Nonorthogonal projective positive-operator-value measurement of photon polarization states with unit probability of success

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In this paper we describe a scheme for performing a nonorthogonal projective positive-operator-value measurement of any arbitrary single-photon polarization input state with unit probability of success. While this probability is reached in the limit of infinite cycles of states through the apparatus, only one actual physical setup is required for a feasible implementation. Specifically, our setup implements a set of three nonorthogonal measurement operators at angles of 120° to each other.

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I. INTRODUCTION

The rapidly increasing interest in quantum-information theory and its applications (see [1] for a comprehensive overview) has resulted in a renewed interest in the theory of quantum measurement. In particular, the theory and possible implementations of generalized measurements in the form of positive-operator-value measures (POVMs) have increasingly attracted attention over the past decade, as POVMs are an essential tool in quantum-information processing and especially in quantum cryptography [2–5]. A wide variety of quantum mechanical phenomena such as teleportation [6], interaction-free measurement [7], and nonlocality [8–10] have been demonstrated using single photons [11–14]. Recently, it was shown that the operations necessary for quantum computation can be implemented using linear optics [15,16]. To date, however, proposed optical implementations of nonorthogonal POVM operators [5,17–19] have concentrated on nonprojective sets of nonorthogonal operators, mostly for the purpose of distinguishing nonorthogonal states. In such nonprojective measurements only the probability of detection is of interest, and not the collapse of the state ρ to $M_i \rho M_i$ for a set of measurement operators $\{M_i\}$, which in fact does not occur in these setups.

In this paper we describe a single-photon implementation of a *projective* nonorthogonal positive-operator-value measurement with three measurement operators which measure the polarization of the photon along three axes in a plane separated by angles of 120°. This corresponds to the toy problem proposed by Preskill [20]. Our measurement has unit probability of success for any arbitrary input state and can be generalized to more than three operators.

II. THE SETUP

Our setup is comprised of three modules, each of which implements a measurement operator of the POVM. These modules are in series, meaning that the state exiting one module is fed into the entrance of the next. The basic module, shown in Fig. 1, consists of two polarizing beam splitters and two conventional beam splitters (partially silvered mirrors). The basis of the polarizing beam splitters is in a plane rotated by an angle θ_j relative to an arbitrary fixed vector in the chosen plane. The conventional beam splitters have equal amplitude transmission coefficients α_j.

The first polarizing beam splitter splits the incoming photon into its polarization components. Each of these components is in turn divided by the conventional beam splitters, after which part of the polarization component amplitudes are reunited at the second polarizing beam splitter.

Following the conventional notation used in a quantum-information context for single photons in an interferometric

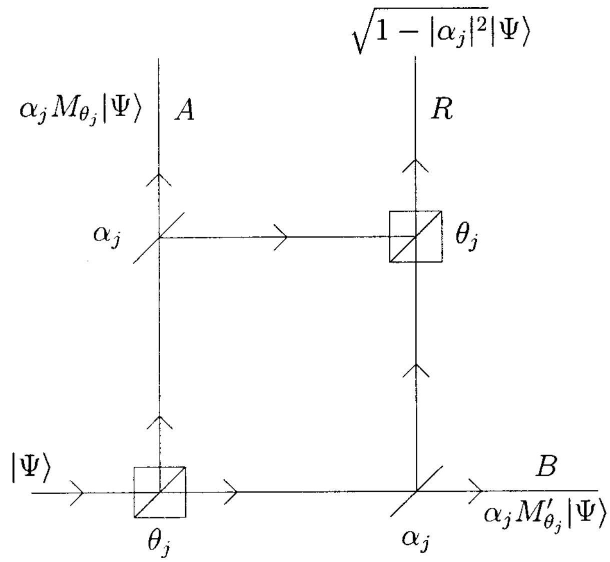

FIG. 1. The basic module which implements a single measurement operator. The input state $|\Psi\rangle$ first passes through a polarizing beam splitter with the polarization basis at angle θ_j with respect to the fixed vector in the chosen plane. The resulting beams are then divided again by conventional beam splitters of equal amplitude transmission coefficients α_j. The reflected components are then reunited, so that the input state $|\Psi\rangle$ reemerges at *R*, diminished in amplitude by a factor of $\sqrt{1-|\alpha_j|^2}$. At *A* and *B*, $\alpha_j M_{\theta_j}|\Psi\rangle$ and $\alpha_j M'_{\theta_j}|\Psi\rangle$ emerge, respectively.

setup [5,10,12,14], the total evolution of a pure photon state $|\Psi\rangle = a|H\rangle + b|V\rangle$ through one module is given by

$$\begin{aligned}|\Psi\rangle &= a|H\rangle + b|V\rangle,\\ &\rightarrow \alpha_j(a\cos\theta_j + b\sin\theta_j)(\cos\theta_j|H\rangle + \sin\theta_j|V\rangle)|s_A\rangle\\ &\quad + \alpha_j(a\sin\theta_j - b\cos\theta_j)(\sin\theta_j|H\rangle - \cos\theta_j|V\rangle)|s_B\rangle\\ &\quad + \sqrt{1-|\alpha_j^2|}(a|H\rangle + b|V\rangle)|s_R\rangle, \end{aligned} \tag{1}$$

where $|H\rangle$ and $|V\rangle$ are states of horizontal and vertical polarization. This can be much more simply written as

$$\begin{aligned}|\Psi\rangle \rightarrow{}& \alpha_j|\psi_{\theta_j}\rangle\langle\psi_{\theta_j}|\Psi\rangle|s_A\rangle + \alpha_j|\psi'_{\theta_j}\rangle\langle\psi'_{\theta_j}|\Psi\rangle|s_B\rangle\\ &+ \sqrt{1-|\alpha_j^2|}|\Psi\rangle|s_R\rangle, \end{aligned} \tag{2}$$

where $|\psi_{\theta_j}\rangle = \cos\theta_j|H\rangle + \sin\theta_j|V\rangle$ and $|\psi'_{\theta_j}\rangle = \sin\theta_j|H\rangle - \cos\theta_j|V\rangle$.

Introducing the projection operators $M_{\theta_j} = |\psi_{\theta_j}\rangle\langle\psi_{\theta_j}| = M_{\theta_j}^2$ and $M'_{\theta_j} = |\psi'_{\theta_j}\rangle\langle\psi'_{\theta_j}| = M'^2_{\theta_j}$ Eq. (2) can be written as

$$|\Psi\rangle \rightarrow \alpha_j M_{\theta_j}|\Psi\rangle|s_A\rangle + \alpha_j M'_{\theta_j}|\Psi\rangle|s_B\rangle + \sqrt{1-|\alpha_j|^2}|\Psi\rangle|s_R\rangle \tag{3}$$

To perform the chosen POVM three such modules are placed in series, with

$$\alpha_1 = \sqrt{\frac{1}{3}}, \quad \alpha_2 = \sqrt{\frac{1}{2}}, \quad \alpha_3 = 1,$$

$$\theta_1 = 0, \quad \theta_2 = \frac{2}{3}\pi, \quad \theta_3 = \frac{4}{3}\pi. \tag{4}$$

Thus a single-photon input state evolves as follows:

$$\begin{aligned}|\Psi\rangle &\rightarrow \alpha_1(M_{\theta_1}|\Psi\rangle|s_A^1\rangle + M'_{\theta_1}|\Psi\rangle|s_B^1\rangle) + \sqrt{1-|\alpha_1|^2}|\Psi\rangle|s_R^1\rangle\\ &\rightarrow \alpha_1(M_{\theta_1}|\Psi\rangle|s_A^1\rangle + M'_{\theta_1}|\Psi\rangle|s_B^1\rangle) + \sqrt{1-|\alpha_1|^2}\alpha_2(M_{\theta_2}|\Psi\rangle|s_A^2\rangle + M'_{\theta_2}|\Psi\rangle|s_B^2\rangle) + \sqrt{1-|\alpha_1|^2}\sqrt{1-|\alpha_2|^2}|\Psi\rangle|s_R^2\rangle,\\ &\rightarrow \alpha_1(M_{\theta_1}|\Psi\rangle|s_A^1\rangle + M'_{\theta_1}|\Psi\rangle|s_B^1\rangle) + \sqrt{1-|\alpha_1|^2}\alpha_2(M_{\theta_2}|\Psi\rangle|s_A^2\rangle + M'_{\theta_2}|\Psi\rangle|s_B^2\rangle)\\ &\quad + \sqrt{1-|\alpha_2|^2}\sqrt{1-|\alpha_1|^2}\alpha_3(M_{\theta_3}|\Psi\rangle|s_A^3\rangle + M'_{\theta_3}|\Psi\rangle|s_B^3\rangle) + \sqrt{1-|\alpha_1|^2}\sqrt{1-|\alpha_2|^2}\sqrt{1-|\alpha_3|^2}|\Psi\rangle|s_R^3\rangle. \end{aligned} \tag{5}$$

Using Eq. (4) we obtain

$$\begin{aligned}|\Psi\rangle \rightarrow{}& \sqrt{\frac{1}{3}}(a|H\rangle|s_A^1\rangle + b|V\rangle|s_B^1\rangle) + \sqrt{\frac{1}{3}}\left[\left(\frac{1}{4}a + \frac{\sqrt{3}}{4}b\right)|H\rangle + \left(\frac{\sqrt{3}}{4}a + \frac{3}{4}b\right)|V\rangle\right]|s_A^2\rangle + \sqrt{\frac{1}{3}}\left[\left(\frac{3}{4}a - \frac{\sqrt{3}}{4}b\right)|H\rangle\right.\\ &\left. + \left(\frac{-\sqrt{3}}{4}a + \frac{1}{4}b\right)|V\rangle\right]|s_B^2\rangle + \sqrt{\frac{1}{3}}\left[\left(\frac{1}{4}a - \frac{\sqrt{3}}{4}b\right)|H\rangle + \left(\frac{-\sqrt{3}}{4}a + \frac{3}{4}b\right)|V\rangle\right]|s_A^3\rangle\\ &+ \sqrt{\frac{1}{3}}\left[\left(\frac{3}{4}a + \frac{\sqrt{3}}{4}b\right)|H\rangle + \left(\frac{\sqrt{3}}{4}a + \frac{1}{4}b\right)|V\rangle\right]|s_B^3\rangle. \end{aligned} \tag{6}$$

Hence, if we place detectors at the A exits of the three modules, we will measure one of three possible outcomes: a state

$$|\phi_1\rangle = |H\rangle \tag{7}$$

emerging from module 1 with probability

$$p_1 = \frac{1}{3}|a|^2, \tag{8}$$

or a second state

$$|\phi_2\rangle = \frac{\left(\frac{1}{4}a + \frac{\sqrt{3}}{4}b\right)|H\rangle + \left(\frac{\sqrt{3}}{4}a + \frac{3}{4}b\right)|V\rangle}{\sqrt{\frac{1}{4}|a|^2 + \frac{\sqrt{3}}{2}|a||b| + \frac{3}{4}|b|^2}} \tag{9}$$

emerging from module 2 with probability

$$\begin{aligned}p_2 &= \frac{1}{3}\left|\left(\frac{1}{4}a + \frac{\sqrt{3}}{4}b\right)\right|^2 + \frac{1}{3}\left|\left(\frac{\sqrt{3}}{4}a + \frac{3}{4}b\right)\right|^2\\ &= \frac{1}{3}\left(\frac{1}{4}|a|^2 + \frac{\sqrt{3}}{2}|a||b| + \frac{3}{4}|b|^2\right), \end{aligned} \tag{10}$$

or a third state

$$|\phi_3\rangle = \frac{\left(\frac{1}{4}a - \frac{\sqrt{3}}{4}b\right)|H\rangle + \left(-\frac{\sqrt{3}}{4}a + \frac{3}{4}b\right)|V\rangle}{\sqrt{\frac{1}{4}|a|^2 - \frac{\sqrt{3}}{2}|a||b| + \frac{3}{4}|b|^2}} \tag{11}$$

emerging from module 3 with probability

$$\begin{aligned} p_3 &= \frac{1}{3}\left|\left(\frac{1}{4}a - \frac{\sqrt{3}}{4}b\right)\right|^2 + \frac{1}{3}\left|\left(-\frac{\sqrt{3}}{4}a + \frac{3}{4}b\right)\right|^2 \\ &= \frac{1}{3}\left(\frac{1}{4}|a|^2 - \frac{\sqrt{3}}{2}|a||b| + \frac{3}{4}|b|^2\right). \end{aligned} \tag{12}$$

Note that the probability of any of these measurements occurring is $p = p_1 + p_2 + p_3 = 1/2$. This means that the three possible measurement outcomes constitute a projective, nonorthogonal POVM with overall probability $p = 1/2$ and measurement operators

$$M_{\theta_1} = \sqrt{\frac{1}{3}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$M_{\theta_2} = \sqrt{\frac{1}{3}}\begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix},$$

$$M_{\theta_3} = \sqrt{\frac{1}{3}}\begin{pmatrix} 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 \end{pmatrix}. \tag{13}$$

This, in itself, is of course not particularly interesting, since we have simply chosen to perform measurements on a subset of a larger set of orthogonal pairs of POVM operators $\{M_{\theta_j}, M'_{\theta_j}\}$ (for $j = 1, \ldots, 3$). Of greater importance is the fact that the other three operators M'_{θ_j} form a similar POVM with probability $p = 1/2$. Therefore, if we can reunify the beams emerging from the B exits we will reconstruct our original input state. Thus, if this reunification is possible, by feeding the reconstructed state into the measurement apparatus we can perform another POVM on this new input state with a probability of success of 1/2, increasing the total probability of a successful POVM to 3/4. In the limit of infinite cycles the probability of performing a successful POVM tends to unity. We have found a technique for performing the reunification starting with the three mutually nonorthogonal states $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$, which is achieved by the setup illustrated in Fig. 2.

To perform the reunification, first $|\psi_1\rangle$ and $|\psi_2\rangle$ enter a conventional (50%) beam splitter simultaneously, each occupying one of the two possible entrance path states, in the basis of which the following vectors are to be understood:

$$\begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} |\psi_1\rangle + |\psi_2\rangle \\ -|\psi_1\rangle + |\psi_2\rangle \end{pmatrix}. \tag{14}$$

The third input state is passed through a beam splitter by itself, and then each of the resulting states is passed through another beam splitter together with one of the above mixtures of $|\psi_1\rangle$ and $|\psi_2\rangle$, giving the four output states

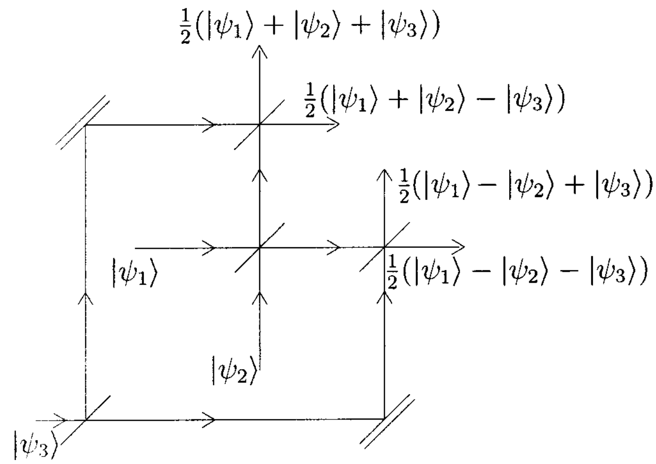

FIG. 2. The reunification setup for three states. States $|\psi_1\rangle$ and $|\psi_2\rangle$ are brought together and then each output is combined with a branch of the split $|\psi_3\rangle$ state. All beam splitters have transmission coefficients of 50%.

$$\begin{aligned} \frac{1}{\sqrt{2}}\begin{pmatrix} |\psi_1\rangle + |\psi_2\rangle \\ |\psi_3\rangle \end{pmatrix} &\rightarrow \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} |\psi_1\rangle + |\psi_2\rangle \\ |\psi_3\rangle \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} |\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle \\ -|\psi_1\rangle - |\psi_2\rangle + |\psi_3\rangle \end{pmatrix} \end{aligned} \tag{15}$$

and

$$\begin{aligned} \frac{1}{\sqrt{2}}\begin{pmatrix} |\psi_3\rangle \\ -|\psi_1\rangle + |\psi_2\rangle \end{pmatrix} &\rightarrow \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} |\psi_3\rangle \\ -|\psi_1\rangle + |\psi_2\rangle \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -|\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle \\ -|\psi_1\rangle + |\psi_2\rangle - |\psi_3\rangle \end{pmatrix}. \end{aligned} \tag{16}$$

One of the four final states [the first state in Eq. (15)] is the original input state $|\psi'_0\rangle = |\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle$. This state can be fed back into the measurement apparatus. The three other states produced are, however, different superpositions of the original input states, namely,

$$|\psi'_1\rangle = \frac{1}{\sqrt{3}}(|\psi_1\rangle - |\psi_2\rangle + |\psi_3\rangle),$$

$$|\psi'_2\rangle = \frac{1}{\sqrt{3}}(|\psi_1\rangle + |\psi_2\rangle - |\psi_3\rangle),$$

$$|\psi'_3\rangle = \frac{1}{\sqrt{3}}(-|\psi_1\rangle + |\psi_2\rangle + |\psi_3\rangle), \tag{17}$$

where we have flipped the sign of the first two states. However, if these three states are fed back into this reunification setup, one obtains the following four states:

$$\frac{1}{2}(|\psi_1'\rangle+|\psi_2'\rangle+|\psi_3'\rangle)=\frac{1}{\sqrt{12}}(|\psi_1\rangle+|\psi_2\rangle+|\psi_3\rangle),$$

$$\frac{1}{2}(|\psi_1'\rangle-|\psi_2'\rangle+|\psi_3'\rangle)=\frac{1}{\sqrt{12}}(-|\psi_1\rangle-|\psi_2\rangle+3|\psi_3\rangle),$$

$$\frac{1}{2}(|\psi_1'\rangle+|\psi_2'\rangle-|\psi_3'\rangle)=\frac{1}{\sqrt{12}}(3|\psi_1\rangle-|\psi_2\rangle-|\psi_3\rangle),$$

$$\frac{1}{2}(-|\psi_1'\rangle+|\psi_2'\rangle+|\psi_3'\rangle)=\frac{1}{\sqrt{12}}(-|\psi_1\rangle+3|\psi_2\rangle-|\psi_3\rangle). \tag{18}$$

Thus this cycle yields another copy of the state $|\psi_0'\rangle=|\psi_1\rangle+|\psi_2\rangle+|\psi_3\rangle$. As can easily be seen, feeding the last three states of Eq. (18) into the setup again will give another copy of $|\psi_0'\rangle$. In the limit of an infinite number of cycles the three states can be successfully reunited to form $|\psi_0'\rangle$ with unit probability. Hence, for the POVM setup states emerging from the B exits

$$|\psi_1\rangle=\sqrt{\frac{1}{3}}b|V\rangle,$$

$$|\psi_2\rangle=\sqrt{\frac{1}{3}}\left[\left(\frac{3}{4}a-\frac{\sqrt{3}}{4}b\right)|H\rangle+\left(\frac{-\sqrt{3}}{4}a+\frac{1}{4}b\right)|V\rangle\right],$$

$$|\psi_3\rangle=\sqrt{\frac{1}{3}}\left[\left(\frac{3}{4}a+\frac{\sqrt{3}}{4}b\right)|H\rangle+\left(\frac{\sqrt{3}}{4}a+\frac{1}{4}b\right)|V\rangle\right], \tag{19}$$

we get

$$\begin{aligned}|\psi_0'\rangle&=|\psi_1\rangle+|\psi_2\rangle+|\psi_3\rangle\\ &=\sqrt{\frac{1}{3}}b|V\rangle+\sqrt{\frac{1}{3}}\left[\left(\frac{3}{4}a-\frac{\sqrt{3}}{4}b\right)|H\rangle\right.\\ &\quad\left.+\left(\frac{-\sqrt{3}}{4}a+\frac{1}{4}b\right)|V\rangle\right]+\sqrt{\frac{1}{3}}\left[\left(\frac{3}{4}a+\frac{\sqrt{3}}{4}b\right)|H\rangle\right.\\ &\quad\left.+\left(\frac{\sqrt{3}}{4}a+\frac{1}{4}b\right)|V\rangle\right]\\ &=\frac{\sqrt{3}}{4}\bigg(a|H\rangle+b|V\rangle\bigg),\end{aligned} \tag{20}$$

which, when normalized, is just our initial input state $|\Psi\rangle$ to the entire measurement apparatus. The complete arrangement for performing the POVM, including the reunification, is shown in Fig. 3.

Given the recent progress in experiments dealing with the manipulation of single photons [11–14], our reunification is experimentally feasible if we consider narrow pulses (or single-photon states) and the possibility of delaying and reunifying the amplitudes from A, B, and C arising in subsequent cycles of $|\Psi\rangle$, before collectively feeding them into the reunification setup. This allows for the sequential use of only one physical setup for an arbitrary number of cycles as there is no overlap of pulses. Thus, in the limit of an infinite number of cycles in both the POVM and reunification setups, the POVM is performed with unit probability of success. For cascades of photons the setup would still be possible but rather more difficult to implement. Further to this work, we have found a generalization to a setup for projective nonorthogonal POVMs with N measurement operators, where the restrictions on the choices of angles lessen considerably for $N>6$. This work is to be published separately.

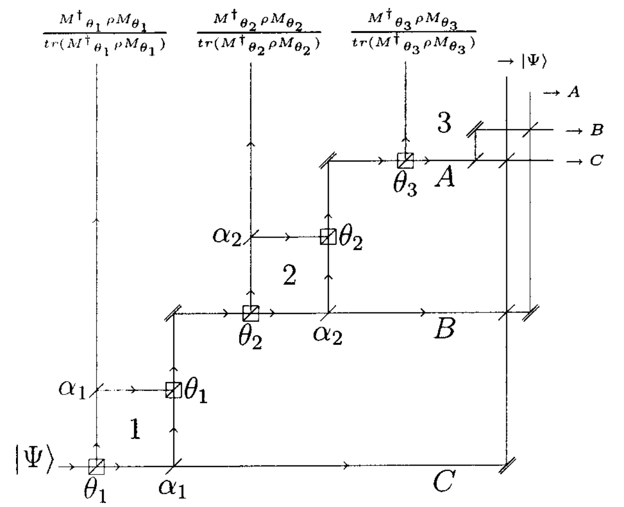

FIG. 3. The complete POVM setup including measurement and reunification for three measurement operators with $\theta_1=0$, $\theta_2=2\pi/3$, $\theta_3=4\pi/3$, $\alpha_1=\sqrt{1/3}$, $\alpha_2=\sqrt{1/2}$, and $\alpha_3=1$. Note the cyclic reunification scheme where the three undesired output states of Fig. 2 are in turn fed back into the entrances A,B,C of the reunification setup, while the desired state $|\Psi\rangle$ is fed back into the entrance of the entire POVM (bottom left). Measurements are performed at the three outputs at the top of the diagram, which yield the normalized states $M^\dagger_{\theta_i}\rho M_{\theta_i}/\mathrm{tr}(M^\dagger_{\theta_i}\rho M_{\theta_i})$ with probabilities $p_i=\mathrm{tr}(M^\dagger_{\theta_i}\rho M_{\theta_i})$, respectively, where $\rho=|\Psi\rangle\langle\Psi|$. This scheme is possible because we are considering single photons or narrow pulses. For photon cascades the setup would still be possible, but experimental realization would be more difficult.

III. CONCLUSIONS

We have proposed a scheme for performing a nonorthogonal projective POVM measurement of the polarization state of a single photon with unit probability of success. Unlike the work that has been published on optical implementations of POVMs, which focuses on nonprojective POVMs, our projective setup provides an outcome not only in terms of a detection probability, but also in the form of one of three possible output states to which the input state is projected.

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