Scheme for direct measurement of the Wigner characteristic function in cavity QED

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We propose a simple scheme for the reconstruction of the single-mode cavity field by considering the resonant atom-cavity interaction in the presence of a strong classical field. With the aid of the strong classical field, it is easy to realize the displacement operator for the cavity field correlated to the internal state of the atom. It is shown that the measurement of the population of the lower internal state directly yields the Wigner characteristic function of the cavity field.

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In recent years there has been great interest in the preparation and measurement of quantum states [1]. Cavity QED, with Rydberg atoms crossing superconducting cavities, offer an almost ideal system for the generation and measurement of nonclassical states and implementation of small scale quantum information processing [2]. In the context of cavity QED, numerous theoretical schemes [3] for generating various nonclassical states were proposed, which led to experimental realization of Schrödinger cat state [4] and Fock state [5] in a cavity mode. Thus, it is desirable to have a powerful tool to prove that the cavity field has indeed been prepared in the desired state. Several measurement schemes for the cavity fields have been proposed by probing quantum states with two-level atoms and subsequently measuring the atomic state [6]. But only a few of the proposals have a strikingly simple data analysis. Wilkens and Meystre proposed a scheme for directly measuring the Wigner characteristic function of a cavity field via the nonlinear atomic honodyne detection [7]. In Ref. [8], Kim *et al.* made a similar proposal based on current experimental conditions. In Ref. [9], Lutterbach and Davidivich presented a scheme for direct measurement of Wigner function of cavity field, which has been experimentally demonstrated in a cavity [10]. This scheme is based on the dispersive interaction of a single atom with the cavity field. However, dispersive interaction requires that the detuning between the atoms and the cavity is much bigger than the atom-cavity coupling strength. Thus, the quantum dynamics operates at a low speed. In Ref. [11], Bardroff *et al.* proposed a simple and fast scheme for direct measurement of the Wigner characteristic function of the motion state of a trapped ion. This scheme can be applied to measure the quantum state of a cavity field via realizing a displacement operator for the cavity field correlated to the internal state of the atom. Such displacement operation has been suggested by Davidovich *et al.* in the context of quantum switches using a dispersive atom-cavity interaction, but the experimental realization of the scheme is difficult.

In this paper, we propose an alternative scheme to directly measure the characteristic function of the Wigner function of single-mode cavity field. The physical system is a two-level atom interacting with a single-mode cavity field in the presence of a strong classical field. Recently, Solano *et al.* studied such physical model for generating atom-field entangled states and field superposition states [12]. These authors showed that, with the aid of the strong classical field, it is easy to realize the displacement operator for the cavity field correlated to the internal state of the atom. In this paper, we show that such physical system can be used to measure the Wigner characteristic function of single-mode cavity field, and the phase of classical field acting as a tunable parameter is important for the measurement of cavity field.

We consider a two-level atom interacting with a single-mode cavity field and driven additionally by an external classical field. In the rotating-wave approximation, the Hamiltonian is (assuming $\hbar=1$) [12]

$$H=\frac{\omega_{at}}{2}\sigma_z+\omega_{cav}a^\dagger a+g(\sigma_- e^{i\varphi+i\omega_L t}+\sigma_+ e^{-i\varphi-i\omega_L t}) + \Omega(a^\dagger\sigma_- + a\sigma_+), \tag{1}$$

where $\sigma_z=|e\rangle\langle e|-|g\rangle\langle g|, \sigma_+=|e\rangle\langle g|$, and $\sigma_-=|g\rangle\langle e|$, with $|g\rangle$ and $|e\rangle$ being the ground and excited states of the two-level atom. ω_{at} is the atomic transition frequency. a and $a^\dagger$ are the annihilation and creation operator of the single-mode cavity field of frequency ω_{cav}. Ω is the atom-cavity interaction strength. g and φ are the amplitude and phase of the classical field. ω_L is the frequency of the classical field. Here we should mention the physical system proposed by Alsing and Carmichael [13] on the single atom cavity QED system with a strongly driven cavity field, which was later studied by Mabuchi and Wiseman [14]. Although these authors consider a strongly driven cavity rather than a strongly driven atom, there is a canonical mapping between the two cases, so both systems are in fact essentially identical.

In a frame, which rotates with the classical wave frequency ω_L, the associated Hamiltonian of the system becomes

$$H=\frac{\Delta}{2}\sigma_z+\delta a^\dagger a+g(\sigma_- e^{i\varphi}+\sigma_+ e^{-i\varphi})+\Omega(a^\dagger\sigma_- + a\sigma_+), \tag{2}$$

where $\Delta=\omega_0-\omega_L$ and $\delta=\omega-\omega_L$.

In the following we assume that the atom, the cavity, and the driving classical field are all resonant: $\Delta=\delta=0$. In this case, the Hamiltonian (2) can be written as

$$H=g(\sigma_- e^{i\varphi}+\sigma_+ e^{-i\varphi})+\Omega(a^\dagger\sigma_- + a\sigma_+). \tag{3}$$

To see how quantum dynamics is modified by a strong classical field, we first move into the dressed state picture obtained by rotating atomic states with transformation

$$R=\exp\left[\frac{\pi}{4}(\sigma_+-\sigma_-)\right]\exp\left(\frac{i\varphi}{2}\sigma_z\right). \tag{4}$$

Using the relation

$$R\sigma_\pm R^\dagger=\tfrac{1}{2}e^{\pm i\varphi}[\sigma_z\pm(\sigma_+-\sigma_-)], \tag{5}$$

we can see that, in this picture, the Hamiltonian (3) becomes

$$H'=RHR^\dagger=g\sigma_z+\frac{\Omega}{2}\{a^\dagger e^{-i\varphi}[\sigma_z-(\sigma_+-\sigma_-)] + ae^{i\varphi}[\sigma_z+(\sigma_+-\sigma_-)]\}. \tag{6}$$

Making a further interaction picture transformation of the Hamiltonian by the unitary operator $T(t)=\exp(ig\sigma_z t)$, we have

$$H''=\frac{\Omega}{2}\{a^\dagger e^{-i\varphi}[\sigma_z-(\sigma_+e^{2igt}-\sigma_-e^{-2igt})] + ae^{i\varphi}[\sigma_z+(\sigma_+e^{2igt}-\sigma_-e^{-2igt})]\}. \tag{7}$$

Assuming $g\gg\Omega$, one can eliminate the term that oscillates with high frequencies in Hamiltonian (7) and obtain the effective interaction

$$H_{eff}=\frac{\Omega}{2}(a^\dagger e^{-i\varphi}+ae^{i\varphi})\sigma_z, \tag{8}$$

which is precisely as in Ref. [12] except that it retains the phase φ of the classical field. In Ref. [14], Mabuchi *et al.* have also obtained the same effective Hamiltonian by using same procedure as here.

The time evolution operator of Hamiltonian (8) is

$$U_{eff}(t)=\exp\left[-i\frac{\Omega t}{2}(a^\dagger e^{-i\varphi}+ae^{i\varphi})\sigma_z\right]. \tag{9}$$

In the standard picture [i.e., the one corresponding to the Hamiltonian (2)], the time evolution operator is

$$\begin{aligned}U(t)&=R^\dagger T^\dagger(t)U_{eff}(t)T^\dagger(0)R\\&=\exp[-igt(\sigma_-e^{i\varphi}+\sigma_+e^{-i\varphi})]\exp\Big[-i\frac{\Omega t}{2}(a^\dagger e^{-i\varphi}\\&\quad+ae^{i\varphi})(\sigma_-e^{i\varphi}+\sigma_+e^{-i\varphi})\Big].\end{aligned} \tag{10}$$

One possible application of the scheme is to reconstruct the unknown quantum state ρ_c of single-mode cavity field. We assume that the atomic state $|\Psi_a\rangle$ is initially prepared in a superposition of the ground and excited states

$$|\Psi_a\rangle=A|g\rangle+Be^{i\theta}|e\rangle, \tag{11}$$

where A, B, and θ are real numbers, and $A^2+B^2=1$. Thus the state of the whole system (atom+cavity) is

$$\rho(0)=|\Psi_a\rangle\langle\Psi_a|\rho_c. \tag{12}$$

After an interaction time t, the density operator for the whole system is

$$\begin{aligned}\rho(t)=\tfrac{1}{2}[&(Ae^{-i\varphi/2}+Be^{i\varphi/2+i\theta})\exp(-igt-ikX_\varphi/2)|+\rangle\\&+(Ae^{-i\varphi/2}-Be^{i\varphi/2+i\theta})\exp(igt+ikX_\varphi/2)|-\rangle]\rho_c\\&\times[(Ae^{i\varphi/2}+Be^{-i\varphi/2-i\theta})\exp(igt+ikX_\varphi/2)\langle+|\\&+(Ae^{i\varphi/2}-Be^{-i\varphi/2-i\theta})\exp(-igt-ikX_\varphi/2)\langle-|],\end{aligned} \tag{13}$$

where

$$|\pm\rangle=\frac{1}{\sqrt{2}}(e^{i\varphi/2}|g\rangle\pm e^{-i\varphi/2}|e\rangle), \tag{14}$$

$$X_\varphi=a^\dagger e^{-i\varphi}+ae^{i\varphi}, \tag{15}$$

and $k=\Omega t$. In Eq. (13), the cavity field is entangled with the atomic state. We can remove this entanglement by detecting the atomic state. Measuring the population of the lower internal state $|g\rangle$, we find the probability

$$P_g(\theta)=\frac{1}{2}+\frac{A^2-B^2}{2}\mathrm{Tr}[\cos(gt+kX_\varphi)\rho_c]+AB\sin(\theta+\varphi)\mathrm{Tr}[\sin(gt+kX_\varphi)\rho_c]. \tag{16}$$

We see that the probability $P_g(\theta)$ is directly related to the Wigner characteristic function of cavity field [15]

$$\chi(k,\varphi)=\mathrm{Tr}[\exp(ikX_\varphi)\rho_c] \tag{17}$$

through

$$\begin{aligned}P_g(\theta)=&\frac{1}{2}+\frac{A^2-B^2}{2}\cos(kg/\Omega)+ABAB\sin(\theta\\&+\varphi)\sin(kg/\Omega)\mathrm{Re}[\chi(k,\varphi)]+\Big[AB\sin(\theta\\&+\varphi)\sin(kg/\Omega)-\frac{A^2-B^2}{2}\cos(kg/\Omega)\Big]\mathrm{Im}[\chi(k,\varphi)].\end{aligned} \tag{18}$$

Therefore, we obtain the Wigner characteristic function [16]

$$\chi(k,\varphi)=e^{-igk/\Omega}\left\{\frac{P_g(-\varphi-\pi/2)+P_g(-\varphi+\pi/2)-1}{A^2-B^2}+i\frac{P_g(-\varphi+\pi/2)-P_g(-\varphi+\pi/2)}{2AB}\right\}. \tag{19}$$

Thus a measurement of $P_g(\theta)$ for two phases $-\varphi\pm\pi/2$ directly yields the Wigner characteristic function of the initial state at the point (k,φ). With regard to the argument k, it can be varied by changing the evolution time. The parameter φ is phase of classical field, which is a tunable parameter.

In summary, we proposed a scheme to directly measure the Wigner characteristic function of the single-mode cavity field. The scheme is based on the displacement operation for the cavity field correlated to the internal state of the atom. With the aid of strong classical field, it is easy to realize such a displacement operation for the cavity field. It is shown that the phase φ of classical field acting as a tunable parameter is important for the measurement of cavity field. The scheme only requires the resonant atom-cavity interaction so that the quantum dynamics operates at a high speed, which is important in view of decoherence.

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