Entanglement and purity of two-mode Gaussian states in noisy channels

Alessio Serafini,[1,*] Fabrizio Illuminati,[1,†] Matteo G. A. Paris,[2,3,‡] and Silvio De Siena[1,§]
[1]*Dipartimento di Fisica "E. R. Caianiello," Università di Salerno, INFM UdR Salerno, INFN Sezione di Napoli, Gruppo Collegato Salerno, Via S. Allende, 84081 Baronissi (SA), Italy*
[2]*Dipartimento di Fisica, Università di Milano, Milano, Italy*
[3]*Dipartimento di Fisica "A. Volta," Università di Pavia, Pavia, Italy*
(Received 13 October 2003; published 25 February 2004)

We study the evolution of purity, entanglement, and total correlations of general two-mode continuous variable Gaussian states in arbitrary uncorrelated Gaussian environments. The time evolution of purity, von Neumann entropy, logarithmic negativity, and mutual information is analyzed for a wide range of initial conditions. In general, we find that a local squeezing of the bath leads to a faster degradation of purity and entanglement, while it can help to preserve the mutual information between the modes.

DOI: 10.1103/PhysRevA.69.022318 PACS number(s): 03.67.Pp, 42.50.Dv

I. INTRODUCTION

In recent years, it has been increasingly realized that Gaussian states and Gaussian channels are essential ingredients of continuous variable quantum information [1]. Indeed, entangled Gaussian states have been successfully exploited in realizations of quantum key distribution [2] and teleportation [3] protocols.

In such experimental settings, the entanglement of a bipartite state is usually distilled locally, and then distributed over space, letting the entangled subsystems evolve independently and move to separated spatial regions. In the course of this process, interaction with the external environment is unavoidable and must be properly understood. Therefore, the analysis of the evolution of quantum correlations and decoherence of Gaussian states in noisy channels is of crucial interest, and has spurred several theoretical works [4–10].

The evolution of fidelity of generic bosonic fields in noisy channels has been addressed in Ref. [4]. Indeed, the relevant instance of initial two-mode squeezed vacua (possessing nontrivial entanglement properties) has drawn most of the attention in the field. Decoherence and entanglement degradation of such states in thermal baths have been analyzed in Refs. [6,7], whereas phase damping and the effects of squeezed reservoirs are dealt with in Refs. [5,8,9]. In Ref. [10] the author studies the evolution of a two-mode squeezed vacuum in a common bath endowed with cross correlations and asymptotic entanglement. Decoherence and entanglement degradation in continuous variable systems have been experimentally investigated in Ref. [11].

In this paper we address the general case of an arbitrary two-mode Gaussian state dissipating in arbitrary local Gaussian environments. The resulting dynamics is governed by a two-mode master equation describing losses and thermal hopping in presence of (local) nonclassical fluctuations of the environment.

We study the evolution of quantum and total correlations and the behavior of decoherence in noisy channels. Quantum and total correlations of a state will be quantified by, respectively, its logarithmic negativity [12] and its mutual information, while the rates of decoherence will be determined by following the evolution of the purity (conjugate to the linear entropy) and of the von Neumann entropy. We present explicit analytical results, as well as numerical studies, on the optimization of the relevant physical quantities along the nonunitary evolution. Our analysis provides an answer to the question whether possible effective schemes to mimic general Gaussian environments [13,14] are able to delay the decay of quantum coherence and correlations. We mention that, among such schemes, the most interesting for applications to bosonic fields is based on quantum nondemolition (QND) measurements and feedback dynamics [13,15]. We finally remark that the optimization of the quantities we are going to study with respect to phenomenological parameters turns out to be particularly relevant at ''small times,'' before decoherence has irreversibly corrupted the quantum features of the state, crucial for applications in quantum information.

This paper is structured as follows. In Sec. II we provide a self-contained description of the general structure of two-mode Gaussian states, including the characterization of their mixedness and entanglement. In Sec. III we review the evolution of Gaussian states in general Gaussian environments. In Sec. IV we focus on the evolution of purity and entanglement, determining the optimal regimes that can help preserve these quantities from environmental corruption. Finally, in Sec. V we summarize our results and discuss some outlook on future research.

II. TWO-MODE GAUSSIAN STATES: GENERAL PROPERTIES

Let us consider a two-mode continuous variable system, described by a Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ resulting from the tensor product of the Fock spaces $\mathcal{H}_i$'s. We denote by a_i the annihilation operator acting on the space $\mathcal{H}_i$, and by $\hat{x}_i$

*Electronic address: serale@sa.infn.it
†Electronic address: illuminati@sa.infn.it
‡Electronic address: matteo.paris@fisica.unimi.it
§Electronic address: desiena@sa.infn.it

$=(a_i+a_i^\dagger)/\sqrt{2}$ and $\hat{p}_i=(a_i-a_i^\dagger)/\sqrt{2}$ the quadrature phase operators related to the mode i of the field. The corresponding phase-space variables will be denoted by x_i and p_i.

The set of Gaussian states is, by definition, the set of states with Gaussian characteristic functions and quasi–probability distributions. Therefore a Gaussian state is completely characterized by its first and second statistical moments, which will be denoted, respectively, by the vector of first moments $\overline{X}\equiv(\langle\hat{x}_1\rangle,\langle\hat{p}_1\rangle,\langle\hat{x}_2\rangle,\langle\hat{p}_2\rangle)$ and by the covariance matrix $\boldsymbol{\sigma}$:

$$\sigma_{ij}\equiv\tfrac{1}{2}\langle\hat{x}_i\hat{x}_j+\hat{x}_j\hat{x}_i\rangle-\langle\hat{x}_i\rangle\langle\hat{x}_j\rangle. \tag{1}$$

First moments can be arbitrarily adjusted by local unitary operations, which do not affect any quantity related to entanglement or mixedness. Moreover, as we will see in Sec. III, they do not influence the evolution of second moments in the instances we will deal with. Therefore they will be unimportant to our aims and we will set them to 0 in the following, without any loss of generality for our subsequent results. Throughout the paper, $\boldsymbol{\sigma}$ will stand both for the covariance matrix and the Gaussian state ϱ itself.

It is convenient to express $\boldsymbol{\sigma}$ in terms of the three 2×2 matrices $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$:

$$\boldsymbol{\sigma}\equiv\begin{pmatrix}\boldsymbol{\alpha} & \boldsymbol{\gamma}\\ \boldsymbol{\gamma}^T & \boldsymbol{\beta}\end{pmatrix}. \tag{2}$$

Positivity of ϱ and the canonical commutation relations impose the following constraint for $\boldsymbol{\sigma}$ to be a *bona fide* covariance matrix [16]:

$$\boldsymbol{\sigma}+\frac{i}{2}\boldsymbol{\Omega}\geqslant0, \tag{3}$$

where $\boldsymbol{\Omega}$ is the standard symplectic form

$$\boldsymbol{\Omega}\equiv\begin{pmatrix}\boldsymbol{\omega} & 0\\ 0 & \boldsymbol{\omega}\end{pmatrix},\quad \boldsymbol{\omega}\equiv\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Inequality (3) is a useful and elegant way to express Heisenberg uncertainty principle.

In the following, we will make use of the Wigner quasiprobability representation W, defined as the Fourier transform of the symmetrically ordered characteristic function [17]. In Wigner phase-space picture, the tensor product $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ of the Hilbert spaces H_i's of the two modes results in the direct sum $\Gamma=\Gamma_1\oplus\Gamma_2$ of the associated phase spaces Γ_i's. A symplectic transformation acting on the global phase space Γ corresponds to a unitary operator acting on the global Hilbert space $\mathcal{H}$ [18]. In what follows we will refer to a transformation $S_l=S_1\oplus S_2$, with each $S_i\in Sp_{(2,\mathbb{R})}$ acting on Γ_i, as to a ''local symplectic operation.'' The corresponding unitary transformation is the local unitary transformation $U_l=U_1\otimes U_2$, with each U_i acting on $\mathcal{H}_i$.

The Wigner function of a Gaussian state can be written as follows in terms of phase-space quadrature variables:

$$W(X)=\frac{\mathrm{e}^{-(1/2)X\sigma^{-1}X^T}}{\pi\sqrt{\mathrm{Det}[\boldsymbol{\sigma}]}}, \tag{4}$$

where X stands for the vector $(x_1,p_1,x_2,p_2)\in\Gamma$.

It is well known that for any covariance matrix $\boldsymbol{\sigma}$ there exists a local canonical operation $S_l=S_1\oplus S_2$ which transforms $\boldsymbol{\sigma}$ to the so-called standard form $\boldsymbol{\sigma}_{sf}$ [19],

$$S_l^T\boldsymbol{\sigma}S_l=\boldsymbol{\sigma}_{sf}\equiv\begin{pmatrix}a & 0 & c_1 & 0\\ 0 & a & 0 & c_2\\ c_1 & 0 & b & 0\\ 0 & c_2 & 0 & b\end{pmatrix}. \tag{5}$$

States whose standard form fulfills $a=b$ are said to be symmetric. Let us recall that any pure state is symmetric and fulfills $c_1=-c_2=\sqrt{a^2-1/4}$. The correlations a, b, c_1, and c_2 are determined by the four local symplectic invariants $\mathrm{Det}\,\boldsymbol{\sigma}=(ab-c_1^2)(ab-c_2^2)$, $\mathrm{Det}\,\boldsymbol{\alpha}=a^2$, $\mathrm{Det}\,\boldsymbol{\beta}=b^2$, $\mathrm{Det}\,\boldsymbol{\gamma}=c_1c_2$. Therefore, the standard form corresponding to any covariance matrix is unique.

Inequality (3) can be recast as a constraint on the $Sp_{(4,\mathbb{R})}$ invariants $\mathrm{Det}\,\boldsymbol{\sigma}$ and $\Delta(\boldsymbol{\sigma})=\mathrm{Det}\,\boldsymbol{\alpha}+\mathrm{Det}\,\boldsymbol{\beta}+2\,\mathrm{Det}\,\boldsymbol{\gamma}$:

$$\Delta(\boldsymbol{\sigma})\leqslant\tfrac{1}{4}+4\,\mathrm{det}\boldsymbol{\sigma}. \tag{6}$$

Finally, let us recall that a centered two-mode Gaussian state can always be written as [20,21]

$$\boldsymbol{\sigma}=S^T\boldsymbol{\nu}S, \tag{7}$$

where $S\in Sp_{(4,\mathbb{R})}$ and $\boldsymbol{\nu}$ is the tensor product of thermal states with covariance matrix

$$\boldsymbol{\nu}=\mathrm{diag}(n_-,n_-,n_+,n_+). \tag{8}$$

The quantities $n_\mp$ form the symplectic spectrum of the covariance matrix $\boldsymbol{\sigma}$. They can be easily computed in terms of the $Sp_{(4,\mathbb{R})}$ invariants:

$$2n_\mp^2=\Delta(\boldsymbol{\sigma})\mp\sqrt{\Delta(\boldsymbol{\sigma})^2-4\,\mathrm{Det}\,\boldsymbol{\sigma}}. \tag{9}$$

The symplectic eigenvalues $n_\mp$ encode essential informations about the Gaussian state $\boldsymbol{\sigma}$ and provide powerful, simple ways to express its fundamental properties. For instance, the Heisenberg uncertainty relation (3) can be recast in the compact, equivalent form

$$n_-\geqslant\tfrac{1}{2}. \tag{10}$$

A relevant subclass of Gaussian states we will make use of is constituted by the two-mode squeezed thermal states. Let $S_r=\exp(\frac{1}{2}ra_1a_2-\frac{1}{2}ra_1^\dagger a_2^\dagger)$ be the two-mode squeezing operator with real squeezing parameter r, and let $\boldsymbol{\nu}_\mu=1/(2\sqrt{\mu})\mathbb{1}$ be the tensor product of identical thermal states, where $\mu=\mathrm{Tr}(\varrho^2)$ is the purity of the state. Then, for a two-mode squeezed thermal state $\boldsymbol{\xi}_{\mu,r}$ we can write $\boldsymbol{\xi}_{\mu,r}=S_r\boldsymbol{\nu}_\mu S^\dagger$. The covariance matrix of $\boldsymbol{\xi}_{\mu,r}$ is a symmetric standard form satisfying

$$a=\frac{\cosh 2r}{2\sqrt{\mu}},\quad c_1=-c_2=\frac{\sinh 2r}{2\sqrt{\mu}}, \tag{11}$$

and in the instance $\mu=1$ one recovers the pure two-mode squeezed vacuum states. Two-mode squeezed states are endowed with remarkable properties related to entanglement [22,23]; their dynamics in noisy channels will be analyzed in detail.

A. Characterization of mixedness

Let us briefly recall that the degree of purity of a quantum state can be properly characterized either by the von Neumann entropy S_V or by the linear entropy S_l. Such quantities are defined as follows for continuous variable systems:

$$S_V \equiv -\mathrm{Tr}(\varrho \ln \varrho), \tag{12}$$

$$S_l \equiv 1-\mathrm{Tr}(\varrho^2) \equiv 1-\mu, \tag{13}$$

where the purity $\mu \equiv \mathrm{Tr}(\varrho^2)$ has already been introduced. We first point out that μ can be easily computed for Gaussian states. In fact, in the Wigner phase-space picture the trace of a product of operators corresponds to the integral of the product of their Wigner representations (when existing) over the whole phase space. Because the representation of a state ϱ is just W, for an n-mode Gaussian state we have, taking into account the proper normalization factor,

$$\mu(\boldsymbol{\sigma})=\frac{\pi}{2^n}\int_{\mathbb{R}^{2n}} W^2 d^n x d^n p=\frac{1}{2^n\sqrt{\mathrm{Det}\,\boldsymbol{\sigma}}}. \tag{14}$$

For Gaussian states, the von Neumann entropy can be computed as well, determining their symplectic spectra. For single-mode Gaussian states, one has [24]

$$S_V(\boldsymbol{\sigma})=\frac{1-\mu}{2\mu}\ln\left(\frac{1+\mu}{1-\mu}\right)-\ln\left(\frac{2\mu}{1+\mu}\right), \tag{15}$$

where μ can be computed from Eq. (14) for $n=1$. S_V is in this case an increasing function of the linear entropy, so that both quantities provide the same characterization of mixedness. This is no longer true for two-mode Gaussian states: in this case the von Neumann entropy reads [20,21]

$$S_V(\boldsymbol{\sigma})=f(\tilde{n}_-(\boldsymbol{\sigma}))+f(\tilde{n}_+(\boldsymbol{\sigma})), \tag{16}$$

where

$$f(x)\equiv(x+\tfrac{1}{2})\ln(x+\tfrac{1}{2})-(x-\tfrac{1}{2})\ln(x-\tfrac{1}{2}),$$

and the symplectic eigenvalues $n_\mp(\boldsymbol{\sigma})$ are given by Eq. (9).

Knowledge of the von Neumann entropy S_V allows for the determination of the mutual information I defined, for a general bipartite quantum state ϱ, as $I(\varrho)=S_V(\varrho_1)+S_V(\varrho_2)-S_V(\varrho)$, where ϱ_i refers to the reduced state obtained tracing over the variables of subsystem $j\neq i$. The mutual information $I(\boldsymbol{\sigma})$ of a two-mode Gaussian state $\boldsymbol{\sigma}$ reads [21]

$$I(\boldsymbol{\sigma})=f(a)+f(b)-f(n_-)-f(n_+). \tag{17}$$

One can make use of such a quantity to estimate the amount of total (quantum plus classical) correlations contained in a state $\boldsymbol{\sigma}$ [25].

B. Characterization of entanglement

We now review some properties of entanglement for two-mode Gaussian states. The necessary and sufficient separability criterion for such states is positivity of the partially transposed state $\tilde{\boldsymbol{\sigma}}$ (PPT criterion) [16]. It can be easily seen from the definition of $W(X)$ that the action of partial transposition amounts, in phase space, to a mirror reflection of one of the four canonical variables. In terms of $Sp_{2,\mathbb{R}}\oplus Sp_{2,\mathbb{R}}$ invariants, this results in flipping the sign of Det $\boldsymbol{\gamma}$. Therefore the invariant $\Delta(\boldsymbol{\sigma})$ is changed into $\tilde{\Delta}(\boldsymbol{\sigma})=\Delta(\tilde{\boldsymbol{\sigma}})=\mathrm{Det}\,\boldsymbol{\alpha}+\mathrm{Det}\,\boldsymbol{\beta}-2\,\mathrm{Det}\,\boldsymbol{\gamma}$. Now, the symplectic eigenvalues $\tilde{n}_\mp$ of $\tilde{\boldsymbol{\sigma}}$ read

$$\tilde{n}_\mp=\sqrt{\frac{\tilde{\Delta}(\boldsymbol{\sigma})\mp(\tilde{\Delta}(\boldsymbol{\sigma})^2-4\,\mathrm{Det}\,\boldsymbol{\sigma})^{1/2}}{2}}. \tag{18}$$

The PPT criterion then reduces to a simple inequality that must be satisfied by the smallest symplectic eigenvalue $\tilde{n}_-$ of the partially transposed state,

$$\tilde{n}_- \geqslant \tfrac{1}{2}, \tag{19}$$

which is equivalent to

$$\tilde{\Delta}(\boldsymbol{\sigma})\leqslant 4\,\mathrm{Det}\,\boldsymbol{\sigma}+\tfrac{1}{4}. \tag{20}$$

The above inequalities imply Det $\boldsymbol{\gamma}=c_1c_2<0$ as a necessary condition for a two-mode Gaussian state to be entangled. The quantity $\tilde{n}_-$ encodes all the qualitative characterization of the entanglement for arbitrary (pure or mixed) two-mode Gaussian states. Note that $\tilde{n}_-$ takes a particularly simple form for entangled symmetric states, whose standard form has $a=b$:

$$\tilde{n}_-=\sqrt{(a-|c_1|)(a-|c_2|)}. \tag{21}$$

As for the quantification of entanglement, no fully satisfactory measure is known at present for arbitrary mixed two-mode Gaussian states. However, a quantification of entanglement which can be computed for general two-mode Gaussian states is provided by the negativity $\mathcal{N}$, introduced by Vidal and Werner for continuous variable systems [12]. The negativity of a quantum state ϱ is defined as

$$\mathcal{N}(\varrho)=\frac{\|\tilde{\varrho}\|_1-1}{2}, \tag{22}$$

where $\tilde{\varrho}$ is the partially transposed density matrix and $\|\hat{o}\|_1 \equiv \mathrm{Tr}\sqrt{\hat{o}^\dagger\hat{o}}$ stands for the trace norm of an operator $\hat{o}$. The quantity $\mathcal{N}(\varrho)$ is equal to $|\Sigma_i\lambda_i|$, the modulus of the sum of the negative eigenvalues of $\tilde{\varrho}$, and it quantifies the extent to which $\tilde{\varrho}$ fails to be positive. Strictly related to $\mathcal{N}$ is the

logarithmic negativity $E_{\mathcal{N}}$, defined as $E_{\mathcal{N}} \equiv \ln\|\tilde{\varrho}\|_1$. The negativity has been proved to be convex and monotone under local operations and classical communications [26], but fails to be continuous in trace norm on infinite-dimensional Hilbert spaces. Anyway, this problem can be somehow eluded by restricting to states with finite mean energy [28]. For two-mode Gaussian states it can be easily shown that the negativity is a simple function of $\tilde{n}_-$, which is thus itself an (increasing) entanglement monotone; one has in fact [12]

$$E_{\mathcal{N}}(\boldsymbol{\sigma}) = \max\{0, -\ln 2\tilde{n}_-\}. \tag{23}$$

This is a decreasing function of the smallest partially transposed symplectic eigenvalue $\tilde{n}_-$, quantifying the amount by which inequality (19) is violated. Thus, for our aims, the eigenvalue $\tilde{n}_-$ completely qualifies and quantifies the quantum entanglement of a two-mode Gaussian state $\boldsymbol{\sigma}$.

We finally mention that, as far as symmetric states are concerned, another measure of entanglement, the entanglement of formation E_F [29], can be actually computed [23]. Fortunately, since E_F turns out to be, again, a decreasing function of $\tilde{n}_-$, it provides for symmetric states a quantification of entanglement fully equivalent to the one provided by the logarithmic negativity $E_{\mathcal{N}}$. Therefore, from now on we will adopt $E_{\mathcal{N}}(\boldsymbol{\sigma})$ as the entanglement measure of Gaussian states, recalling that this quantity constitutes an upper bound to the *distillable entanglement* of quantum states [12].

III. EVOLUTION IN GENERAL GAUSSIAN ENVIRONMENTS

We now consider the local evolution of an arbitrary two-mode Gaussian state in noisy channels, in the presence of arbitrarily squeezed (phase-sensitive) environments. In general, the two channels related to the two different modes could be different from one another, each mode evolving independently in its channel. We will refer to the channel (bath) in which mode i evolves as to channel (bath) i. The system is governed, in interaction picture, by the following master equation [30]:

$$\dot{\varrho} = \sum_{i=1,2} \frac{\Gamma}{2} N_i L[a_i^\dagger]\varrho + \frac{\Gamma}{2}(N_i+1)L[a_i]\varrho - \frac{\Gamma}{2}(\overline{M}_i D[a_i]\varrho + M_i D[a_i^\dagger]\varrho), \tag{24}$$

where the dot stands for time derivative and the Lindblad superoperators are defined by $L[O]\varrho \equiv 2O\varrho O^\dagger - O^\dagger O\varrho - \varrho O^\dagger O$ and $D[O]\varrho \equiv 2O\varrho O - OO\varrho - \varrho OO$. The complex parameter M_i is the correlation function of bath i; it is usually referred to as the ''squeezing'' of the bath. N_i is instead a phenomenological parameter related to the purity of the asymptotic stationary state. Positivity of the density matrix imposes the constraint $|M_i|^2 \leqslant N_i(N_i+1)$. At thermal equilibrium; i.e., for $M_i = 0$, the parameter N_i coincides with the average number of thermal photons in bath i.

A squeezed environment, leading to the master equation (24), may be modeled as the interaction with a bath of oscillators excited in squeezed thermal states [31]. Several effective realizations of squeezed baths have been proposed in recent years [13,14]. In particular, in Ref. [13] the authors show that a squeezed environment can be obtained, for a mode of the radiation field, by means of feedback schemes relying on QND ''intracavity'' measurements, capable of affecting the master equation of the system. More specifically, an effective squeezed reservoir is shown to be the result of a continuous homodyne monitoring of a field quadrature, with the addition of a feedback driving term, coupling the homodyne output current with another field quadrature of the mode.

Let $\varrho_i = S(r_i,\varphi_i)\nu_{\overline{n}_i} S(r_i,\varphi_i)^\dagger$ be the environmental Gaussian state of mode i [32]. Here $\overline{n}_i$ denotes the mean number of photons in the thermal state $\nu_{\overline{n}_i}$. Its knowledge allows us to determine the purity of the state via the relation $\mu_i = 1/(2\overline{n}_i + 1)$. The operator $S(r,\varphi) = \exp(\frac{1}{2} r e^{-i2\varphi} a^2 - \frac{1}{2} r\, e^{i2\varphi} a^2)$ is the one-mode squeezing operator. A more convenient parametrization of the channel, endowed with a direct phenomenological interpretation, can be achieved by expressing N_i and M_i in terms of the three real variables μ_i, r_i, and φ_i [33]:

$$\mu_i = \frac{1}{\sqrt{(2N_i+1)^2 - 4|M_i|^2}}, \tag{25}$$

$$\cosh(2r_i) = \sqrt{1 + 4\mu_i^2 |M_i|^2}, \tag{26}$$

$$\tan(2\varphi_i) = -\tan(\arg M_i). \tag{27}$$

Note that the Gaussian state of the environment in bath i coincides with the asymptotic state of mode i, the global asymptotic state being an uncorrelated product of the states ϱ_i's, irrespective of the initial state.

With standard techniques, it can be shown that the master equation (24) corresponds to a Fokker-Planck equation for the Wigner function of the system [30]. In compact notation, one has

$$\dot{W}(X,t) = \frac{\Gamma}{2}[\partial_X X^T + \partial_X \boldsymbol{\sigma}_\infty \partial_X^T] W(X,t), \tag{28}$$

with $\partial_X \equiv (\partial_{x_1}, \partial_{p_1}, \partial_{x_2}, \partial_{p_2})$ and with a diffusion matrix

$$\boldsymbol{\sigma}_\infty = \boldsymbol{\sigma}_{1\infty} \oplus \boldsymbol{\sigma}_{2\infty} = \begin{pmatrix} \boldsymbol{\sigma}_{1\infty} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_{2\infty} \end{pmatrix}, \tag{29}$$

resulting from the tensor product of the asymptotic Gaussian states $\boldsymbol{\sigma}_{i\infty}$'s, given by

$$\boldsymbol{\sigma}_{i\infty} = \begin{pmatrix} \frac{1}{2} + N_i + \mathrm{Re}\, M_i & \mathrm{Im}\, M_i \\ \mathrm{Im}\, M_i & \frac{1}{2} + N_i - \mathrm{Re}\, M_i \end{pmatrix}. \tag{30}$$

For an initial Gaussian state of the form, Eq. (4), the Fokker-Planck equation (28) corresponds to a set of decoupled equations for the second moments and can be easily solved. Note that the drift term always damps to 0 the first statistical moments, and it may thus be neglected for our aims. The evo-

lution in the bath preserves the Gaussian form of the initial condition and is described by the following equation for the covariance matrix [4,33,34]:

$$\boldsymbol{\sigma}(t)=\boldsymbol{\sigma}_{\infty}(1-e^{-\Gamma t})+\boldsymbol{\sigma}(0)e^{-\Gamma t}. \tag{31}$$

This is a simple Gaussian completely positive map, and $\boldsymbol{\sigma}(t)$ satisfies the uncertainty relation, Eq. (3), if and only if the latter is satisfied by both $\boldsymbol{\sigma}_{\infty}$ and $\boldsymbol{\sigma}_0$. The compliance of $\boldsymbol{\sigma}_{\infty}$ with inequality (3) is equivalent to the conditions $|M_i| \leqslant N_i(N_i+1)$.

It is easy to see that Eq. (31) describes the evolution of an initial Gaussian state $\boldsymbol{\sigma}_0$ in an arbitrary Gaussian environment $\boldsymbol{\sigma}_{\infty}$, which can in general be different from that defined by Eq. (29). It would be interesting to find systems whose dynamics could be effectively described by the dissipation in a correlated Gaussian environment (recall that the instance we are analyzing involves a completely uncorrelated environment). Some perspectives in this direction, which lie outside the scopes of the present paper, could come from feedback and conditional measurement schemes.

The initial Gaussian state is described, in general, by a set of ten covariances. To simplify the problem and to better point out the relevant features of the nonunitary evolution, we will choose an initial state already brought in standard form $\boldsymbol{\sigma}_0=\boldsymbol{\sigma}_{sf}$. With this choice the parametrization of the initial state is completely determined by the four parameters a, b, c_1, and c_2, defined in Eq. (5). This choice is not restrictive as far as the dynamics of purity and entanglement are concerned. In fact, let us consider the most general initial Gaussian state $\boldsymbol{\sigma}$ evolving in the most general Gaussian uncorrelated environment $\oplus_i\boldsymbol{\sigma}_{i\infty}$. The state $\boldsymbol{\sigma}$ can always be put in standard form by means of some local transformation $S_l=\oplus_i S_i$. Under the same transformation, the state of the environment $\oplus_i\boldsymbol{\sigma}'_{i\infty}$ remains uncorrelated, with $\boldsymbol{\sigma}'_{i\infty}=S_i^T\boldsymbol{\sigma}_{i\infty}S_i$. All the properties of entanglement and mixedness for the evolving state are invariant under local operations. Therefore, we can state that the evolution of the mixedness and the entanglement of any initial Gaussian state $\boldsymbol{\sigma}$ in any uncorrelated Gaussian environment $\boldsymbol{\sigma}_{\infty}$ is equivalent to the evolution of the initial state in standard form $S_l^T\boldsymbol{\sigma}S_l$ in the uncorrelated Gaussian environment $S_l^T\boldsymbol{\sigma}_{\infty}S_l$.

Finally, to further simplify the dynamics of the state without loss of generality, we can set $\varphi_1=0$ (corresponding to $\mathrm{Im}\,M_1=0$) as a ''reference choice'' for phase-space rotations.

Quite obviously, the standard form of the state is not preserved in an arbitrary channel, as can be seen from Eqs. (30) and (31).

IV. EVOLUTION OF MIXEDNESS AND ENTANGLEMENT

Let us now consider the evolution of mixedness and entanglement of a generic state in standard form (parametrized by a, b, c_1, and c_2) in a generic channel (parametrized by μ_1, r_1, μ_2, r_2, and φ_2). Knowledge of the exact evolution of the state in the channel, given by Eq. (31), allows us to apply the results reviewed in Sec. II and to keep track of the quantities $\mu(t)$, $S_V(t)$, $I(t)$, and $E_{\mathcal{N}}(t)$ during the nonunitary evolution in the channel. However the explicit dependence of such quantities on the nine parameters characterizing the initial state and the environment is quite involved. We provide the explicit expressions in Appendix A. They give a systematic recipe to compute the evolution of mixedness, correlations, and entanglement for any given Gaussian state in standard form (and, therefore, for any Gaussian state).

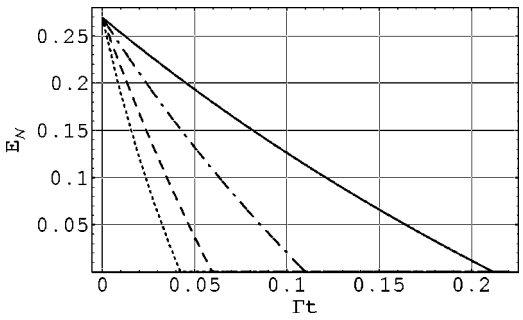

FIG. 1. Time evolution of logarithmic negativity of a nonsymmetric Gaussian state with $a=2$, $b=1$, $c_1=1$, and $c_2=-1$ in several noncorrelated environments. The solid line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=0$; the dashed line refers to the case $\mu_1=1/2$, $\mu_2=1/6$, $r_1=r_2=0$; the dot-dashed line refers to the case $\mu_1=1/6$, $\mu_2=1/2$, $r_1=r_2=0$; the dotted line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=1$. In all cases the squeezing angle $\varphi_2=0$. All the plotted quantities are dimensionless.

We now investigate the duration and robustness of entanglement during the evolution of the field modes in the channels. Let us consider an initial entangled state $\boldsymbol{\sigma}_e$ evolving in the bath. Making use of the separability criterion, Eq. (20), one finds that the state $\boldsymbol{\sigma}_e$ becomes separable at a certain time t if

$$ue^{-4\Gamma t}+ve^{-3\Gamma t}+we^{-2\Gamma t}+ye^{-\Gamma t}+z=0. \tag{32}$$

The coefficients u, v, w, y, and z are functions of the nine parameters characterizing the initial state and the channel (see Appendix A). Equation (32) is an algebraic equation of fourth degree in the unknown $k=e^{-\Gamma t}$. The solution k_{ent} of such an equation closest to one, and satisfying $k_{ent}\leqslant 1$ can be found for any given initial entangled state. Its knowledge promptly leads to the determination of the ''entanglement time'' t_{ent} of the initial state in the channel, defined as the time interval after which the initial state becomes separable,

$$t_{ent}=-\frac{1}{\Gamma}\ln k_{ent}. \tag{33}$$

The results of the numerical analysis of the evolution of entanglement and mixedness for several initial states are reported in Figs. 1–8. In general, one can see that, trivially, a less mixed environment better preserves both purity and entanglement by prolonging the entanglement time. More remarkably, Fig. 1 shows that local squeezing of the two uncorrelated channels does not help preserve quantum correlations between the evolving modes. Moreover, as can be seen from Fig. 1, states with greater uncertainties on, say, mode 1 ($a>b$) better preserves its entanglement if bath 1 is more mixed than bath 2 ($\mu_1<\mu_2$). A quite interesting fea-

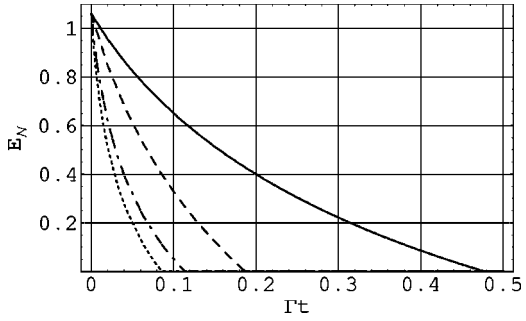

FIG. 2. Time evolution of logarithmic negativity of a symmetric Gaussian state with $a=1.5$, $b=1.5$, $c_1=1.2$, and $c_2=-1.4$ in several noncorrelated environments. The solid line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=0$; the dashed line refers to the case $\mu_1=\mu_2=1/4$, $r_1=r_2=0$; the dot-dashed line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=1$; the dotted line refers to the case $\mu_1=\mu_2=1/2$, $r_1=0$, $r_2=1.5$. In all cases the squeezing angle $\varphi_2=0$. All the plotted quantities are dimensionless.

ture is shown in Fig. 8: the mutual information is better preserved in squeezed channels, especially at long times. This property has been tested as well on nonentangled states, endowed only with classical correlations, see Fig. 8, and on two-mode squeezed states, see Fig. 7, and seems to hold quite generally. In Fig. 2, the behavior of some initially symmetric states is considered. In this instance we can see that, in squeezed baths, the entanglement of the initial state is better preserved if the squeezing of the two channels is balanced.

The analytic optimization of the relevant quantities characterizing mixedness and correlations in the channel turn out to be difficult in the general case. Thus it is convenient to proceed with our analysis by dealing with particular instances of major phenomenological interest.

A. Standard form states in thermal channels

In this section, we deal with the case of states in generic standard form (parametrized by a,b,c_i) evolving in two thermal channels (parametrized by two—generally different—

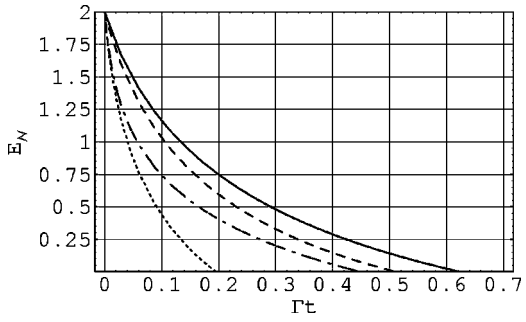

FIG. 3. Time evolution of logarithmic negativity of a two-mode squeezed state with $r=1$ in several noncorrelated environments. The solid line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=0$, $\varphi_2=0$; the dashed line refers to the case $\mu_1=4$, $\mu_2=1$, $r_1=r_2=0$, $\varphi_2=0$; the dot-dashed line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=1$, $\varphi_2=0$; the dotted line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=1$, $\varphi_2=\pi/4$. All the plotted quantities are dimensionless.

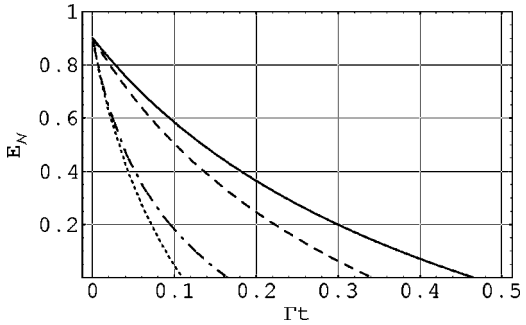

FIG. 4. Time evolution of logarithmic negativity of a two-mode squeezed thermal state with initial purity $\mu=1/9$, $r=1$ in several noncorrelated environments. The solid line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=0$, $\varphi_2=0$; the dashed line refers to the case $\mu_1=4$, $\mu_2=1$, $r_1=r_2=0$, $\varphi_2=0$; the dot-dashed line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=1$, $\varphi_2=0$; the dotted line refers to the case $\mu_1=\mu_2=1/2$, $r_1=r_2=1$, $\varphi_2=\pi/4$. All the plotted quantities are dimensionless.

mean photon numbers N_i's). This instance is particularly relevant, because it gives a basic description of actual experimental settings involving, for instance, fiber-mediated communications.

The purity μ of the global quantum state turns out to be a decreasing function of N_i's at any given time. The symplectic eigenvalue $\tilde{n}_-$ is also in general an increasing function of N_i's. Therefore, the entanglement of the evolving state is optimal for ideal vacuum environments, which is quite trivial, recalling the well-understood synergy between entanglement and purity for general quantum states.

B. Entanglement time of symmetric states

We have already provided a way of computing the entanglement time of an arbitrary two-mode state in arbitrary channels. The expression of such a quantity is, unfortunately, rather involved in the general case. However, focusing on

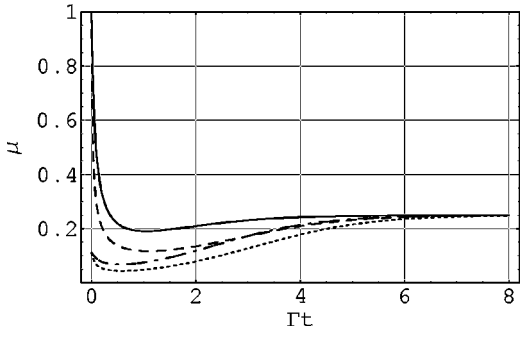

FIG. 5. Time evolution of the purity of two-mode squeezed thermal states. The solid line refers to a two-mode squeezed vacuum state with $r=1$ in an environment with $\mu_1=\mu_2=1/2$, $r_1=r_2=0$; the dashed line shows the behavior of the same state for $\mu_1=\mu_2=1/2$, $r_1=r_2=1$. The dot-dashed line refers to a mixed state with $\mu=1/9$, $r=1$ in an environment with $\mu_1=\mu_2=1/2$, $r_1=r_2=0$; the dotted line refers to the same state for $\mu_1=\mu_2=1/2$, $r_1=r_2=1$. The squeezing angle φ_2 has always been set to 0. All the plotted quantities are dimensionless.

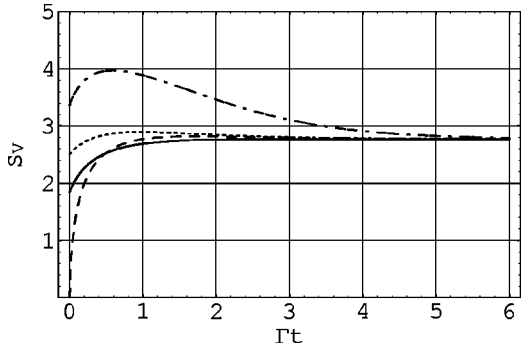

FIG. 6. Time evolution of the von Neumann entropy of several Gaussian states in a thermal environment with $\mu_1=\mu_2=1/3$. the solid line refers to a state with $a=1$, $b=c_1=-c_2=1$; the dashed line refers to a two-mode squeezed vacuum state with $r=1$; the dot-dashed line refers to a squeezed thermal state with $\mu=1/16$, $r=1$; the dotted line refers to a nonentangled state with $a=b=2$, $c_1=-c_2=1.5$. The squeezing angle φ_2 has always been set to 0. All the plotted quantities are dimensionless.

symmetric states (which satisfy $a=b$), some simple analytic results can be found, thanks to the simple form taken by $\tilde{n}_-$ for these states. An initially symmetric entangled state maintains its symmetric standard form if evolving in equal, independent environments (with $N_1=N_2\equiv N_B$). This is the instance we will consider in the following.

Let us suppose that $|c_1|\leqslant|c_2|$, then Eqs. (19) and (21) provide the following bounds for the entanglement time:

$$\ln\left(1+\frac{|c_1|-a+\frac{1}{2}}{N_B}\right)\leqslant\Gamma t_{ent}\leqslant\ln\left(1+\frac{|c_2|-a+\frac{1}{2}}{N_B}\right). \tag{34}$$

Imposing the additional property $c_1=-c_2$ we obtain standard forms which can be written as squeezed thermal states [see Eqs. (11)]. For such states, inequality (34) reduces to

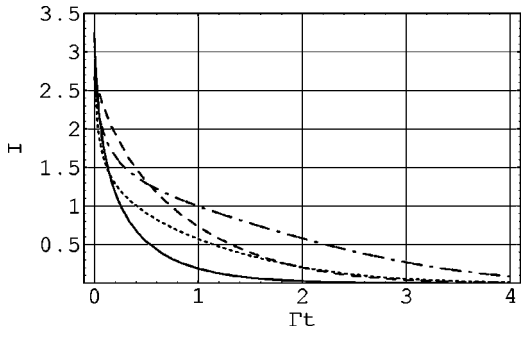

FIG. 7. Time evolution of the mutual information of two-mode squeezed thermal states in an environment with $\mu_1=\mu_2=1/3$. The solid line refers to a pure state with $r=1$ in a nonsqueezed environment; the dotted line refers to the same state in an environment with $r_1=r_2=1$; the dashed line refers to a squeezed thermal state with $\mu=1/16$, $r=1$ in a nonsqueezed environment; the dot-dashed line refers to the same state in a squeezed environment with $r_1=r_2=1$. The squeezing angle φ_2 has always been set to 0. All the plotted quantities are dimensionless.

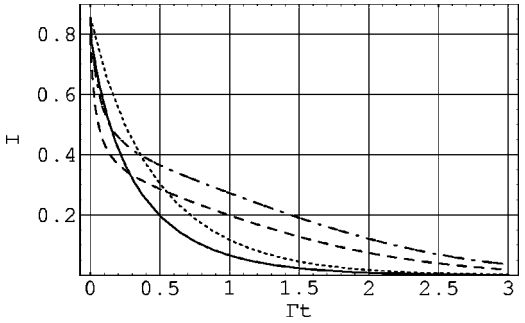

FIG. 8. Time evolution of the mutual information of Gaussian states in an environment with $\mu_1=\mu_2=1/3$. The solid line refers to state with $a=2$, $b=c_1=-c_2=1$ in a nonsqueezed environment; the dotted line refers to the same state in an environment with $r_1=r_2=1$; the dashed line refers to a nonentangled state with $a=b=2$, $c_1=-c_2=1.5$ in a nonsqueezed environment; the dot-dashed line refers to the same state in a squeezed environment with $r_1=r_2=1$. The squeezing angle φ_2 has always been set to 0. All the plotted quantities are dimensionless.

$$t_{ent}=\frac{1}{\Gamma}\ln\left(1+\frac{1-e^{-2r}}{2\sqrt{\mu}N_B}\right). \tag{35}$$

In particular, for $\mu=1$, one recovers the entanglement time of a two-mode squeezed vacuum state in a thermal channel [7,10,19]. Note that two-mode squeezed vacuum states encompass all the possible standard forms of pure Gaussian states.

C. Two-mode squeezed thermal states

As we have seen, two-mode squeezed thermal states constitute a relevant class of Gaussian states, parametrized by their purity μ and by the squeezing parameter r according to Eqs. (11). In particular, two-mode squeezed vacuum states (or twin beams), which can be defined as squeezed thermal states with $\mu=1$, correspond to maximally entangled symmetric states for fixed marginal purity. Therefore, they constitute a crucial resource for possible applications of Gaussian states in quantum information engineering.

For squeezed thermal states (chosen as initial conditions in the channel), it can be shown analytically that the partially transposed symplectic eigenvalue $\tilde{n}_-$ is at any time an increasing function of the bath squeezing angle φ_2: ''parallel'' squeezing in the two channels optimizes the preservation of entanglement. Both in the instance of two equal squeezed baths (i.e., with $r_1=r_2=r$) and of a thermal bath joined to a squeezed one (i.e., $r_1=r$ and $r_2=0$), it can be shown that $\tilde{n}_-$ is an increasing function of r. These analytical results agree with those provided in Ref. [8] in the study of the qualitative degradation of entanglement for pure squeezed states. The proofs of the above statements are given in Appendix B.

Such analytical considerations, supported by direct numerical analysis, clearly show that a local squeezing of the environment faster degrades the entanglement of the initial state. The same behavior occurs for purity. The time evolution of the logarithmic negativity of two-mode squeezed states—thermal and pure—is shown in Figs. 3 and 4. The evolution of the global purity is reported in Fig. 5, while the evolutions of the von Neumann entropy and of the mutual information are shown, respectively, in Figs. 6 and 7.

V. SUMMARY AND CONCLUSIONS

We studied the evolution of mixedness, entanglement, and mutual information of initial two-mode Gaussian states evolving in uncorrelated Gaussian environments. We derived exact general relations that allowed us to determine the time evolution of such quantities, and provided analytical estimates on the entanglement time. The optimal bath parameters for the preservation of quantum correlations and purity have been determined for thermal baths and for two-mode squeezed states in more general baths. A detailed numerical analysis has been performed for the most general cases.

We found that, in general, a local squeezing of the baths does not help to preserve purity and quantum correlations of the evolving state, both at small times (i.e., for $\Gamma t \lesssim 1$) and asymptotically. On the other hand, local squeezing of the baths can improve the preservation of the mutual information in uncorrelated channels. Besides, coherence and correlations are better maintained in environments with lower average number of photons.

The present study may be be extended to the case of n-mode Gaussian states. This generalization would be desirable, since the practical implementation of quantum information protocols usually requires some redundancy. For three-mode Gaussian states, separability conditions analogous to inequality (6) have been determined [35], and could be exploited to provide a qualitative picture of the evolution of three-mode entanglement in noisy channels.

ACKNOWLEDGMENTS

A.S., F.I., and S.D.S. thank INFM, INFN, and MIUR under National Project No. PRIN-COFIN 2002 for financial support. The work of M.G.A.P. was supported in part by UE programs ATESIT (Contract No. IST-2000-29681).

APPENDIX A: EXPLICIT DETERMINATION OF MIXEDNESS AND ENTANGLEMENT IN THE GENERAL CASE

Here we provide explicit expressions which allow one to determine the exact evolution in uncorrelated channels of a generic initial state in standard form. The relevant quantities $E_{\mathcal{N}}, \mu, S_V, I$, as we have seen, are all functions of the four $\mathrm{Sp}_{(2,\mathbb{R})} \oplus \mathrm{Sp}_{(2,\mathbb{R})}$ invariants. Let us then write such quantities as follows:

$$\mathrm{Det}\,\boldsymbol{\sigma} = \sum_{k=0}^{4} \Sigma_k e^{-k\Gamma t}, \tag{A1}$$

$$\mathrm{Det}\,\boldsymbol{\alpha} = \sum_{k=0}^{2} \alpha_k e^{-k\Gamma t}, \tag{A2}$$

$$\mathrm{Det}\,\boldsymbol{\beta} = \sum_{k=0}^{2} \beta_k e^{-k\Gamma t}, \tag{A3}$$

$$\mathrm{Det}\,\boldsymbol{\gamma} = \gamma_2 e^{-2\Gamma t}, \tag{A4}$$

defining the sets of coefficients $\Sigma_i, \alpha_i, \beta_i, \gamma_i$. One has

$$\begin{aligned}
\Sigma_4 = {} & a^2 b^2 + \frac{a^2}{4\mu_2^2} + \frac{b^2}{4\mu_1^2} - a^2 b \frac{\cosh 2r_2}{\mu_2} - a b^2 \frac{\cosh 2r_1}{\mu_1} \\
& + ab \frac{\cosh 2r_1 \cosh 2r_2}{\mu_1 \mu_2} - a \frac{\cosh 2r_1}{4\mu_1 \mu_2^2} - b \frac{\cosh 2r_2}{4\mu_1^2 \mu_2} \\
& + (c_1^2 + c_2^2) \left(a \frac{\cosh 2r_2}{2\mu_2} + \frac{b \cosh 2r_1}{2\mu_1} \right. \\
& \left. - \frac{\cosh 2r_1 \cosh 2r_2}{4\mu_1 \mu_2} - \frac{\sinh 2r_1 \sinh 2r_2 \cos 2\varphi_2}{4\mu_1 \mu_2} - ab \right) \\
& + (c_1^2 - c_2^2) \left(a \frac{\sinh 2r_2 \cos 2\varphi_2}{2\mu_2} + b \frac{\sinh 2r_1}{2\mu_1} \right. \\
& \left. - \frac{\sinh 2r_1 \cosh 2r_2}{4\mu_1 \mu_2} - \frac{\cosh 2r_1 \sinh 2r_2 \cos 2\varphi_2}{4\mu_1 \mu_2} \right) \\
& + c_1^2 c_2^2 + \frac{1}{16 \mu_1^2 \mu_2^2},
\end{aligned} \tag{A5}$$

$$\begin{aligned}
\Sigma_3 = {} & -2\frac{a^2}{4\mu_2^2} - 2\frac{b^2}{4\mu_1^2} + a^2 b \frac{\cosh 2r_2}{\mu_2} + a b^2 \frac{\cosh 2r_1}{\mu_1} \\
& - 2ab \frac{\cosh 2r_1 \cosh 2r_2}{\mu_1 \mu_2} + 3a \frac{\cosh 2r_1}{4\mu_1 \mu_2^2} + 3b \frac{\cosh 2r_2}{4\mu_1^2 \mu_2} \\
& - (c_1^2 - c_2^2) \left(a \frac{\sinh 2r_2 \cos 2\varphi_2}{2\mu_2} + b \frac{\sinh 2r_1}{2\mu_1} \right. \\
& \left. - 2\frac{\sinh 2r_1 \cosh 2r_2}{4\mu_1 \mu_2} - 2\frac{\cosh 2r_1 \sinh 2r_2 \cos 2\varphi_2}{4\mu_1 \mu_2} \right) \\
& - (c_1^2 + c_2^2) \left(a \frac{\cosh 2r_2}{2\mu_2} + \frac{b \cosh 2r_1}{2\mu_1} \right. \\
& \left. - 2\frac{\cosh 2r_1 \cosh 2r_2}{4\mu_1 \mu_2} - 2\frac{\sinh 2r_1 \sinh 2r_2 \cos 2\varphi_2}{4\mu_1 \mu_2} \right) \\
& - \frac{1}{4\mu_1^2 \mu_2^2},
\end{aligned} \tag{A6}$$

$$\Sigma_2=\frac{a^2}{4\mu_2^2}+\frac{b^2}{4\mu_1^2}+ab\frac{\cosh 2r_1\cosh 2r_2}{\mu_1\mu_2}-3a\frac{\cosh 2r_1}{4\mu_1\mu_2^2}$$
$$-3b\frac{\cosh 2r_2}{4\mu_1^2\mu_2}-(c_1^2+c_2^2)\left(\frac{\cosh 2r_1\cosh 2r_2}{4\mu_1\mu_2}\right.$$
$$\left.+\frac{\sinh 2r_1\sinh 2r_2\cos 2\varphi_2}{4\mu_1\mu_2}\right)-(c_1^2-c_2^2)$$
$$\times\left(\frac{\sinh 2r_1\cosh 2r_2}{4\mu_1\mu_2}+\frac{\cosh 2r_1\sinh 2r_2\cos 2\varphi_2}{4\mu_1\mu_2}\right)$$
$$+\frac{1}{16\mu_1^2\mu_2^2}, \tag{A7}$$

$$\Sigma_1=+a\frac{\cosh 2r_1}{4\mu_1\mu_2^2}+b\frac{\cosh 2r_2}{4\mu_1^2\mu_2}-\frac{1}{4\mu_1^2\mu_2^2}, \tag{A8}$$

$$\Sigma_0=\frac{1}{16\mu_1^2\mu_2^2}, \tag{A9}$$

$$\alpha_2=a^2-a\frac{\cosh 2r_1}{\mu_1}+\frac{1}{4\mu_1^2}, \tag{A10}$$

$$\alpha_1=a\frac{\cosh 2r_1}{\mu_1}-2\frac{1}{4\mu_1^2}, \tag{A11}$$

$$\alpha_0=\frac{1}{4\mu_1^2}, \tag{A12}$$

$$\beta_2=b^2-b\frac{\cosh 2r_2}{\mu_2}+\frac{1}{4\mu_2^2}, \tag{A13}$$

$$\beta_1=b\frac{\cosh 2r_2}{\mu_2}-2\frac{1}{4\mu_2^2}, \tag{A14}$$

$$\beta_0=\frac{1}{4\mu_2^2}, \tag{A15}$$

$$\gamma_2=c_1c_2. \tag{A16}$$

The coefficients of Eq. (32), whose solution k_{ent} allows one to determine the entanglement time of an arbitrary two-mode Gaussian state, read

$$u=\Sigma_4, \tag{A17}$$

$$v=\Sigma_3, \tag{A18}$$

$$w=\Sigma_2-\alpha_2-\beta_2-|\gamma_2|, \tag{A19}$$

$$y=\Sigma_1-\alpha_1-\beta_1, \tag{A20}$$

$$z=\Sigma_0-\alpha_0-\beta_0+\tfrac{1}{4}. \tag{A21}$$

APPENDIX B: PROOFS FOR TWO-MODE SQUEEZED STATES

In this appendix we consider a two-mode squeezed thermal state of the form of Eq. (11) as the initial input in the noisy channels.

We first deal with the dependence of entanglement and mixedness on the squeezing angle φ_2 of bath 2. It can be easily shown (see Appendix A) that $\Delta(\boldsymbol{\sigma})$ does not depend on φ_2, whereas Det $\boldsymbol{\sigma}$ turns out to be a decreasing function of $\cos\varphi_2$. Therefore, since the symplectic eigenvalue $\tilde{n}_-$ increases with Det $\boldsymbol{\sigma}$, one has that $\varphi_2=0$ is the optimal choice for maximizing both entanglement and purity of the evolving state.

We now address the instance of two equally squeezed baths, with $N_i\equiv N_B$, $r_i\equiv r_B$, and $\varphi_2=0$. The time dependent covariance matrix $\boldsymbol{\sigma}_{2m}$ can be written in the form

$$\boldsymbol{\sigma}_{2m}=\begin{pmatrix} j_- & 0 & k & 0\\ 0 & j_+ & 0 & -k\\ k & 0 & j_- & 0\\ 0 & -k & 0 & j_+\end{pmatrix},$$

with

$$j_\mp=\frac{\cosh 2r}{2\sqrt{\mu}}e^{-\Gamma t}+(N_B+\tfrac{1}{2})e^{\mp 2r_B}(1-e^{-\Gamma t}),$$

$$k=\frac{\sinh 2r}{2\sqrt{\mu}}e^{-\Gamma t}.$$

The standard form of $\boldsymbol{\sigma}_{2m}$ is easily found just by squeezing the field in the two modes of the same quantity $\sqrt{j_+/j_-}$. The result is a symmetric standard form, whose smallest partially transposed symplectic eigenvalue $\tilde{n}_-$ can be computed according to Eq. (21):

$$\tilde{n}_-=(j_--k)(j_+-k)=d\cosh 2r_B+\cdots,$$

where the terms that do not depend on r_B are irrelevant to our discussion and have thus been neglected. The coefficient d is a positive function of t, r, and N_B, so that the best choice to maximize entanglement at any given time is given by $r_B=0$. Quite obviously, $\tilde{n}_-$ turns out to be an increasing function of N_B as well.

Finally, we deal with the instance in which bath 1 is squeezed while bath 2 is thermal, with $r_2=0$. For ease of notation we define $|\boldsymbol{\sigma}|=\text{Det}\,\boldsymbol{\sigma}$. We recall that $2\tilde{n}_-^2=\tilde{\Delta}-\sqrt{\tilde{\Delta}^2-4|\boldsymbol{\sigma}|}$. Thus, for entangled states (for which $\tilde{n}_-<1/2$), one finds

$$\partial_{|\boldsymbol{\sigma}|}(2\tilde{n}_-^2) > -4\partial_{\tilde{\Delta}}(2\tilde{n}_-^2) > 0.$$

The sign of the quantity $4\partial_{r_1}|\boldsymbol{\sigma}| - \partial_{r_1}\tilde{\Delta}$ for the case of the initial two-mode squeezed can be shown, after some algebra, to be determined by

$$4(e^{\Gamma t}-1)\cosh 2r n_2^2 + (3+\cosh 4r)n_2 - (e^{\Gamma t}+1)\cosh 2r.$$

This second degree polynomial is positive for $n_2 \equiv N_2 + 1/2 \geqslant 1/2$. This proves that the entanglement decreases as the squeezing of bath 1 increases.

[1] *Quantum Information Theory with Continuous Variables*, edited by S.L. Braunstein and A.K. Pati (Kluwer Academic, Dordrecht, 2002).

[2] H.P. Yuen and A. Kim, Phys. Lett. A **241**, 135 (1998); F. Grosshans, G. Van Assche, J. Wenger, R. Brouri, N.J. Cerf, and P. Grangier, Nature (London) **421**, 238 (2003).

[3] A. Furusawa, J.L. Sorensen, S.L. Braunstein, C.A. Fuchs, H.J. Kimble, and E.S. Polzik, Science **282**, 706 (1998); T.C. Zhang, K.W. Goh, C.W. Chou, P. Lodahl, and H.J. Kimble, Phys. Rev. A **67**, 033802 (2003).

[4] L.-M. Duan and G.-C. Guo, Quantum Semiclassic. Opt. **9**, 953 (1997).

[5] T. Hiroshima, Phys. Rev. A **63**, 022305 (2001).

[6] S. Scheel and D.-G. Welsch, Phys. Rev. A **64**, 063811 (2001).

[7] M.G.A. Paris, in *Entangled Light and Applications* in *Progress in Quantum Physics Research*, edited by V. Krasnoholovets (Nova Science, New York, in press).

[8] D. Wilson, J. Lee, and M.S. Kim, J. Opt. Soc. Am. B **50**, 1809 (2003).

[9] S. Olivares, M.G.A. Paris, and A.R. Rossi, Phys. Lett. A **319**, 32 (2003).

[10] J.S. Prauzner-Bechcicki, e-print quant-ph/0211114.

[11] W.P. Bowen, R. Schnabel, P.K. Lam, and T.C. Ralph, Phys. Rev. Lett. **90**, 043601 (2003).

[12] G. Vidal and R.F. Werner, Phys. Rev. A **65**, 032314 (2002); K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, *ibid.* **58**, 883 (1998).

[13] P. Tombesi and D. Vitali, Phys. Rev. A **50**, 4253 (1994); Appl. Phys. B: Lasers Opt. **60**, S69 (1995).

[14] N. Lütkenhaus, J.I. Cirac, and P. Zoller, Phys. Rev. A **57**, 548 (1998).

[15] H.M. Wiseman and G.J. Milburn, Phys. Rev. Lett. **70**, 548 (1993); H.M. Wiseman and G.J. Milburn, Phys. Rev. A **49**, 1350 (1994).

[16] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).

[17] See, e.g., S.M. Barnett and P.M. Radmore, *Methods in Theoretical Quantum Optics* (Clarendon Press, Oxford, 1997).

[18] R. Simon, N. Mukunda, and B. Dutta, Phys. Rev. A **49**, 1567 (1994).

[19] L.-M. Duan, G. Giedke, J.I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).

[20] A.S. Holevo, M. Sohma, and O. Hirota, Phys. Rev. A **59**, 1820 (1999); A.S. Holevo and R.F. Werner, *ibid.* **63**, 032312 (2001).

[21] A. Serafini, F. Illuminati, and S. De Siena, J. Phys. B **37**, L21 (2004).

[22] S.M. Barnett and S.J.D. Phoenix, Phys. Rev. A **40**, 2404 (1989); **44**, 535 (1991); M.G.A. Paris, *ibid.* **59**, 1615 (1999).

[23] G. Giedke, M.M. Wolf, O. Krüger, R.F. Werner, and J.I. Cirac, Phys. Rev. Lett. **91**, 107901 (2003).

[24] G.S. Agarwal, Phys. Rev. A **3**, 828 (1971).

[25] L. Henderson and V. Vedral, J. Phys. A **34**, 6899 (2001).

[26] The proof of such properties can be found in Ref. [12] for finite-dimensional Hilbert spaces. Anyway, since the proof essentially relies on the use of the Jordan decomposition, it still holds in infinite dimension as shown in Ref. [27].

[27] J. Eisert, Ph.D. thesis, University of Potsdam, 2001 (unpublished).

[28] J. Eisert, C. Simon, and M.B. Plenio, J. Phys. A **35**, 3911 (2002).

[29] $E_F(\varrho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \Sigma p_i E(|\psi_i\rangle\langle\psi_i|)$, where $E(|\psi\rangle\langle\psi|)$ is the entropy of entanglement of the pure state $|\psi\rangle$, defined as the Von Neumann entropy of its reduced density matrix, and the min is taken over all the pure states realization of $\varrho = \Sigma p_i |\psi_i\rangle\langle\psi_i|$.

[30] See, e.g., D. Walls and G. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).

[31] Possible squeezed reservoirs are treated in M.-A. Dupertuis and S. Stenholm, J. Opt. Soc. Am. B **4**, 1094 (1987); M.-A. Dupertuis, S.M. Barnett, and S. Stenholm, *ibid.* **4**, 1102 (1987); Z. Ficek and P.D. Drummond, Phys. Rev. A **43**, 6247 (1991); K.S. Grewal, *ibid.* **67**, 022107 (2003); see also C.W. Gardiner and P. Zoller, *Quantum Noise* (Springer Verlag, Berlin, 1999), and Ref. [34].

[32] Any centered single-mode Gaussian state can be written in this form, see P. Marian and T.A. Marian, Phys. Rev. A **47**, 4474 (1993).

[33] M.G.A. Paris, F. Illuminati, A. Serafini, and S. De Siena, Phys. Rev. A **68**, 012314 (2003).

[34] M.S. Kim and N. Imoto, Phys. Rev. A **52**, 2401 (1995).

[35] G. Giedke, B. Kraus, M. Lewenstein, and J.I. Cirac, Phys. Rev. A **64**, 052303 (2001).