Distinguishability and nonclassicality of one-mode Gaussian states

Paulina Marian,[1,2] Tudor A. Marian,[1,3] and Horia Scutaru[4]
[1]*Institute for Theoretical Atomic, Molecular, and Optical Physics, Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138, USA*
[2]*Department of Chemistry, University of Bucharest, Boulevard Regina Elisabeta 4-12, R-030018 Bucharest, Romania*
[3]*Department of Physics, University of Bucharest, P.O. Box MG-11, R-077125 Bucharest-Măgurele, Romania*
[4]*Center for Advanced Studies in Physics of the Romanian Academy, Calea 13 Septembrie 13, R-050711 Bucharest, Romania*
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We consider the class of the one-mode Gaussian states of the quantum radiation field. The relative entropy of such a state with respect to a similar one is derived. We analyze the entropic amount of nonclassicality of a Gaussian state defined as the minimal relative entropy of any classical Gaussian state with respect to it. A similar quantity built with the Hilbert-Schmidt distance is also calculated. Both nonclassicality measures are then compared with the Bures-metric degree of nonclassicality evaluated previously. The properties of the closest classical Gaussian state, as well as the decrease of the nonclassicality under thermal noise mappings are carefully examined in each case. For mixed states we find that only the Bures-distance amount of nonclassicality is equivalent to the nonclassical depth.

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I. INTRODUCTION

Quantifying the amount of nonclassicality contained in a quantum state of light is still an important problem in both quantum optics and quantum information theory. Interest in the nonclassical properties of Gaussian states was recently renewed by the experimental realization of the teleportation of a one-mode coherent state [1]. For a potential experiment using a nonclassical state it has become important to evaluate the extent to which the nonclassicality can survive imperfect teleportation [2].

A direct approach for describing how much the distributions of observable quantities in a given quantum state differ from classical distributions is due to Lee [3]. Lee defined a *nonclassical depth* τ_m of an arbitrary one-mode field state as follows: it is the minimum average number of thermal photons added to destroy all the nonclassical properties of the state. By this thermalization process [4], a state possessing a well-behaved P representation of the density operator [5] is obtained. We recall that a well-behaved P representation is either a non-negative regular function or a distribution no more singular than Dirac's δ. As examples, Lee found the nonclassical depth of Gaussian states to depend on the squeezing and that of Fock states to have the maximal value 1. An analysis of nonclassicality using the Wigner function of the state was carried out by Lütkenhaus and Barnett [6]. Nonclassical depth of a phase state was recently studied in Ref. [7].

It was first pointed out by Hillery [8] that a measure of the nonclassicality of a state could be a suitable distance between the state and the set of all classical ones. Hillery termed this measure of nonclassicality a *nonclassical distance*. His choice to employ the trace metric in defining the nonclassical distance was motivated by the fact that the expectation value of an observable is evaluated as the trace of its product with the density operator. However, the trace metric is difficult to deal with analytically. Therefore, Hillery gave upper and lower bounds of the nonclassical trace distance. Meanwhile, the Hilbert-Schmidt metric was quite extensively used in building a nonclassical distance [9,10]. More recently, the present authors [11] gave a measure of nonclassicality for an arbitrary one-mode Gaussian state in terms of its Bures distance [12] to the set of all classical one-mode Gaussian states. Note finally that an analysis of the sensitivity of the two measures of nonclassicality, namely, the nonclassical depth and a specially defined nonclassical distance applied to superpositions of two states, can be found in Ref. [13]. Some recent papers on nonclassicality criteria for quantum states are listed in Ref. [14].

By analyzing this recent progress in quantifying nonclassicality, we consider that two main problems regarding the definition of the nonclassical distance still need to be discussed.

1. Identifying the reference set of classical states. Although the Hillery-type definition is a theoretically appealing measure of nonclassicality, one is unable to use it in practice due to the lack of a parametrization for the whole set of classical one-mode (Gaussian and non-Gaussian) states. A reference set of classical states as large and relevant as possible should therefore be used in practical calculations.

2. Choosing a convenient metric. It is desirable to build a measure of nonclassicality making use of a "distance" which has the ability to distinguish between quantum states, by analyzing the results of any quantum measurement in those states. Opting for one or another of the known "distances" should thus be connected with the description of a general measurement in quantum mechanics. Strong candidates for such an approach are the relative entropy [15,16] and the Bures distance.

In the paper [11], we formulated three requirements that any measure of nonclassicality has to satisfy.

(Q1) The amount of nonclassicality vanishes if and only if the state is classical.

(Q2) Classical transformations preserve the amount of nonclassicality. We have termed *classical* those unitary transformations in Hilbert space which map the classical states into classical states [17].

(Q3) Nonclassicality does not increase under a positive operator-valued measure (POVM) mapping [18].

In this paper we present and compare explicit calculations of the distance-type amount of nonclassicality for single-mode Gaussian states of the radiation field. As a matter of fact, it is convenient to describe these states as displaced squeezed thermal states (DSTS's). In all our calculations we take as reference set the whole class $\mathcal{C}_0$ of the one-mode Gaussian states which have a regular P representation. These states can readily be recognized within the framework of the DSTS parametrization. We define several amounts of *Gaussian nonclassicality* of a Gaussian state ρ as the minimum over the set $\mathcal{C}_0$ either of the relative entropy $S(\rho'/\rho)$ [19] or of a squared distance $d^2(\rho,\rho')$, where $\rho'\in\mathcal{C}_0$. We find it important to stress that our limitation to studying the nonclassicality of solely the Gaussian states has, besides their experimental significance, also pragmatic reasons. What is possible to do in the Gaussian case, for which all the pure and mixed states are described by a simple parametrization, seems to be impossible for other states. In principle, there is no parametrization for the whole set of classical states. In addition, analytic expressions for distances between non-Gaussian mixed states are not yet available.

The paper is organized as follows. Some properties of the one-mode Gaussian states are shortly reviewed in Sec. II. We build then two measures of nonclassicality employing the relative entropy in Sec. III and the Hilbert-Schmidt metric in Sec. IV. Section V is devoted to a comparison of the results with our previous ones based on the Bures metric [11]. Here we also analyze the behavior of the nonclassicality under a thermal noise mapping [20,21]. For this important nonorthogonal POVM mapping, we check the requirement (Q3). Our conclusions are drawn in Sec. VI. The Appendix presents a derivation of the relative entropy of a one-mode Gaussian state with respect to a similar state, which is required in Sec. III.

II. ONE-MODE GAUSSIAN STATES

The one-mode Gaussian states of the radiation field are of the greatest importance both theoretically and experimentally. This broad class includes pure states such as coherent and squeezed coherent ones and mixed states such as displaced thermal and squeezed thermal ones. Moreover, superposition of a thermal field on a Gaussian one yields a Gaussian mixed state of the field. The characteristic function (CF) of a Gaussian state ρ has the form [22]

$$\begin{aligned}\chi(\lambda) &:= \mathrm{Tr}(\rho D(\lambda))\\ &= \exp[-(A+\tfrac{1}{2})|\lambda|^2-\tfrac{1}{2}B^*\lambda^2-\tfrac{1}{2}B(\lambda^*)^2+C^*\lambda\\ &\quad -C\lambda^*]\quad (A\geqslant 0),\end{aligned} \tag{2.1}$$

where

$$D(\alpha):=\exp(\alpha a^\dagger-\alpha^* a) \tag{2.2}$$

is a Weyl displacement operator with the coherent-state amplitude $\alpha\in\mathbb{C}$. The CF is the weight function in the Weyl expansion of the density operator,

$$\rho=\frac{1}{\pi}\int \mathrm{d}^2\lambda\,\chi(\lambda)D(-\lambda). \tag{2.3}$$

A meaningful expression of the CF Eq. (2.1) is obtained by using the real variables x_1,x_2 defined by $\lambda=1/\sqrt{2}(x_2-ix_1)$. We readily get the alternative formula

$$\chi(x_1,x_2)=\exp\{-\tfrac{1}{2}x^T\mathcal{V}x-ix_1\langle q\rangle-ix_2\langle p\rangle\}, \tag{2.4}$$

where x^T is the row vector $(x_1\, x_2)$. q and p are the coordinate and momentum operators, respectively,

$$q=\frac{1}{\sqrt{2}}(a+a^\dagger),\quad p=\frac{1}{\sqrt{2}i}(a-a^\dagger). \tag{2.5}$$

In Eq. (2.4), $\mathcal{V}$ is the 2×2 real symmetric *covariance matrix*

$$\mathcal{V}:=\begin{pmatrix}\sigma_{qq} & \sigma_{qp}\\ \sigma_{pq} & \sigma_{pp}\end{pmatrix}. \tag{2.6}$$

The elements of the matrix (2.6) are determined by the parameters A and B of the CF Eq. (2.1) as

$$\sigma_{qq}:=\langle(\Delta q)^2\rangle=A+\frac{1}{2}-\mathrm{Re}(B), \tag{2.7a}$$

$$\sigma_{pp}:=\langle(\Delta p)^2\rangle=A+\frac{1}{2}+\mathrm{Re}(B), \tag{2.7b}$$

$$\sigma_{qp}=\sigma_{pq}:=\frac{1}{2}\langle\Delta q\Delta p+\Delta p\Delta q\rangle=-\mathrm{Im}(B). \tag{2.7c}$$

Note also the mean photon number in a Gaussian state,

$$\mathrm{Tr}(\rho a^\dagger a)=A+|C|^2. \tag{2.8}$$

As a consequence of the fundamental commutation relation $[q,p]=iI$, the generalized Heisenberg uncertainty relation holds:

$$\sigma_{qq}\sigma_{pp}-(\sigma_{pq})^2\geqslant\frac{1}{4}, \tag{2.9}$$

or equivalently

$$\det\mathcal{V}\geqslant\frac{1}{4}. \tag{2.10}$$

At the same time, $\det\mathcal{V}$ determines the purity of the state. Indeed, from Eq. (2.3) it follows that the degree of purity of the state ρ is the integral

$$\mathrm{Tr}(\rho^2)=\frac{1}{\pi}\int \mathrm{d}^2\lambda|\chi(\lambda)|^2. \tag{2.11}$$

By inserting the CF Eq. (2.4) we find

$$\mathrm{Tr}(\rho^2) = \frac{1}{2\sqrt{\det \mathcal{V}}} \leqslant 1. \tag{2.12}$$

For a pure Gaussian state $\mathrm{Tr}(\rho^2) = 1$, so that the uncertainty relation (2.9) is an equality. The pure Gaussian states are therefore minimum uncertainty states for the generalized uncertainty relation (2.9).

It is useful to recall that any Gaussian state can be written as a DSTS, so that it has a very simple parametrization of the coefficients A, B, C [22]. Indeed, a DSTS ρ is defined as the unitary transform of a thermal state (TS) ρ_T by an ordered pair of one-mode squeeze and displacement operators, $\mathcal{S}(r,\varphi)$ and $D(\alpha)$, respectively,

$$\rho = D(\alpha)\mathcal{S}(r,\varphi)\rho_T \mathcal{S}^\dagger(r,\varphi)D^\dagger(\alpha). \tag{2.13}$$

In Eq. (2.13), the TS

$$\rho_T = \frac{1}{\bar{n}+1}\exp(-\eta a^\dagger a) \tag{2.14}$$

is determined by the dimensionless positive parameter

$$\eta := \frac{\hbar\omega}{k_B T}.$$

Indeed, according to the Planck law, the thermal average photon number is

$$\bar{n} = (e^\eta - 1)^{-1}.$$

The density operator ρ_T, Eq. (2.14), has the spectral decomposition

$$\rho_T = \sum_{n=0}^{\infty} \lambda_n |n\rangle\langle n|, \tag{2.15}$$

with the positive nondegenerate eigenvalues

$$\lambda_n = \frac{\bar{n}^n}{(\bar{n}+1)^{n+1}} \quad (n = 0,1,2,3, \ldots). \tag{2.16}$$

Further,

$$\mathcal{S}(r,\varphi) := \exp\{\tfrac{1}{2}r[e^{i\varphi}(a^\dagger)^2 - e^{-i\varphi}a^2]\} \tag{2.17}$$

is a Stoler operator with the squeeze factor r and the squeeze angle $\varphi \in (-\pi, \pi]$.

The CF of the DSTS (2.13) has the form (2.1) with the coefficients

$$A = (\bar{n} + \tfrac{1}{2})\cosh(2r) - \tfrac{1}{2}, \tag{2.18a}$$

$$B = -(\bar{n} + \tfrac{1}{2})e^{i\varphi}\sinh(2r), \tag{2.18b}$$

$$C = \alpha. \tag{2.18c}$$

Note that, in the DSTS parametrization, the variances (2.7) of the canonical variables are phase dependent,

$$\sigma_{qq} = (\bar{n} + \tfrac{1}{2})[\cosh(2r) + \cos(\varphi)\sinh(2r)], \tag{2.19a}$$

$$\sigma_{pp} = (\bar{n} + \tfrac{1}{2})[\cosh(2r) - \cos(\varphi)\sinh(2r)], \tag{2.19b}$$

$$\sigma_{qp} = (\bar{n} + \tfrac{1}{2})\sin(\varphi)\sinh(2r). \tag{2.19c}$$

Employing Eqs. (2.19) we find that the purity (2.12) of a Gaussian state is determined by its thermal mean occupancy $\bar{n}$ as

$$\mathrm{Tr}(\rho^2) = \frac{1}{2\bar{n}+1}. \tag{2.20}$$

Besides, the squeeze factor can be obtained from the covariance matrix (2.6) via the identity

$$\cosh(2r) = \frac{\mathrm{tr}\mathcal{V}}{2\sqrt{\det\mathcal{V}}}. \tag{2.21}$$

Recall also that the condition for squeezing in coordinate is the reduction of the variance σ_{qq} under its value for a coherent state, $\sigma_{qq} < 1/2$. A DSTS is *effectively* squeezed provided the squeeze factor exceeds the threshold value [22]

$$r_c := \frac{1}{2}\ln(2\bar{n}+1). \tag{2.22}$$

We stress that higher-order squeezing takes place equally if and only if the condition $r > r_c$ is fulfilled. The value (2.22) of the squeeze factor is a *nonclassicality threshold* as well. Indeed, for $r = r_c$, the P representation of a DSTS ceases to exist as a well-behaved Gaussian probability distribution. In other words, a single-mode Gaussian state is classical if $r \leqslant r_c$ and nonclassical if $r > r_c$. Moreover, Lee's nonclassical depth $\tau_m \in [0, \tfrac{1}{2})$ of a Gaussian state is simply [3]

$$\tau_m = |B| - A \quad (A \leqslant |B|), \tag{2.23}$$

or

$$\tau_m = \tfrac{1}{2}\{1 - \exp[-2(r - r_c)]\} \quad (r \geqslant r_c). \tag{2.24}$$

III. NONCLASSICALITY MEASURED BY RELATIVE ENTROPY

The relative entropy of a state σ' with respect to the state σ'' of a quantum system is defined as

$$S(\sigma'/\sigma'') := \mathrm{Tr}\{\sigma''[\ln(\sigma'') - \ln(\sigma')]\}. \tag{3.1}$$

Although the relative entropy is not a metric, it is acceptable as a measure of distinguishability due to the quantum Sanov theorem [16]: the probability of confusing the states σ'' and σ' after performing N measurements on σ' is for $N \gg 1$

$$P_N(\sigma' \to \sigma'') = \exp[-NS(\sigma'/\sigma'')].$$

For a review of the properties of the relative entropy the reader is referred to the classic paper of Wehrl [15] and the recent ones of Vedral *et al.* [16].

We define the entropic amount of Gaussian nonclassicality of a Gaussian state ρ as

$$Q_S(\rho) := \min_{\rho' \in \mathcal{C}_0} S(\rho'/\rho), \tag{3.2}$$

where $\mathcal{C}_0$ is the set of all classical one-mode Gaussian states. If the state ρ is classical ($r \leqslant r_c$), then the minimal value (3.2) is reached for $\rho' = \rho$, because this is the unique state for which the relative entropy vanishes. Hence

$$Q_S(\rho) = 0 \quad (r \leqslant r_c), \tag{3.3}$$

as required by condition (Q1).

To start on the program of Eq. (3.2), we need to: (i) evaluate the relative entropy of a one-mode DSTS with respect to a similar state; (ii) minimize it over the set $\mathcal{C}_0$ of all classical one-mode DSTS's.

The Appendix is devoted to step (i). Thus, the relative entropy $S(\rho'/\rho)$ of the Gaussian state ρ', with the parameters $\overline{n'}, r', \varphi', \alpha'$, with respect to the Gaussian state ρ, having the parameters $\overline{n}, r, \varphi, \alpha$, is given by Eq. (A14). We discuss here step (ii). If ρ is a nonclassical state ($r > r_c$), then minimization of the relative entropy $S(\rho'/\rho)$ under the condition $r' \leqslant r'_c$ is achieved for a DSTS having the displacement parameter

$$\tilde{\alpha} = \alpha \tag{3.4}$$

and the squeeze angle

$$\tilde{\varphi} = \varphi. \tag{3.5}$$

We are left to analyze the minimum of the relative entropy $S(\tilde{\rho}'/\rho)$ over the class of the DSTS's $\tilde{\rho}'$ with two fixed parameters,

$$\tilde{\alpha}' = \alpha, \quad \tilde{\varphi}' = \varphi, \tag{3.6}$$

and, in addition, with the squeeze factor at the nonclassicality threshold (2.22),

$$\tilde{r}' = \tilde{r}'_c. \tag{3.7}$$

Accordingly,

$$2\widetilde{\overline{n'}} + 1 = \exp(2\tilde{r}'), \tag{3.8}$$

so that we may choose $\tilde{r}'$ as the only independent variable. By insertion of Eqs. (3.4), (3.5), and (3.8) into Eq. (A14) written for $S(\rho'/\rho)$, we find the following function:

$$S(\tilde{\rho}'/\rho) = \tilde{r}' - r_c + \exp(r_c)\{\sinh(r_c)\ln[\sinh(r_c)] - \cosh(r_c)\ln[\cosh(r_c)]\} + \tfrac{1}{2}\ln[\tfrac{1}{2}\sinh(2\tilde{r}')] - \tfrac{1}{2}\exp(2r_c)\cosh[2(\tilde{r}' - r)]\ln[\tanh(\tilde{r}')]. \tag{3.9}$$

Its first- and second-order derivatives with respect to $\tilde{r}'$ are, respectively,

$$\frac{\partial}{\partial \tilde{r}'} S(\tilde{\rho}'/\rho) = \exp(2r_c)\left\{\frac{\exp[2(\tilde{r}' - r_c)]}{\sinh(2\tilde{r}')} - \frac{\cosh[2(\tilde{r}' - r)]}{\sinh(2\tilde{r}')} - \sinh[2(\tilde{r}' - r)]\ln[\tanh(\tilde{r}')]\right\}, \tag{3.10}$$

$$\frac{\partial^2}{\partial \tilde{r}'^2} S(\tilde{\rho}'/\rho) = \frac{2\exp(2r_c)}{[\sinh(2\tilde{r}')]^2}\{-\exp(-2r_c) + \cosh(2r) - \sinh(2\tilde{r}')\sinh[2(\tilde{r}' - r)] - \cosh[2(\tilde{r}' - r)][\sinh(2\tilde{r}')]^2\ln[\tanh(\tilde{r}')]\}. \tag{3.11}$$

Therefore, in order to find the absolute minimum of the relative entropy (3.9), one has to solve a transcendental equation, which does not have an exact analytic solution. However, the opposite signs of the limits of the first-order derivative (3.10) for the extreme values of the squeeze factor $\tilde{r}'$,

$$\lim_{\tilde{r}' \to 0} \frac{\partial}{\partial \tilde{r}'} S(\tilde{\rho}'/\rho) = -\infty, \quad \lim_{\tilde{r}' \to \infty} \frac{\partial}{\partial \tilde{r}'} S(\tilde{\rho}'/\rho) = 2, \tag{3.12}$$

prove the existence of at least one minimum of the relative entropy (3.9). Such a minimum could be found graphically, as in Fig. 1, where we plot the function $S(\tilde{\rho}'/\rho)$ versus the variable $\tilde{r}'$ for several DSTS's having the same nonclassicality threshold $r_c = 1$ and different squeeze factors. The graphs in Fig. 1 suggest that the value

$$\tilde{r}_S := \tfrac{1}{2}(r + r_c) \tag{3.13}$$

of the variable $\tilde{r}'$ may be an approximate analytic solution for the minimum conditions

$$\frac{\partial}{\partial \tilde{r}'} S(\tilde{\rho}'/\rho) = 0, \quad \frac{\partial^2}{\partial \tilde{r}'^2} S(\tilde{\rho}'/\rho) > 0. \tag{3.14}$$

We proved that this really happens by writing the simplest lower and upper bounds for the logarithm occurring in the first-order derivative (3.10) taken at the point $\tilde{r}' = \tilde{r}_S$, as well as an adequate lower bound for the logarithm term in the second-order derivative (3.11). The approximation consisting in the use of the value (3.13) as the absolute minimum point $\tilde{r}'_m$ of the relative entropy (3.9) becomes more and more

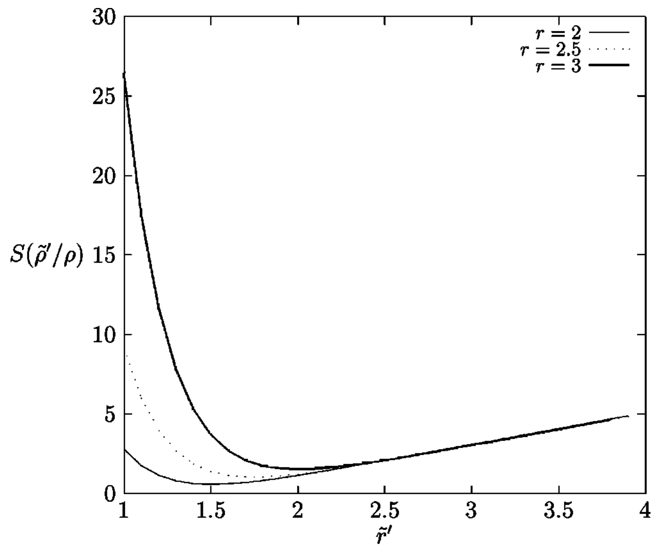

FIG. 1. Displaying the minimum of the relative entropy. The function $S(\widetilde{\rho}'/\rho)$, Eq. (3.9), is plotted vs the squeeze factor $\widetilde{r}'$ for several nonclassical Gaussian states having the nonclassicality threshold $r_c=1$. The approximation $\widetilde{r}'_m \approx \widetilde{r}_S := \frac{1}{2}(r+r_c)$ is seen to be reasonably good in all the shown cases.

accurate in so far as the squeezing of the given nonclassical state ρ is stronger, i.e., the inequality $\widetilde{r}'_m - r_c \gg 1$ is better and better fulfilled. In this case of strong nonclassicality, the DSTS $\widetilde{\rho}_S$ with the parameters

$$\widetilde{\bar{n}}_S = \tfrac{1}{2}[\exp(2\widetilde{r}_S)-1], \quad \widetilde{r}_S, \varphi, \alpha \tag{3.15}$$

can be considered the closest classical Gaussian state. Thus the entropic amount of nonclassicality, which for $r > r_c$ is only *approximated* as

$$Q_S(\rho) = S(\widetilde{\rho}_S/\rho) \quad (r > r_c), \tag{3.16}$$

has the explicit expression

$$Q_S(\rho) = 0 \quad (r \leqslant r_c)$$

$$\begin{aligned} Q_S(\rho) = {} & \tfrac{1}{2}(r-r_c) + \exp(r_c)\{\sinh(r_c)\ln[\sinh(r_c)] \\ & - \cosh(r_c)\ln[\cosh(r_c)]\} \\ & + \tfrac{1}{2}[\exp(2r_c)\cosh(r-r_c)+1]\ln\left[\cosh\left(\frac{r+r_c}{2}\right)\right] \\ & - \tfrac{1}{2}[\exp(2r_c)\cosh(r-r_c)-1]\ln\left[\sinh\left(\frac{r+r_c}{2}\right)\right] \\ & (r > r_c). \end{aligned} \tag{3.17}$$

IV. HILBERT-SCHMIDT DISTANCE AS A MEASURE OF NONCLASSICALITY

Any density operator σ is of trace class and *a fortiori* has a finite Hilbert-Schmidt norm,

$$\|\sigma\|_2 = \sqrt{\mathrm{Tr}(\sigma^2)}. \tag{4.1}$$

The Hilbert-Schmidt distance between two quantum states, σ' and σ'', is introduced as the usual norm metric,

$$d_{HS}(\sigma', \sigma'') = \|\sigma' - \sigma''\|_2 = \sqrt{\mathrm{Tr}[(\sigma' - \sigma'')^2]}. \tag{4.2}$$

In the case of the single-mode field states, making use of the Weyl expansion (2.3), one can express the Hilbert-Schmidt inner product $\mathrm{Tr}(\sigma'\sigma'')$ in terms of their CF's as

$$\mathrm{Tr}(\sigma'\sigma'') = \frac{1}{\pi}\int d^2\lambda\, \chi'^*(\lambda)\chi''(\lambda). \tag{4.3}$$

Specifically, for a pair of one-mode Gaussian states, ρ' and ρ'', we employ the Gaussian integral recollected in Appendix A of the first paper of Ref. [22] to get from Eq. (4.3) the explicit formula

$$\begin{aligned} \mathrm{Tr}(\rho'\rho'') = {} & \frac{1}{\sqrt{\Delta}}\exp\bigg(-\frac{1}{\Delta}\{(A'+A''+1)|C'-C''|^2 \\ & + \mathrm{Re}[(B'+B'')^*(C'-C'')^2]\}\bigg). \end{aligned} \tag{4.4}$$

In Eq. (4.4), we have used the coefficients (2.18) for both states and introduced the determinant of the sum of the corresponding covariance matrices, $\mathcal{V}'$ and $\mathcal{V}''$:

$$\begin{aligned} \Delta := {} & \det(\mathcal{V}'+\mathcal{V}'') = (A'+A''+1)^2 - |B'+B''|^2 \\ = {} & \left(\bar{n}'+\frac{1}{2}\right)^2 + \left(\bar{n}''+\frac{1}{2}\right)^2 + 2\left(\bar{n}'+\frac{1}{2}\right)\left(\bar{n}''+\frac{1}{2}\right)\cosh(2\check{r}), \end{aligned} \tag{4.5}$$

where $\cosh(2\check{r})$ is given by Eq. (A11). In particular, Eq. (4.4) yields the degree of purity (2.20). Consequently, the squared Hilbert-Schmidt distance between two one-mode Gaussian states is

$$\begin{aligned} d^2_{HS}(\rho,\rho') = {} & \frac{1}{2\bar{n}+1} + \frac{1}{2\bar{n}'+1} \\ & - \frac{2}{\sqrt{\Delta}}\exp\bigg(-\frac{1}{\Delta}\{(A'+A''+1)|C'-C''|^2 \\ & + \mathrm{Re}[(B'+B'')^*(C'-C'')^2]\}\bigg). \end{aligned} \tag{4.6}$$

We define the Hilbert-Schmidt-metric amount of Gaussian nonclassicality of a one-mode DSTS ρ as

$$Q_{HS}(\rho) := \min_{\rho' \in \mathcal{C}_0} d^2_{HS}(\rho,\rho'). \tag{4.7}$$

In the same way as in Sec. III, we denote by $\bar{n}, r, \varphi, \alpha$ the parameters of the state ρ and by $\bar{n}', r', \varphi', \alpha'$, those of the classical state ρ'. The minimum of the quantity (4.6) under the assumptions $r > r_c$ and $r' \leqslant r'_c$ is achieved by a classical Gaussian state belonging to the class of DSTS's $\widetilde{\rho}'$ with the parameters given by Eqs. (3.6) and (3.7), as in the case of the relative entropy. We insert therefore the values (3.6) and the threshold relation (3.8) into Eq. (4.6) to find the function $d^2_{HS}(\rho,\tilde{\rho}')$ of a single variable $\tilde{r}'$,

$$d^2_{HS}(\rho,\tilde{\rho}')=\exp(-2r_c)\Bigg[1-\exp[-(r-r_c)]\times\left(-x+4\{1-\exp[-2(r-r_c)]\}^{-1/2}\times\left(1+\frac{1}{x^2}\right)^{-1/2}\right)\Bigg] \tag{4.8}$$

with

$$x:=\exp(r+r_c-2\tilde{r}'). \tag{4.9}$$

The function (4.8) has a unique minimum at the exact value

$$\tilde{r}_{HS}:=\tfrac{1}{2}(r+r_c)-\tfrac{1}{4}\ln(2^{4/3}\{1+\exp[-2(r-r_c)]\}^{-1/3}-1) \tag{4.10}$$

of the squeeze factor $\tilde{r}'$. The closest classical state $\tilde{\rho}_{HS}$ has thus the parameters

$$\tilde{\bar{n}}_{HS}=\tfrac{1}{2}[\exp(2\tilde{r}_{HS})-1],\quad \tilde{r}_{HS},\,\varphi,\,\alpha. \tag{4.11}$$

Accordingly, the Hilbert-Schmidt-metric amount of nonclassicality,

$$Q_{HS}(\rho)=d^2_{HS}(\rho,\tilde{\rho}_{HS})\quad (r>r_c), \tag{4.12}$$

reads explicitly

$$Q_{HS}(\rho)=0\quad (r\leqslant r_c),$$

$$Q_{HS}(\rho)=\exp(-2r_c)\{1-\exp[-(r-r_c)]\times(2^{4/3}\{1+\exp[-2(r-r_c)]\}^{-1/3}-1)^{3/2}\}\quad (r>r_c). \tag{4.13}$$

V. DISCUSSION OF THE RESULTS

A. Nonclassicality measured by the Bures distance

We are now in a position to compare the distance-type measures of nonclassicality evaluated above to our previous result concerning the Bures-distance amount of nonclassicality [11]. Recall that the Bures distance between two mixed states described by the density operators σ' and σ'' on a Hilbert space $\mathcal{H}_A$ was originally introduced on mathematical grounds [12]. Its square is

$$d^2_B(\sigma',\sigma''):=\min\|\,|\Psi'\rangle-|\Psi''\rangle\|^2=2(1-\max|\langle\Psi'|\Psi''\rangle|), \tag{5.1}$$

where $|\Psi'\rangle$ and $|\Psi''\rangle$ are vectors in an enlarged Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$, describing pure states whose reductions over the auxiliary Hilbert space $\mathcal{H}_B$ are precisely the given mixed states: $\sigma'=\mathrm{Tr}_B(|\Psi'\rangle\langle\Psi'|)$ and $\sigma''=\mathrm{Tr}_B(|\Psi''\rangle\langle\Psi''|)$. Since this definition can obviously be extended to pure states, the set of all quantum states (pure and mixed) may be equipped with the Bures distance to become a metric space. Note that the "transition probability" between the mixed states σ' and σ'', defined later by Uhlmann [23],

$$\mathcal{F}(\sigma',\sigma''):=\max|\langle\Psi'|\Psi''\rangle|^2, \tag{5.2}$$

is closely related to the Bures metric:

$$d_B(\sigma',\sigma'')=\sqrt{2-2\sqrt{\mathcal{F}(\sigma',\sigma'')}}. \tag{5.3}$$

Uhlmann [23] succeeded in deriving an explicit expression of the quantity (5.2), now called *fidelity* [24]:

$$\mathcal{F}(\sigma',\sigma'')=\{Tr[(\sqrt{\sigma'}\sigma''\sqrt{\sigma'})^{1/2}]\}^2. \tag{5.4}$$

It was proved [25] that $\sqrt{\mathcal{F}(\sigma',\sigma'')}$ equals the minimal overlap of the probability distributions for the outcomes of any POVM. The Bures metric exhibits therefore the best probabilistic distinguishability between quantum states. As a consequence, it cannot increase under *any* POVM. We also mention that a metric on the set of density operators revealing quantum distinguishability has been defined by Braunstein and Caves [26]. For neighboring quantum states, this *statistical distance* coincides up to a factor of 2 with the Bures distance. There is therefore a strong distinguishability reason to use the Bures distance as a measure of nonclassicality. In Ref. [11] we defined an amount of nonclassicality of any one-mode Gaussian state ρ in terms of its Bures distance to the set $\mathcal{C}_0$ of all classical one-mode Gaussian states:

$$Q_B(\rho):=\frac{1}{2}\min_{\rho'\in\mathcal{C}_0} d^2_B(\rho,\rho'). \tag{5.5}$$

By employing in Ref. [11], for the Bures metric, the method applied here as well, in Secs. III and IV, but making use of the relative entropy and the Hilbert-Schmidt distance, respectively, we found the closest classical Gaussian state $\tilde{\rho}_B$ to a nonclassical DSTS ρ whose parameters are denoted again by $\bar{n},r,\varphi,\alpha$. The DSTS $\tilde{\rho}_B$ is specified by its parameters,

$$\tilde{\bar{n}}_B=\tfrac{1}{2}[\exp(2\tilde{r}_B)-1],$$

$$\tilde{r}_B=\tfrac{1}{4}\ln[1+2\sinh(2r_c)\exp(2r)],\quad \varphi,\,\alpha. \tag{5.6}$$

The Bures-metric degree of nonclassicality,

$$Q_B(\rho)=\frac{1}{2}d^2_B(\rho,\tilde{\rho}_B)\quad (r>r_c), \tag{5.7}$$

has the simple expression

$$Q_B(\rho)=0\quad (r\leqslant r_c),$$

$$Q_B(\rho)=1-[\mathrm{sech}(r-r_c)]^{1/2}\quad (r>r_c). \tag{5.8}$$

TABLE I. Purity $\mathrm{Tr}(\tilde{\rho}^2)$ of the closest classical DSTS to a pure Gaussian state having the squeeze factor r.

Metric	Amount of nonclassicality $Q(\rho)$	$\mathrm{Tr}(\tilde{\rho}^2)$
Bures[a]	$1-[\mathrm{sech}(r)]^{1/2}$	1
Relative-entropy[b]	$\frac{1}{2}\left\{r+\left[\cosh\left(\frac{r}{2}\right)\right]^2\ln\left(\left[\cosh\left(\frac{r}{2}\right)\right]^2\right)-\left[\sinh\left(\frac{r}{2}\right)\right]^2\ln\left(\left[\sinh\left(\frac{r}{2}\right)\right]^2\right)\right\}$	e^{-r}
Hilbert-Schmidt[c]	$1-e^{-r}[2^{4/3}(1+e^{-2r})^{-1/3}-1]^{3/2}$	$e^{-r}[2^{4/3}(1+e^{-2r})^{-1/3}-1]^{1/2}$

[a]Equations (5.8) and (5.6).
[b]Equations (3.17), (3.13), and (3.15), obtained as an approximate solution in the strong-squeezing case.
[c]Equations (4.13), (4.10), and (4.11).

B. The closest classical Gaussian state

It is instructive to compare the features of the classical Gaussian state closest to a given nonclassical one for the three "distances" studied in Ref. [11] and in the present paper.

Shared features. (a) The closest classical DSTS has the same displacement parameter and rotation angle as the given nonclassical one. It is easy to explain this property. The translations and rotations in phase space generate the only classical one-mode unitary transformations, $D(\alpha)$, Eq. (2.2), and $R(\theta)$, Eq. (A9), respectively. In addition, they leave invariant the class of the Gaussian states. As all three distances employed above are invariant under any unitary transformation, the corresponding measures of nonclassicality are preserved by translations and rotations. Therefore, they meet the demand (Q2) of being invariant under all classical one-mode unitary transformations. (b) The closest classical Gaussian state is on the boundary of the reference set $\mathcal{C}_0$ given by the condition (3.7).

Different features. A striking difference is displayed by the degree of purity (2.20) of the closest classical DSTS in the three cases. According to Eqs. (3.13), (3.15), (4.10), (4.11), and (5.6), the following inequalities hold:

$$\mathrm{Tr}(\tilde{\rho}_S^2)<\mathrm{Tr}(\tilde{\rho}_{HS}^2)<\mathrm{Tr}(\rho^2),\quad \mathrm{Tr}(\tilde{\rho}_S^2)<\mathrm{Tr}(\tilde{\rho}_B^2)\leqslant\mathrm{Tr}(\rho^2). \tag{5.9}$$

Note that the last inequality in Eq. (5.9) turns into an equality if and only if the given nonclassical state is pure. This case ($r_c=0$) is studied in detail in Table I, where the three purities are written down explicitly. We see that the closest classical Gaussian state is a pure one, namely a coherent state, when using the Bures distance to quantify nonclassicality, and a mixed one for the other two measures of nonclassicality.

C. Thermal noise mapping

The above-mentioned property of the Bures distance of not increasing under any POVM mapping [18] results in a not increasing of the Bures-metric degree of nonclassicality $Q_B(\rho)$, Eq. (5.5), under any POVM mapping that transforms the DSTS's into DSTS's. As a consequence of a similar property of the relative entropy [15,16], the entropic amount of nonclassicality $Q_S(\rho)$, Eq. (3.2), does not increase under POVM mappings preserving the Gaussian form of the states. Both measures of the nonclassicality of a Gaussian state satisfy the condition (Q3).

A significant example of a continuous-variable POVM mapping is the thermal noise mapping. For one-mode field states, this is defined [20,21] as follows:

$$\Gamma_{\bar{m}}(\rho):=\frac{1}{\pi\bar{m}}\int d^2\beta\,\exp\left(-\frac{|\beta|^2}{\bar{m}}\right)D(\beta)\rho D^\dagger(\beta)\quad(\bar{m}\geqslant 0), \tag{5.10}$$

where $\bar{m}$ is the thermal mean number of added photons. The Gaussian mapping (5.10) transforms DSTS's into DSTS's with the only modification of the CF Eq. (2.1) consisting in the addition of $\bar{m}$ to $A:A\rightarrow A':=A+\bar{m}$. Correspondingly, thermalization raises the nonclassicality threshold Eq. (2.22) of a Gaussian state ρ,

$$r_c(\bar{m})=\frac{1}{2}\ln\left(\frac{2\bar{n}+1}{1-2\bar{m}}\right)\quad\left(\bar{m}<\frac{1}{2}\right). \tag{5.11}$$

In Ref. [11] we pointed out that the Bures-metric degree of nonclassicality $Q_B[\Gamma_{\bar{m}}(\rho)]$ of the thermalized state (5.10) *decreases* with the thermal noise $\bar{m}$, as required by the demand (Q3). This statement is equally true for the Hilbert-Schmidt measure of nonclassicality $Q_{HS}[\Gamma_{\bar{m}}(\rho)]$, Eq. (4.13). Since it does not increase under any thermal noise mapping (5.10), the function $Q_{HS}(\rho)$, Eq. (4.13), is acceptable as a measure of nonclassicality, even if there is no evidence for the fulfillment of condition (Q3) for all POVM mappings that leave invariant the class of the Gaussian states. In order to illustrate this important point, we plot in Fig. 2 the amounts of Gaussian nonclassicality $Q_B[\Gamma_{\bar{m}}(\rho)]$, Eq. (5.8), $Q_S[\Gamma_{\bar{m}}(\rho)]$, Eq. (3.17), and $Q_{HS}[\Gamma_{\bar{m}}(\rho)]$, Eq. (4.13), versus the thermal noise $\bar{m}$, where ρ is a nonclassical DSTS with the parameters $r=3$, $r_c=1$. All three nonclassical distances decrease with the Gaussian noise $\bar{m}$, in accordance with the requirement (Q3).

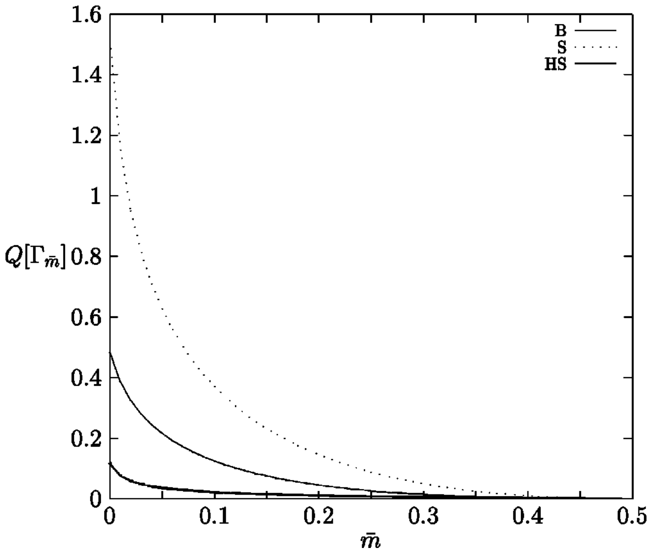

FIG. 2. Nonclassicality vs thermal noise $\overline{m}$ for a DSTS with the squeeze factor $r=3$ and the nonclassicality threshold $r_c=1$. We employ the Bures-metric [curve B, Eq. (5.8)], the entropic [curve S, Eq. (3.17)], and the Hilbert-Schmidt-metric [curve HS, Eq. (4.13)] amounts of nonclassicality.

VI. CONCLUSIONS

The discussion of the above-presented results favors the Bures distance in comparison with the relative entropy and the Hilbert-Schmidt distance in finding an adequate measure of nonclassicality for one-mode Gaussian states.

First, the degree of Gaussian nonclassicality $Q_B(\rho)$, built with the Bures distance, has an *exact* analytic expression, Eq. (5.8), which is simple and insightful. This is not the case when employing the *approximate* entropic amount of nonclassicality $Q_S(\rho)$, Eq. (3.17).

Second, the distance $Q_B(\rho)$ fulfills the demand (Q3) for *any* POVM preserving the Gaussian character of the state. The situation is different with the Hilbert-Schmidt measure of nonclassicality $Q_{HS}(\rho)$, Eq. (4.13), which, however, does not increase under any thermal noise mapping (5.10).

Third, we stress that, besides satisfying the conditions (Q1)–(Q3), the Bures-metric degree of nonclassicality (5.8) is a bijective function of the Lee nonclassical depth τ_m, Eq. (2.24): $Q_B(\rho)$ increases from 0 to 1 when τ_m increases from 0 to 1/2. Explicitly,

$$Q_B(\rho)=1-\left[1-\left(\frac{\tau_m}{1-\tau_m}\right)^2\right]^{1/4} \quad (r>r_c). \tag{6.1}$$

Equation (6.1) therefore proves that the amount of Gaussian nonclassicality $Q_B(\rho)$ and the nonclassical depth τ_m are *equivalent* measures of the nonclassicality of a single-mode Gaussian state. This happens neither for the relative entropy nor for the Hilbert-Schmidt metric. In the mixed-state case, the quantities $Q_S(\rho)$ and $Q_{HS}(\rho)$ are not equivalent to the nonclassical depth, which is a genuine measure of nonclassicality. Hence we draw the conclusion that making use of the reference set $\mathcal{C}_0$ of all classical DSTS's in order to build a distance-type measure of nonclassicality is suitable only in the case of the Bures distance.

Furthermore, the Bures metric could analogously be employed to calculate a Gaussian amount of entanglement of two-mode Gaussian states. Specifically, we recently succeeded to carry out an approximate analytic evaluation of the Bures-metric degree of entanglement for two-mode squeezed thermal states [27].

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APPENDIX: RELATIVE ENTROPY OF AN ONE-MODE GAUSSIAN STATE WITH RESPECT TO ANOTHER ONE

To evaluate the relative entropy of an one-mode Gaussian field state with respect to a similar state, one could either take advantage of the exponential form of a Gaussian density operator or employ the eigenvalue problem for the Gaussian density operator in the DSTS parametrization.

Here we present the second derivation pointing out its simplicity. Consider a pair of one-mode DSTS's, ρ' and ρ'', having the DSTS-parameters $\overline{n'},r',\varphi',\alpha'$, and $\overline{n''},r'',\varphi'',\alpha''$, respectively. With Eqs. (2.13) and (2.15) we get the spectral decompositions of their density operators:

$$\rho'=\sum_{n=0}^{\infty}\lambda'_n D(\alpha')\mathcal{S}(r',\varphi')|n\rangle\langle n|\mathcal{S}^{\dagger}(r',\varphi')D^{\dagger}(\alpha') \tag{A1a}$$

and

$$\rho''=\sum_{n=0}^{\infty}\lambda''_n D(\alpha'')\mathcal{S}(r'',\varphi'')|n\rangle\langle n|\mathcal{S}^{\dagger}(r'',\varphi'')D^{\dagger}(\alpha''). \tag{A1b}$$

The corresponding eigenvalues, λ'_n and λ''_n, are of the form (2.16). The operators $\ln(\rho')$ and $\ln(\rho'')$ have similar spectral expansions, with the only difference that the eigenvalues λ'_n and λ''_n are replaced by their logarithms, $\ln(\lambda'_n)$ and $\ln(\lambda''_n)$, respectively. We evaluate the relative entropy of the state ρ' with respect to the state ρ'':

$$S(\rho'/\rho'')=\mathrm{Tr}\{\rho''[\ln(\rho'')-\ln(\rho')]\}. \tag{A2}$$

The first term in Eq. (A2) is the negative of the familiar von Neumann entropy

$$S(\rho''):=-\mathrm{Tr}[\rho''\ln(\rho'')]=-\sum_{n=0}^{\infty}\lambda''_n\ln(\lambda''_n) \tag{A3}$$

of the DSTS ρ'',

$$S(\rho'') = (\overline{n''}+1)\ln(\overline{n''}+1) - \overline{n''}\ln(\overline{n''}). \tag{A4}$$

We evaluate the second term in Eq. (A2) in the eigenvector basis $\{D(\alpha')\mathcal{S}(r',\varphi')|n\rangle\}$ of the density operator ρ',

$$-\mathrm{Tr}[\rho''\ln(\rho')] = -\sum_{n=0}^{\infty} \ln(\lambda'_n)\langle n|\check{\rho}|n\rangle. \tag{A5}$$

In Eq. (A5), $\check{\rho}$ is the Gaussian state

$$\check{\rho} := \mathcal{S}^{\dagger}(r',\varphi')D(\alpha''-\alpha')\mathcal{S}(r'',\varphi'')\rho''_T\mathcal{S}^{\dagger}(r'',\varphi'') \times D^{\dagger}(\alpha''-\alpha')\mathcal{S}(r',\varphi'). \tag{A6}$$

Its DSTS parameters are specified by the multiplication rules

$$\mathcal{S}^{\dagger}(r',\varphi')D(\alpha''-\alpha') = D(\beta)\mathcal{S}^{\dagger}(r',\varphi'),$$

$$\beta := (\alpha''-\alpha')\cosh(r') - (\alpha''-\alpha')^* e^{i\varphi'}\sinh(r'), \tag{A7}$$

and [28]

$$S^{\dagger}(r',\varphi')S(r'',\varphi'') = e^{-i\check{\theta}/2}S(\check{r},\check{\varphi})R(\check{\theta}). \tag{A8}$$

In Eq. (A8), $R(\check{\theta})$ is a rotation operator with the angle $\check{\theta}$,

$$R(\theta) := \exp(-i\theta a^{\dagger}a) \quad (\theta \in (-\pi,\pi]). \tag{A9}$$

The parameters $\check{r},\check{\varphi}$, and $\check{\theta}$ are given by the following equations:

$$e^{i\check{\theta}}\cosh(\check{r}) = \cosh(r')\cosh(r'') - e^{-i(\varphi'-\varphi'')}\sinh(r')\sinh(r''),$$
$$e^{i(\check{\varphi}+\check{\theta})}\sinh(\check{r}) = e^{i\varphi''}\cosh(r')\sinh(r'') - e^{i\varphi'}\sinh(r')\cosh(r''), \tag{A10}$$

which yield the composition formula

$$\cosh(2\check{r}) = \cosh(2r')\cosh(2r'') - \sinh(2r')\sinh(2r'')\cos(\varphi'-\varphi''). \tag{A11}$$

Accordingly, the one-mode Gaussian state $\check{\rho}$, defined by Eq. (A6), has the explicit DSTS form

$$\check{\rho} = D(\beta)\mathcal{S}(\check{r},\check{\varphi})\rho''_T\mathcal{S}^{\dagger}(\check{r},\check{\varphi})D^{\dagger}(\beta). \tag{A12}$$

Owing to the structure (2.16) of the eigenvalues λ'_n, the summation in Eq. (A5) is readily performed to obtain

$$-\mathrm{Tr}[\rho''\ln(\rho')] = [1+\mathrm{Tr}(\check{\rho}a^{\dagger}a)]\ln(\overline{n'}+1) - \mathrm{Tr}(\check{\rho}a^{\dagger}a)\ln(\overline{n'}). \tag{A13}$$

A straightforward calculation based on the expression of the mean photon number in a DSTS, Eq. (2.8), gives the final expression of the relative entropy (A2):

$$S(\rho'/\rho'') = -[(\overline{n''}+1)\ln(\overline{n''}+1) - \overline{n''}\ln(\overline{n''})] + \frac{1}{2}\ln[\overline{n'}(\overline{n'}+1)] + \frac{2}{2\overline{n'}+1}\Bigg\{\left(A'+\frac{1}{2}\right)\left(A''+\frac{1}{2}\right) - \mathrm{Re}[B'(B'')^*] + \left(A'+\frac{1}{2}\right)|C'-C''|^2 + \mathrm{Re}[(B')^*(C'-C'')^2]\Bigg\}\ln\left(\frac{\overline{n'}+1}{\overline{n'}}\right). \tag{A14}$$

In Eq. (A14), we have employed the coefficients (2.18) of the corresponding CF's: A',B',C' for the state ρ' and A'',B'',C'' for the state ρ''. It appears that the expression (A14) of the relative entropy of a Gaussian state with respect to another one is a new result in the extensively studied area of single-mode field states.

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$$a_s' := W a_s W^{\dagger} = \sum_{r=1}^{n} U_{rs} a_r + \beta_s I \quad (s=1, \ldots, n),$$

where $U \in U(n)$ and $\beta_s \in \mathbb{C}$, then W transforms any classical n-mode state into a similar state. Otherwise said, the unitary transformation $\rho \to \rho' := W\rho W^{\dagger}$ is a bijective mapping on the convex set of the classical n-mode states. For a proof of this theorem in an illustrative particular case, namely the mixing of two field modes in a lossless beam splitter, see Wang Xiang-bin, Phys. Rev. A **66**, 024303 (2002).

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