq. (82) formally reduces precisely to the expression in Eq. (79) of Ref. [23]. As shown in Ref. [23], this latter expression for the condensate mode damping is equivalent to the result derived by the method of Williams and Griffin [25] in the collisionless regime, under the assumption that the thermal cloud always remained in static thermal equilibrium. This makes sense since our assumptions of $\mathbf{u}_n=0$ and $\boldsymbol{\nabla} T=0$ are equivalent to assuming a static thermal cloud.

In Fig. 5, we plot the temperature dependence of the damping of this condensate mode. The damping of this condensate mode is extremely small, simply because we are always in the extreme hydrodynamic limit $\omega_z\tau_\mu\ll 1$.

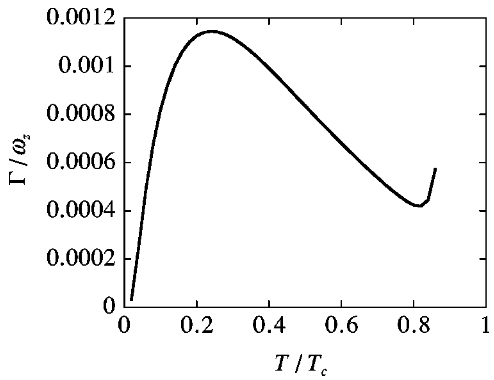

FIG. 5. Damping rate of the $m=0$ condensate mode.

VIII. DISCUSSION AND CONCLUDING REMARKS

Recently, the Landau-Khalatnikov two-fluid equations were derived [4,5] for a trapped Bose gas in the collision-dominated local equilibrium domain. These involve the transport coefficients [8] (thermal conductivity, shear viscosity, and four second viscosity coefficients) describing the processes leading to local equilibrium in a superfluid, from which various relaxation times τ_i can be extracted. In the present paper, we used these equations to derive a general expression for the damping of hydrodynamic modes in a trapped Bose gas. This formula makes use of the variational solution of the two-fluid hydrodynamic equations in the Landau limit ($\omega\tau_i \ll 1$, where ω is the frequency of a hydrodynamic mode). We hope that our work will stimulate further experimental studies of the collision-dominated hydrodynamic regime in trapped Bose gases.

As illustration, we used our formalism to evaluate the hydrodynamic damping of the $m=0$ monopole-quadrupole mode of a cigar-shaped trap, using parameters appropriate to the recent ENS experiments on metastable ^{4}He* [10–12]. We presented results for both $T>T_{\mathrm{BEC}}$ and $T<T_{\mathrm{BEC}}$. We also used the moment method developed by Guéry-Odelin *et al.* [22] for a trapped classical gas and give results (see Appendix B) for a degenerate trapped Bose gas above T_{BEC}. The advantage of the moment approach [22] is that the resulting equations of motion for various moments can be solved over the entire frequency domain, including both the collisionless ($\omega\tau>1$) and hydrodynamic regions ($\omega\tau<1$).

For temperatures characteristic of the ENS experiments ($T\sim 3T_c$), the effect of Bose statistics is almost negligible. Thus, as expected, our calculated values of the damping of the coupled $m=0$ monopole-quadrupole mode (due entirely to the shear viscosity) are in good agreement with the moment calculations of Ref. [22] for a classical gas. Indeed, our calculation shows that the difference remains small down to the superfluid transition T_c. However, as discussed in Refs. [11,12], the analysis of the ENS experimental data exhibits a puzzling discrepancy. The damping and frequency of the $m=0$ mode is consistent with a maximum value of the collision rate [defined in Eq. (B9)] being given by $\Gamma_{\mathrm{coll}} \simeq 2\times 10^3\ \mathrm{s}^{-1}$, which is achieved at $T=3T_c$. Lowering the temperature led to an apparent *decrease* in the values of Γ_{coll}, the latter being determined by the measured changes in the frequency and damping of the $m=0$ mode. This result seems inconsistent with the calculated value of $\Gamma_{\mathrm{coll}}=\sqrt{2}\bar{n}\sigma\bar{v}$ at T_c, using the measured values at T_c of the average density and velocity of atoms, and the collision cross section $\sigma=8\pi a^2$. Our calculated value is $\Gamma_{\mathrm{coll}}(T_c)\simeq 10^4\ \mathrm{s}^{-1}$ (see Fig. 7), considerably larger than the value $\Gamma_{\mathrm{coll}}\simeq 10^3\ \mathrm{s}^{-1}$ as estimated from the frequency and damping of the $m=0$ mode [11,12].

Thus the ENS experiment on the $m=0$ collective mode does not seem to be able to enter deeply into the hydrodynamic regime. Reference [11] tentatively interprets this to being due to increasing inelastic collision processes, which effectively lead to a decreasing collision rate Γ_{coll} for T below $3T_c$. However this does not explain why the estimated value of Γ_{coll} calculated at T_c is an order of magnitude larger, which would correspond to the $m=0$ mode being deeply in the collision-dominated hydrodynamic domain.

In order to clarify this puzzling behavior above T_c, it would be useful to measure the damping and frequency of the monopole-quadrupole collective mode in the Bose-condensed region. As discussed in Ref. [8], the effective value of the relaxation time τ_η associated with the shear viscosity is calculated to become much smaller as one goes below T_c. This is simply because $1/\tau_\eta$ of the thermal cloud atoms is dominated by collisions with condensate atoms. As a result, even if one has a low thermal cloud density $\tilde{n}_0$ (spatially averaged) such that one is in the collisionless region above T_c, one is automatically well inside the hydrodynamic region for $T<T_c$ because of the rapid buildup of the condensate density in the center of the trap. This suggested variant of the ENS experiments effectively uses the formation of the high-density Bose condensate to increase the collision rate which determines the frequency and damping of the monopole-quadrupole mode.

In Sec. VII, we presented the first explicit calculations of the hydrodynamic damping of the monopole-quadrupole mode in the superfluid phase. As can be seen in Fig. 4, the damping of this mode involving the noncondensate thermal cloud is still dominated by the shear viscosity down to $T\sim 0.3T_c$. Comparing the results above (Fig. 1) and below T_c (Fig. 4), one sees the mode damping Γ is fairly smooth going through the transition, with only a slight decrease in magnitude below T_c. This slight decrease in the value of Γ hides the fact that (as discussed above) the effective value of the shear collision relaxation time τ is rapidly decreasing as we go below T_c, putting one deep into the hydrodynamic domain.

In Sec. VII, we also evaluated the hydrodynamic damping of the monopole-quadrupole mode in the condensate. As seen in Fig. 5, this damping is extremely small since one is effectively in the "Landau limit," $\omega\tau_\mu \ll 1$.

Duine and Stoof [26], building on the general theory developed by Stoof [13], have also calculated the damping of collective modes at finite temperatures. However, these authors effectively only calculate the collisional damping in the "collisionless" region. They do not treat the hydrodynamic damping associated with various transport processes, as worked out in detail in the present paper. Moreover, in Ref. [26], it is assumed that the thermal cloud always remains in static thermal equilibrium (see also Ref. [25]). In contrast, our calculations include the coupled dynamics of both the thermal cloud and the condensate in the hydrodynamic region. On the other hand, Duine and Stoof do include the effect of the fluctuations of the condensate order parameter around its mean-field value. This leads to the generalized GP equation (such as we use in this paper and derived in Ref. [3]) being replaced by a Fokker-Planck equation for a time-dependent condensate probability distribution. Our analysis and the LK two-fluid equations do not include this kind of order parameter fluctuations. Their importance, if any, in the collisional hydrodynamic region discussed in the present paper remains to be investigated.

ACKNOWLEDGMENTS

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APPENDIX A: FREQUENCY-DEPENDENT TRANSPORT COEFFICIENTS

In this appendix, we give some details on the frequency-dependent coefficients of the shear viscosity and thermal conductivity, starting from the kinetic equation for the noncondensate atoms. As in the usual Chapman-Enskog procedure described in Ref. [5], we insert the local equilibrium distribution function $f_{\rm leq}$ in the left hand side of the kinetic equation, where $f_{\rm leq}$ is given by

$$f_{\rm leq}(\mathbf{r},\mathbf{p},t)=\frac{1}{z^{-1}(\mathbf{r},t)e^{\beta(\mathbf{r},t)[\mathbf{p}-m\mathbf{v}_n(\mathbf{r},t)]^2/2m}-1}, \quad \text{(A1)}$$

where $z(\mathbf{r},t)=e^{\beta(\mathbf{r},t)[\tilde{\mu}(\mathbf{r},t)-U(\mathbf{r},t)]}$ is the local fugacity. As shown in Appendix A of Ref. [5], the kinetic equation is then given by

$$\frac{\partial f}{\partial t}+\left[\frac{1}{z}\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}z+\frac{(\mathbf{p}-m\mathbf{v}_n)^2}{2mk_{\rm B}T^2}\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}T+\frac{\mathbf{p}-m\mathbf{v}_n}{k_{\rm B}T}\cdot\left(\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\right)\mathbf{v}_n + \frac{\boldsymbol{\nabla}U}{mk_{\rm B}T}\cdot(\mathbf{p}-m\mathbf{v}_n)\right]f_{\rm leq}(1+f_{\rm leq})=C_{12}+C_{22}. \quad \text{(A2)}$$

In contrast to Ref. [5], we keep the time derivative of f explicitly. Since we are interested in small-amplitude collective oscillations, in the following we always expand the theory to first order in the fluctuations around static equilibrium.

We first consider the shear viscosity, which is associated with the anisotropic pressure tensor. In a linearized theory, this is given by

$$\begin{aligned}P'_{\mu\nu}&\equiv P_{\mu\nu}-\delta_{\mu\nu}\tilde{P}\\ &=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)f(\mathbf{r},\mathbf{p},t). \quad \text{(A3)}\end{aligned}$$

The equation of motion for $P'_{\mu\nu}$ can be obtained by taking the moment of Eq. (A2) and linearizing it around static thermal equilibrium. One finds that the term that contributes to this moment is

$$\begin{aligned}&\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)\times\frac{\mathbf{p}}{k_{\rm B}T_0}\cdot\left(\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\right)\mathbf{v}_n f_0(1+f_0)\\ &=\tilde{P}_0\left(\frac{\partial v_{n\nu}}{\partial x_\mu}+\frac{\partial v_{n\mu}}{\partial x_\nu}-\frac{2}{3}\delta_{\mu\nu}\boldsymbol{\nabla}\cdot\mathbf{v}_n\right). \quad \text{(A4)}\end{aligned}$$

One thus obtains

$$\begin{aligned}&\frac{\partial P'_{\mu\nu}}{\partial t}+\tilde{P}_0\left(\frac{\partial v_{n\nu}}{\partial x_\mu}+\frac{\partial v_{n\mu}}{\partial x_\nu}-\frac{2}{3}\delta_{\mu\nu}\boldsymbol{\nabla}\cdot\mathbf{v}_n\right)\\ &=\left\langle\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}p^2\delta_{\mu\nu}\right)\right\rangle_{\rm coll}, \quad \text{(A5)}\end{aligned}$$

where

$$\begin{aligned}&\left\langle\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)\right\rangle_{\rm coll}\\ &\equiv\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)(C_{12}+C_{22}). \quad \text{(A6)}\end{aligned}$$

The collisional contribution on the right hand side of Eq. (A5) arises from deviation of the distribution f from the local equilibrium solution in Eq. (A1). Following the Chapman-Enskog procedure, we use the ansatz $f=f_{\rm leq}+\delta f$, where

$$\delta f=\sum_{\mu\nu}B_{\mu\nu}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)f_0(1+f_0), \quad \text{(A7)}$$

with $B_{\mu\nu}$ being some momentum-independent symmetric tensor. The relation between $B_{\mu\nu}$ and $P'_{\mu\nu}$ can be found by using Eq. (A7) in Eq. (A3) and carrying out the momentum integral:

$$\begin{aligned}P'_{\mu\nu}&=\sum_{\mu'\nu'}B_{\mu'\nu'}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)\times\left(p_{\mu'}p_{\nu'}-\frac{1}{3}\delta_{\mu'\nu'}p^2\right)f_0(1+f_0)\\ &=\frac{1}{5}\left(B_{\mu\nu}-\frac{1}{3}\delta_{\mu\nu}\,{\rm Tr}\,B\right)\sum_{\mu'\nu'}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{m}\times\left(p_{\mu'}p_{\nu'}-\frac{1}{3}\delta_{\mu'\nu'}p^2\right)^2\end{aligned}$$

$$=2mk_{\mathrm{B}}T_0\tilde{P}_0\left(B_{\mu\nu}-\frac{1}{3}\delta_{\mu\nu}\,\mathrm{Tr}\,B\right). \tag{A8}$$

Using Eq. (A7) in Eq. (A6), we find

$$\begin{aligned}\left\langle\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)\right\rangle_{\mathrm{coll}}\\ &=\sum_{\mu'\nu'}B_{\mu'\nu'}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)\\ &\quad\times\hat{L}\left[p_{\mu'}p_{\nu'}-\frac{1}{3}\delta_{\mu'\nu'}p^2\right]\\ &=\frac{1}{5}\left(B_{\mu\nu}-\frac{1}{3}\delta_{\mu\nu}\,\mathrm{Tr}\,B\right)\sum_{\mu'\nu'}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{1}{m}\\ &\quad\times\left(p_{\mu'}p_{\nu'}-\frac{1}{3}\delta_{\mu'\nu'}p^2\right)\\ &\quad\times\hat{L}\left[p_{\mu'}p_{\nu'}-\frac{1}{3}\delta_{\mu'\nu'}p^2\right],\end{aligned} \tag{A9}$$

where $\hat{L}$ is the linearized collision operator defined in Eqs. (46) and (48) of Ref. [5]. Combining Eq. (A8) and Eq. (A9) and using the definition of τ_η given in Eq. (B5) of Ref. [5], we find that the collision term reduces to

$$\left\langle\frac{1}{m}\left(p_\mu p_\nu-\frac{1}{3}\delta_{\mu\nu}p^2\right)\right\rangle_{\mathrm{coll}}=-\frac{P'_{\mu\nu}}{\tau_\eta}, \tag{A10}$$

where the viscous relaxation time τ_η is defined in Refs. [5,8]. Assuming the harmonic time dependence $P'_{\mu\nu}\propto e^{-i\omega t}$, we finally obtain

$$\begin{aligned}P'_{\mu\nu}&=-\frac{2\tau_\eta\tilde{P}_0}{1-i\omega\tau_\eta}\left(D_{\mu\nu}-\frac{1}{3}\delta_{\mu\nu}\,\mathrm{Tr}\,D\right)\\ &=-2\eta(\omega)\left(D_{\mu\nu}-\frac{1}{3}\delta_{\mu\nu}\,\mathrm{Tr}\,D\right),\end{aligned} \tag{A11}$$

where the frequency-dependent viscosity coefficient $\eta(\omega)$ is defined by

$$\eta(\omega)\equiv\frac{\tau_\eta\tilde{P}_0}{1-i\omega\tau_\eta}=\frac{\eta}{1-i\omega\tau_\eta}. \tag{A12}$$

The frequency-dependent thermal conductivity can also be obtained in the same manner, by considering the linearized heat current

$$\begin{aligned}\mathbf{Q}(\mathbf{r},t)&=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{p^2}{2m}\frac{\mathbf{p}}{m}f(\mathbf{r},\mathbf{p},t)-\frac{5}{2}\mathbf{v}_n\tilde{P}_0(\mathbf{r})\\ &=\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{p^2}{2m}-\frac{5}{2}k_{\mathrm{B}}T_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]\\ &\quad\times\frac{\mathbf{p}}{m}f(\mathbf{r},\mathbf{p},t).\end{aligned} \tag{A13}$$

Taking the moment of the kinetic equation in Eq. (A2), one finds that the relevant contribution is given by

$$\begin{aligned}&\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{p^2}{2m}-\frac{5}{2}k_{\mathrm{B}}T_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]\frac{\mathbf{p}}{m}\frac{p^2}{2mk_{\mathrm{B}}T_0^2}\\ &\quad\times\left(\frac{\mathbf{p}}{m}\cdot\boldsymbol{\nabla}\delta T\right)f_0(1+f_0)\\ &=\frac{5}{2}\frac{\tilde{n}_0k_{\mathrm{B}}^2T_0}{m}\boldsymbol{\nabla}\delta T\left\{\frac{7g_{7/2}(z_0)}{2g_{3/2}(z_0)}-\frac{5}{2}\left[\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]^2\right\}.\end{aligned} \tag{A14}$$

One thus obtains

$$\begin{aligned}\frac{\partial\mathbf{Q}}{\partial t}+\frac{5}{2}\frac{\tilde{n}_0k_{\mathrm{B}}^2T_0}{m}\boldsymbol{\nabla}\delta T\left\{\frac{7g_{7/2}(z_0)}{2g_{3/2}(z_0)}-\frac{5}{2}\left[\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]^2\right\}\\ =\left\langle\left[\frac{p^2}{2m}-\frac{5}{2}k_{\mathrm{B}}T_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]\frac{\mathbf{p}}{m}\right\rangle_{\mathrm{coll}},\end{aligned} \tag{A15}$$

where

$$\begin{aligned}&\left\langle\left[\frac{p^2}{2m}-\frac{5}{2}k_{\mathrm{B}}T_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]\frac{\mathbf{p}}{m}\right\rangle_{\mathrm{coll}}\\ &\equiv\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\left[\frac{p^2}{2m}-\frac{5}{2}k_{\mathrm{B}}T_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]\frac{\mathbf{p}}{m}(C_{12}+C_{22}).\end{aligned} \tag{A16}$$

To evaluate the collisional term Eq. (A16), which arises from deviation from local equilibrium, we use the Chapman-Enskog ansatz $f=f_{\mathrm{leq}}+\delta f$, where [5]

$$\delta f=\mathbf{A}\cdot\frac{\mathbf{p}}{m}\left[\frac{p^2}{2m}-\frac{5}{2}k_{\mathrm{B}}T_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]f_0(1+f_0). \tag{A17}$$

Here $\mathbf{A}$ is a momentum-independent vector which is directly related to the heat current $\mathbf{Q}$ through

$$\mathbf{Q}=\mathbf{A}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\frac{p^2}{3m}\left[\frac{p^2}{2m}-\frac{5}{2}k_{\mathrm{B}}T_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]^2f_0(1+f_0). \tag{A18}$$

Evaluation of the collisional term with using the ansatz Eq. (A17) closely follows the derivation of the thermal conductivity in Ref. [5]. We find

$$\left\langle\left[\frac{p^2}{2m}-\frac{5}{2}k_BT_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]\frac{\mathbf{p}}{m}\right\rangle_{\text{coll}}$$
$$=\frac{\mathbf{A}}{3}\int\left[\frac{p^2}{2m}-\frac{5}{2}k_BT_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]$$
$$\times\frac{\mathbf{p}}{m}\cdot\hat{L}\left[\left\{\frac{p^2}{2m}-\frac{5}{2}k_BT_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right\}\frac{\mathbf{p}}{m}\right]. \tag{A19}$$

Combining Eq. (A18) and Eq. (A19) and using the definition of the thermal relaxation time τ_κ given in Eq. (B1) of Ref. [5], we find

$$\left\langle\left[\frac{p^2}{2m}-\frac{5}{2}k_BT_0\frac{g_{5/2}(z_0)}{g_{3/2}(z_0)}\right]\frac{\mathbf{p}}{m}\right\rangle_{\text{coll}}=-\frac{\mathbf{Q}}{\tau_\kappa}. \tag{A20}$$

Assuming the harmonic time dependence $\mathbf{Q}\propto e^{-i\omega t}$, we obtain

$$\mathbf{Q}=-\frac{\kappa}{1-i\omega\tau_\kappa}\boldsymbol{\nabla}\delta T\equiv-\kappa(\omega)\boldsymbol{\nabla}\delta T, \tag{A21}$$

where we have used the expression for κ given in Eq. (24), and the frequency-dependent thermal conductivity is defined by

$$\kappa(\omega)\equiv\frac{\kappa}{1-i\omega\tau_\kappa}. \tag{A22}$$

APPENDIX B: MOMENT METHOD FOR A DEGENERATE NORMAL BOSE GAS

In the moment method [22], one derives an equation of motion for some dynamical quantity denoted by $\chi(\mathbf{r},t)$ by taking a moment of the kinetic equation in both position and momentum:

$$\langle\chi\rangle\equiv\frac{1}{N}\int d\mathbf{r}\frac{d\mathbf{p}}{(2\pi\hbar)^3}\chi(\mathbf{r},t)f(\mathbf{r},\mathbf{p},t). \tag{B1}$$

The advantage of this method is that it gives results valid in both the collisionless and hydrodynamic region. It should be noted that in the collisionless region, this approach does not include Landau damping, but this is small above T_{BEC}. For the $m=0$ monopole-quadrupole mode, we need moment equations for the following physical quantities:

$$\chi_1=r^2,\quad \chi_2=2z^2-r^2,$$
$$\chi_3=\mathbf{r}\cdot\mathbf{p}/m,\quad \chi_4=2zp_z/m-\mathbf{r}_\perp\cdot\mathbf{p}_\perp/m,$$
$$\chi_5=p^2/m^2,\quad \chi_6=2p_z^2/m^2-p_\perp^2/m^2. \tag{B2}$$

Calculation of these moments gives the following coupled equations:

$$\frac{d\langle\chi_1\rangle}{dt}-2\langle\chi_3\rangle=0,\quad \frac{d\langle\chi_2\rangle}{dt}-2\langle\chi_4\rangle=0,$$
$$\frac{d\langle\chi_3\rangle}{dt}-\langle\chi_5\rangle+\frac{2\omega_\perp^2+\omega_z^2}{3}\langle\chi_1\rangle+\frac{\omega_z^2-\omega_\perp^2}{3}\langle\chi_2\rangle=0,$$
$$\frac{d\langle\chi_4\rangle}{dt}-\langle\chi_6\rangle+\frac{2\omega_z^2-2\omega_\perp^2}{3}\langle\chi_1\rangle+\frac{\omega_\perp^2+2\omega_z^2}{3}\langle\chi_2\rangle=0,$$
$$\frac{d\langle\chi_5\rangle}{dt}+\frac{2\omega_z^2+4\omega_\perp^2}{3}\langle\chi_3\rangle+\frac{2\omega_z^2-\omega_\perp^2}{3}\langle\chi_4\rangle=0,$$
$$\frac{d\langle\chi_6\rangle}{dt}+\frac{4\omega_z^2-4\omega_\perp^2}{3}\langle\chi_3\rangle+\frac{4\omega_z^2+2\omega_\perp^2}{3}\langle\chi_4\rangle=\langle\chi_6\rangle_{\text{coll}}. \tag{B3}$$

The collisional contribution in the last equation in Eq. (B3) is defined as

$$\langle\chi_6\rangle_{\text{coll}}\equiv\frac{1}{N}\int d\mathbf{r}\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\chi_6C_{22}[f]. \tag{B4}$$

This term can be approximately evaluated by using the ansatz $f=f_0+\delta f$, where

$$\delta f=\beta_0f_0(1+f_0)\left[\frac{\delta T}{T_0}\left(\frac{p^2}{2m}+U_0-\mu_{c0}\right)+\mathbf{p}\cdot\mathbf{v}_n+\delta\tilde{\mu}\right.$$
$$\left.+\alpha(2p_z^2-p^2)\right], \tag{B5}$$

where the time-dependent parameter $\alpha(t)$ characterizes the anisotropy in the momentum distribution described by $\langle\chi_6\rangle$. The above ansatz is a generalization of the Gaussian ansatz for the Maxwell-Boltzmann gas used in Ref. [22] to a degenerate Bose gas.

Using the ansatz in Eq. (B5) in Eq. (B4) and linearizing in α, one obtains

$$\langle\chi_6\rangle_{\text{coll}}=-\frac{\langle\chi_6\rangle}{\tau}, \tag{B6}$$

where τ is a quadrupole relaxation time defined by the weighted spatial average of the inverse of the viscous relaxation time (see also Ref. [23])

$$\frac{1}{\tau}\equiv\frac{\int d\mathbf{r}\tilde{P}_0/\tau_\eta}{\int d\mathbf{r}\tilde{P}_0}. \tag{B7}$$

Using the moment equations (B3) together with the approximation Eq. (B6), we see that the only effect of using Bose statistics is in the value for the spatially-averaged relaxation time τ defined in Eq. (B7).

In a nondegenerate (Maxwell-Boltzmann) gas, one finds that $\tau_\eta(\mathbf{r})=\frac{5}{4}\tau_{\text{cl}}(\mathbf{r})$ [5,8,27], where τ_{cl} is the usual elastic collision time for a classical gas

$$\tau_{\text{cl}}^{-1}(\mathbf{r})=\sqrt{2}\sigma\tilde{n}_0(\mathbf{r})\bar{v}=\sqrt{2}(8\pi a^2)\tilde{n}_0(\mathbf{r})(8k_BT/\pi m)^{1/2}. \tag{B8}$$

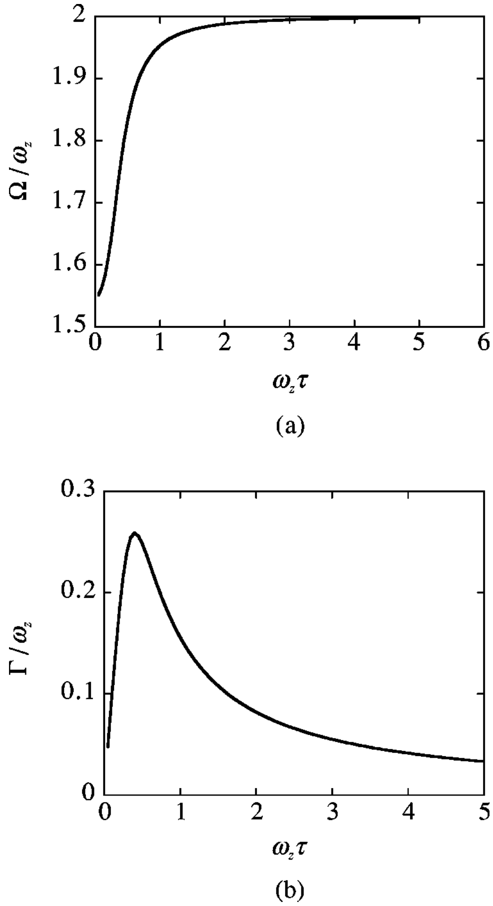

FIG. 6. Moment calculation of the (a) frequency and (b) damping of the $m=0$ mode, both as a function of the quadrupole relaxation time τ. These results are found by solving Eq. (B11). We use the ENS trap frequencies $\omega_\perp/2\pi=988$ Hz and $\omega_z/2\pi=115$ Hz [10–12]..

Using this result and $\tilde{P}_0(\mathbf{r})=k_BT\tilde{n}_0(\mathbf{r})$ in Eq. (B7), the quadrupole relaxation time reduces to

$$\frac{1}{\tau}=\frac{4}{5}\Gamma_{\text{coll}}. \tag{B9}$$

Here $\Gamma_{\text{coll}}\equiv\sqrt{2}\sigma\overline{nv}$, where the spatially averaged density is defined by

$$\overline{n}\equiv\frac{\int d\mathbf{r}\tilde{n}_0^2(\mathbf{r})}{\int d\mathbf{r}\tilde{n}_0(\mathbf{r})}=\frac{\tilde{n}_0(\mathbf{r}=\mathbf{0})}{2\sqrt{2}}. \tag{B10}$$

This result for $1/\tau$ in terms of Γ_{coll} agrees with that of Guéry-Odelin *et al.* [22]. The authors of Ref. [12] analyzed data in terms of the classical gas result of Ref. [22], using the above definition of the averaged collision rate Γ_{coll} valid for a Maxwell-Boltzmann gas.

Assuming the time dependence $e^{-i\omega t}$, the coupled equations in Eq. (B3) with Eq. (B6) can be solved to give

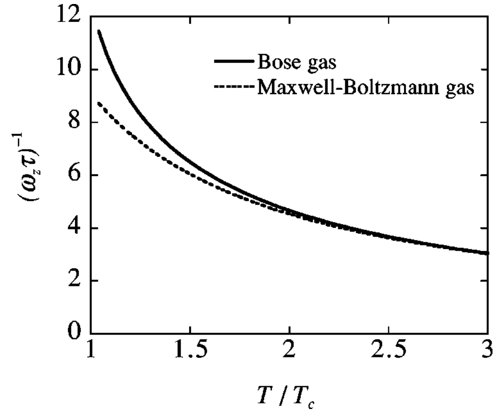

FIG. 7. Temperature dependence of the spatially averaged quadrupole relaxation time τ. The broken line shows the result using the Maxwell-Boltzmann distribution for a classical trapped gas.

$$(\omega^2-4\omega_z^2)(\omega^2-4\omega_\perp^2)+\frac{i}{\omega\tau}\left[\omega^4-\frac{2}{3}\omega^2(5\omega_\perp^2+4\omega_z^2)+8\omega_\perp^2\omega_z^2\right]=0. \tag{B11}$$

Solving Eq. (B11), we obtain the solution $\omega=\Omega-i\Gamma$, describing damped modes. One can see that in the collisionless

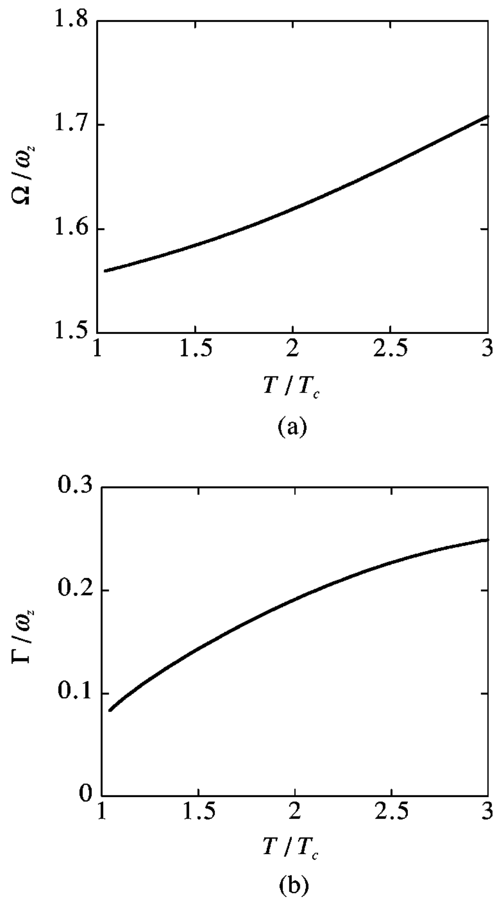

FIG. 8. Temperature dependence of (a) frequency Ω, and (b) damping Γ of the $m=0$ mode, obtained by the moment method.

limit $\omega\tau \gg 1$, the frequency is given by either $2\omega_z$ or $2\omega_\perp$. In the opposite hydrodynamic limit $\omega\tau \ll 1$, the two solutions are given by $\omega = \Omega_\pm$, with $\Omega_\pm$ given by Eq. (69). The frequency Ω and damping Γ obtained by solving Eq. (B11) can be expressed in terms of the spatially averaged quadrupole relaxation time τ, for given trap frequencies. In Fig. 6, we plot Ω and Γ of the low-frequency mode as a function of $\omega_z\tau$. The hydrodynamic domain is the region $\omega_z\tau \lesssim 1$.

In Fig. 7, we plot the temperature dependence of the quadrupole relaxation time τ calculated for the ENS experimental data. For comparison, we also plot the result [12,22] using the Maxwell-Boltzmann distribution. The effect of Bose statistics is clearly very small, down to about $T \simeq 1.5T_c$. We use the result in Fig. 7 to calculate the frequency and damping as functions of the temperature, as shown in Fig. 8.

In the hydrodynamic limit, one can obtain an analytical expression for the damping rate from the moment equations. Assuming $\omega\tau \ll 1$, one can expand the solution to Eq. (B11) to first order in τ. In this limit, the damping Γ_- of the low-frequency mode Ω_- is given by

$$\Gamma_- = \frac{\tau}{2(\Omega_+^2 - \Omega_-^2)}(\Omega_-^2 - 4\omega_z^2)(\Omega_-^2 - 4\omega_\perp^2). \quad \text{(B12)}$$

Apart from a different averaged shear-viscous relaxation time τ, it is satisfying that this moment result is identical to the LL expression in Eq. (75).

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