All quantum observables in a hidden-variable model must commute simultaneously

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Under a standard set of assumptions for a hidden-variable model for quantum events we show that all observables must commute simultaneously. This seems to be an ultimate statement about the inapplicability of the usual hidden-variable model for quantum events. And, despite Bell's complaint that a key condition of von Neumann's was quite unrealistic, we show that these conditions, under which von Neumann produced the first no-go proof, are entirely equivalent to those introduced by Bell and Kochen and Specker. As these conditions are also equivalent to those under which the Bell-Clauster-Horne inequalities are derived, we see that the experimental violations of the inequalities demonstrate only that quantum observables do not commute.

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I. INTRODUCTION

A wide range of no-go proofs for hidden-variable (HV) models for quantum events have been developed and discussed over many years; see [1–8]. Here we derive an alternative no-go proof with a rather striking and informative outcome: under the usual assumptions of an HV model *every* pair of observables must commute. As the conditions for an HV model studied here are known to be entirely equivalent to the conditions under which the usual Bell-Clauser-Horne (BCH) inequalities apply, we see that the numerous experimental violations of the inequalities show only that quantum observables do not commute. And as the initial conditions in the elegant inequality-free, no-go proofs of Peres, and Greenberger, Horne, and Zeilinger (see [5–6]) are exactly those of a deterministic HV model, our argument will show, as well, that these proofs yield only the same conclusion: quantum observables do not commute.

Our paper is organized as follows. We first consider specifications for a deterministic (or factorizable stochastic) hidden-variable models, such as are presented in [1,2]. By extension of a result appearing in [9] we obtain our result on the simultaneous commutativity. We conclude with a short discussion about hybrid HV models, such as those of [10,11], that offer an alternative to the HV assumptions made here. These suggest a more promising route, should one be sought, for characterizations of quantum events as classical statistical schemes.

We briefly anticipate some of this concluding discussion here. Thus, one of the more interesting consequences of our results is that the original conditions under which von Neumann [12] derived the first no-go proof for HV models are entirely equivalent to those introduced much later by Bell [7] and Kochen and Specker [8]. Bell had criticized von Neumann for requiring the deterministic value assignment, under an HV model, to apply to sums over noncommuting (incompatible) observables, as well as for commuting ones; see [6,13]. As an assignment for values across incompatible experiments seemed, to Bell, to be physically quite unrealistic, he introduced the less restrictive condition that the value assignment need apply only to across sums of commuting observables. However, we show that Bell's conditions (and those of Kochen and Specker) are in fact entirely equivalent to those of von Neumann, since, under an HV model of the Bell or Kochen-Specker type, all observables must commute simultaneously.

II. HIDDEN-VARIABLE MODELS

Detailed specifications for a hidden-variable model are given in [1,2,9], some of which we now recall. Let $Q = Q(H,D,\Xi)$ denote a quantum system with Hilbert space H, quantum density operator D, and a family of observables Ξ.

Let $\Omega = \Omega(\Lambda,\sigma(\Lambda),\mu)$ denote a classical probability space, where Λ is a nonempty set, $\sigma(\Lambda)$ is a Boolean σ algebra of subsets of Λ, and μ is a probability measure on $\sigma(\Lambda)$.

As used in this paper, a hidden variable model for a quantum system in a given state D may make one or more of the following assumptions.

HV(a): Given $\omega \in \Lambda$, $A \in \Xi$, there is a mapping f from the pair (ω,A) to $\Re$; it is required that the value of $f(\omega,A)$ be an eigenvalue of A.

HV(b): For any two commuting observables A, B, the mapping f is such that

$$f(\omega,A+B) = f(\omega,A) + f(\omega,B). \tag{2.1}$$

HV(c): The measure μ correctly returns the marginal probabilities for each observable A: that is, for any real Borel set S, μ is such that

$$\mathrm{tr}[DP_A(S)] = \int f(\omega,P_A(S))d\mu, \tag{2.2}$$

where $P_A(S)$ is the projector associated with set S in the spectral resolution for A.

HV(d): For any two commuting observables A, B, the measure μ correctly returns the joint probabilities; that is, for S, T real Borel sets, the measure μ is such that

$$\mathrm{tr}[DP_A(S)P_B(T)] = \int f(\omega,P_A(S)P_B(T))d\mu, \tag{2.3}$$

for $P_A(S)$, $P_B(T)$ the projectors associated with sets S, T in the spectral resolutions of A, B, respectively.

Next we recall a discussion and a definition from [9] on classical and quantum conditional probability. Assume there is a classical probability space such that outcomes for projectors A, B can be described by a joint distribution μ. It is interesting to ask when the conditional distribution derived from μ agrees with the standard definition of quantum conditional probability; see [14] for details of the probability background. For projectors A, B, and any quantum state D, the quantum conditional probability of A, given B, is defined by

$$\Pr[A|B] = \mathrm{tr}[DBAB]/\mathrm{tr}[DB]. \tag{2.4}$$

Consider now the two conditional distributions: that derived from μ and that derived from the standard definition (2.4) above. When these are equal we will say that the *conditional probability rule* holds. For any projector X, let

$$X^{-1}(1) = \{\omega \in \Lambda : X(\omega) = 1\}. \tag{2.5}$$

Then as shown in [9] (Theorem 1) we have the following.

Theorem 1. Assume $\dim H \geqslant 3$ and that HV(a), HV(c), and HV(d) hold. Then for one-dimensional projectors A, B, the conditional probability rule holds:

$$\mu[a|b] = \mu[a \cap b]/\mu[b] = \mathrm{tr}[DBAB]/\mathrm{tr}[DB], \tag{2.6}$$

where $a = A^{-1}(1)$ and $b = B^{-1}(1)$.

We observe that the restriction of this result, to one-dimensional projectors, is not required but the proof in this case can be obtained using straightforward inner product vector space methods; see [9] and Gudder [15] (corollary 5.17). We do not argue here that the no-go proof presented below, based on this restricted case, is in any sense technically simpler than the original Kochen-Specker or Bell proofs—this is partly a matter of taste. However, we will argue that the end point of the proof presented here—namely, commutativity—is more informative and transparent as regarding the problems with local HV models, in particular those studied using the BCH inequalities.

We also note that, as discussed in [9], there are two other conditions equivalent to HV(b): a *Borel function rule* and a *product rule*, both introduced in [2]. Any of these three choices will suit the purposes of our discussion.

In [1] the set of conditions HV(a), HV(c), and HV(d) is called a *deterministic hidden-variable model* (equivalently, a *factorizable stochastic model*). To be more precise, in this paper we take the three conditions {HV(a), HV(c), HV(d)} to jointly define an *HV model*. As shown in [1] the conditions {HV(a), HV(c), HV(d)} are also entirely equivalent to {HV(a), HV(b), HV(d)}, and these are the conditions introduced by Bell [7] and Kochen-Specker [8]. Moreover, as shown by Fine [3] [proposition (2)], a necessary and sufficient condition for the existence of a deterministic HV model is that the usual BCH inequalities must hold. Van Fraassen [16] (pp. 102–105) gives further details of the Fine results, showing how locality, in the form of factorizability, is built into Fine's definition of HV models. Further details concerning how locality might be differently defined can be found in Fine [11] (Appendix to Chap. 4).

III. HIDDEN VARIABLES AND COMMUTATIVITY

Theorem 2. Assume $\dim H \geqslant 3$ and that an HV model holds for quantum events. Then all quantum observables commute.

Proof. Let A, B be two quantum observables. Without loss of generality we may assume they are one-dimensional projectors: A, B commute if and only if all projectors appearing in their spectral resolutions commute, and all the projectors may be reexpressed as (nonunique) sums of one-dimensional ones. From *Theorem 1* we have that

$$\mu[a,b] = \mu[a|b]\mu[b] = \{\mathrm{tr}[DBAB]/\mathrm{tr}[DB]\}\mathrm{tr}[DB] = \mathrm{tr}[DBAB] \tag{3.1}$$

and also that

$$\mu[a,b] = \mu[b|a]\mu[a] = \{\mathrm{tr}[DABA]/\mathrm{tr}[DA]\}\mathrm{tr}[DA] = \mathrm{tr}[DABA]. \tag{3.2}$$

Hence

$$\mathrm{tr}[DBAB] = \mathrm{tr}[DABA] \tag{3.3}$$

for all density operators D. Thus $BAB = ABA$. From this, and using $A^2 = A$, $B^2 = B$, we easily show that

$$(AB - BA)^2 = 0. \tag{3.4}$$

Since $C = AB - BA$ is skew Hermitian, $C^2 = 0$ implies $C = 0$, and the result is proved. ■

We note that Fine [3] (Theorem 7) obtained a commutativity result using a rather different condition, called the *joint distribution (jd) condition*. Briefly, this states that a measure space be given which returns the correct marginal distributions for a set of (not necessarily commuting) observables $A_1, A_2, \ldots, A_k$ and which also reproduces the marginal for any observable of the form $f(A_1, A_2, \ldots, A_k)$, for any Borel measurable f. The joint distribution condition does not by itself reference HV models, but might be considered as useful background to the problems with such models. More precisely, the HV conditions given above, HV(a), HV(c), and HV(d), do not in any obvious way validate the Borel function requirement, just stated, in the *jd condition*. On the other hand, we have from above that an HV model is equivalent to simultaneous commutativity for all observables, so the *jd condition* is now seen as an interesting alternative for the collected assumptions of a deterministic HV model.

IV. DISCUSSION

We have shown that under the standard HV model assumptions ($\dim H \geqslant 3$), all quantum observables must commute. Seemingly no more sharply informative no-go proof is possible, and the conclusion obtains under the Bell, Kochen-Specker, or Fine conditions for an HV model. In particular, we see that the sum rule HV(b) is valid for noncommuting observables, in the presence of the other conditions for an HV model—namely, HV(a), and HV(d). The requirement that HV(b) apply for noncommuting observables was made by von Neumann in his original 1932 no-go proof for HV models. This was declared by Bell to be entirely unphysical for any plausible HV model for quantum events, and he preferred to assume HV(b) only for commuting observables; see the discussion in [6,13]. In fact, we now see that von Neumann's HV assumptions were no more or less unphysical than were Bell's or Kochen-Specker's apparently less restrictive set of assumptions. In this sense von Neumann's original proof is vindicated.

Finally, given the above it seems appropriate to urge consideration instead of models for quantum events that are not tied to these HV conditions. Effectively, the Bell, Kochen-Specker, and the (now equivalent) von Neumann conditions are still too restrictive and truly weaker models could be considered. Such hybrid models appear to be already at hand, as in [10]; see also the discussion of *prism models* in [11] and the references to the literature cited therein. A significant change presented by these prism models is that the hidden variables are not assumed to be factorizable, but do satisfy what Fine calls *Bell locality*, an assumption briefly described as "no outcome-fixing action-at-a-distance;" see [11] (Appendix to Chap. 4). Under this construal, violations of the BCH inequalities do not constitute a failure of Bell locality, and our no-go commutativity result does not extend to a negation of Bell locality.

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