Simple criteria for the implementation of projective measurements with linear optics

Peter van Loock and Norbert Lütkenhaus
Institut für Theoretische Physik I, Institut für Optik, Information und Photonik (Max-Planck-Forschungsgruppe), Staudstr. 7/B3, Universität Erlangen-Nürnberg, 91058 Erlangen, Germany
(Received 16 April 2003; published 9 January 2004)

We derive a set of criteria to decide whether a given projection measurement can be, in principle, exactly implemented solely by means of linear optics. The derivation can be adapted to various detection methods, including photon counting and homodyne detection. These criteria enable one to obtain no-go theorems easily for the exact distinguishability of orthogonal quantum states with linear optics, including the use of auxiliary photons and conditional dynamics.

DOI: 10.1103/PhysRevA.69.012302 PACS number(s): 03.67.Hk, 42.25.Hz, 42.50.Dv

I. INTRODUCTION

Joint orthogonal *projection measurements* are an essential tool in quantum communication. The most prominent example is the Bell measurement that is used, for instance, in quantum teleportation [1]. The canonical way to perform these measurements relies on signal interaction. An example is the optical interaction of light pulses. The latter is particularly relevant for practical applications, since light, traveling at high speed through an optical fiber and allowing for an efficient broadband information encoding, is the most convenient medium for the implementation of quantum communication protocols. In discrete-variable implementations based on single photons, the required strong nonlinear optical interactions are hard to obtain. Alternatively, it is a promising approach to replace interaction by interference, readily available via *linear optics*, and by feedback after detection. There are important cases, however, where linear optics is not sufficient to enable specific projective measurements exactly. For instance, a complete measurement in the qubit polarization Bell basis is not possible within the framework of linear optics, including beam splitters, phase shifters, auxiliary photons, and conditional dynamics utilizing photon counting [2,3]. However, using nontrivial entangled states of n auxiliary photons and conditional dynamics, a perfect projection measurement can be approached asymptotically with a failure rate scaling as $1/n$ [4] or, in a modified version of the scheme of Ref. [4] based on similar resources and tools, with an intrinsic error rate scaling as $1/n^2$ [5]. In any case, no-go statements for exact implementations always indicate whenever finite (and cheaper) resources and less sophisticated tools, such as a fixed array of linear optics, are not sufficient for an arbitrarily good efficiency.

In this article, we propose a different approach to the problem of projective measurements with linear optics and photon counting. Since orthogonal states remain orthogonal after linear-optical mode transformations, the inability to exactly discriminate orthogonal states is due to the measurements in the Fock basis. In our approach, we replace the actual detections by a dephasing of the (linearly transformed) signal states. In other words, the detection mechanism is mimicked by destroying the coherence of the signal states and turning them into mixtures diagonal in the Fock basis. With the resulting density operators, the distinguishability is then expressible in terms of quantum mechanical states. By considering exact distinguishability, this yields a hierarchy of simple conditions for a complete projection measurement. We give a few examples where we employ these conditions in order to make general statements and to derive no-go theorems on linear-optics state discrimination. Moreover, projection measurements based on detection schemes other than photon counting can also be described within the framework of our formalism. In this respect, we include a brief discussion on homodyne-detection based quadrature measurements. However, the essence of our work is the proposal of a universal method. The unified perspective upon which our approach is based will open the path to additional results and applications, including more general measurements than projective ones.

II. THE CRITERIA

Let us define the vectors $\vec{a}=(\hat{a}_1,\hat{a}_2,\ldots,\hat{a}_N)^T$ and $\vec{a}^\dagger=(\hat{a}_1^\dagger,\hat{a}_2^\dagger,\ldots,\hat{a}_N^\dagger)$ representing the annihilation and creation operators of all the electromagnetic modes involved, respectively. A linear-optics circuit can be described via the input-output relations $\vec{c}=U\vec{a}$ or $\vec{c}^\dagger=\vec{a}^\dagger U^\dagger$ with a unitary $N\times N$ matrix U. Conversely, the mixing of N optical modes due to any unitary $N\times N$ matrix is realizable with beam splitters and phase shifters [6]. This excludes linear mixing between annihilation and creation operators, as it results from squeezing transformations. Those require nonlinear optical interactions. On the Hamiltonian level, arbitrary states $|\chi\rangle$ are unitarily transformed via linear optics such that [7]

$$|\chi_H\rangle=\exp(-i\vec{a}^\dagger H\vec{a})|\chi\rangle, \tag{1}$$

where H is an $N\times N$ Hermitian matrix.

We consider projection measurements that operate on subspaces $\mathcal{S}$ of the Hilbert space defined over some signal modes. The orthogonal projection measurement is characterized by one-dimensional projectors $\Pi_k=|s_k\rangle\langle s_k|$ such that $\langle s_k|s_l\rangle=0$ for $k\neq l$, and the completeness relation on the subspace $\mathcal{S}$ is satisfied as $\Sigma_k\Pi_k=\mathbb{1}_{\mathcal{S}}$. In this setting, the problem of implementing the projection measurement is equivalent to the unambiguous discrimination of the orthogonal states $|s_k\rangle$.

The state discrimination may be aided by an auxiliary state $|\psi_{\rm aux}\rangle$ that is supported on auxiliary modes. The states to be distinguished then are $\hat{\rho}_{k,\rm in}=|\chi_k\rangle\langle\chi_k|$ with $|\chi_k\rangle=|s_k\rangle\otimes|\psi_{\rm aux}\rangle$. The entire discrimination process now consists of two steps, $\hat{\rho}_{\rm in}\rightarrow\hat{\rho}_H\rightarrow\hat{\rho}'_H$, where the first step is due to linear optics, $\hat{\rho}_H\equiv|\chi_H\rangle\langle\chi_H|$. In the second step, the detection of the output modes in the Fock basis is mimicked through dephasing,

$$\hat{\rho}_H\rightarrow\hat{\rho}'_H=\frac{1}{(2\pi)^N}\int d\phi^N e^{-i\vec{a}^\dagger D\vec{a}}\hat{\rho}_H e^{i\vec{a}^\dagger D\vec{a}}, \tag{2}$$

with $d\phi^N\equiv d\phi_1 d\phi_2\cdots d\phi_N$ and the diagonal $N\times N$ matrix D, $(D)_{ij}=\delta_{ij}\phi_i$. The distinguishability can then be analyzed on the level of the density operators $\hat{\rho}'_H$. Since exact discrimination is considered, this leads to a huge simplification, as we shall explain now.

In order to decide on the exact distinguishability of any two states $|\chi_k\rangle=|s_k\rangle\otimes|\psi_{\rm aux}\rangle$ and $|\chi_l\rangle=|s_l\rangle\otimes|\psi_{\rm aux}\rangle$, we may use $\mathrm{Tr}(\hat{\rho}'_{k,H}\hat{\rho}'_{l,H})$, where $\hat{\rho}'_{k,H}$ and $\hat{\rho}'_{l,H}$ are the corresponding states after linear optics and dephasing. We obtain the condition for exact distinguishability:

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}'_{k,H}\hat{\rho}'_{l,H})&=\frac{1}{(2\pi)^{2N}}\int d\phi^N d\tilde{\phi}^N|\langle\chi_k|\\&\quad\times e^{i\vec{a}^\dagger H\vec{a}}e^{i\vec{a}^\dagger(D-\tilde{D})\vec{a}}e^{-i\vec{a}^\dagger H\vec{a}}|\chi_l\rangle|^2\\&\stackrel{!}{=}0,\end{aligned} \tag{3}$$

where $d\tilde{\phi}^N\equiv d\tilde{\phi}_1 d\tilde{\phi}_2\cdots d\tilde{\phi}_N$ and $(\tilde{D})_{ij}=\delta_{ij}\tilde{\phi}_i$. Due to the positivity of the integrand, this is equivalent to

$$\begin{aligned}&\langle\chi_k|e^{i\vec{a}^\dagger H\vec{a}}e^{i\vec{a}^\dagger(D-\tilde{D})\vec{a}}e^{-i\vec{a}^\dagger H\vec{a}}|\chi_l\rangle=\langle\chi_k|e^{i\vec{c}^\dagger(D-\tilde{D})\vec{c}}|\chi_l\rangle\\&=0\quad\forall\ \ \phi_j,\tilde{\phi}_j,\end{aligned} \tag{4}$$

where the effect of linear optics is now put into the operators $\vec{c}=e^{i\vec{a}^\dagger H\vec{a}}\vec{a}e^{-i\vec{a}^\dagger H\vec{a}}$ or $\vec{c}=U\vec{a}$. Let us define $y_j\equiv\phi_j-\tilde{\phi}_j$, $j=1,\ldots,N$. Since the derivatives of $\langle\chi_k|e^{i\vec{c}^\dagger(D-\tilde{D})\vec{c}}|\chi_l\rangle$ with respect to any relative phases $y_j, y_{j'}, y_{j''},\ldots$ must also vanish, in particular, at $\vec{y}=(y_1,y_2,\ldots,y_N)=\vec{0}$, we obtain the set of conditions for exact state discrimination:

$$\begin{aligned}\langle\chi_k|\hat{c}_j^\dagger\hat{c}_j|\chi_l\rangle&=0\quad\forall\ \ j,\\ \langle\chi_k|\hat{c}_j^\dagger\hat{c}_j\hat{c}_{j'}^\dagger\hat{c}_{j'}|\chi_l\rangle&=0\quad\forall\ \ j,j',\\ \langle\chi_k|\hat{c}_j^\dagger\hat{c}_j\hat{c}_{j'}^\dagger\hat{c}_{j'}\hat{c}_{j''}^\dagger\hat{c}_{j''}|\chi_l\rangle&=0\quad\forall\ \ j,j',j'',\\ \vdots\quad&=\quad\vdots\quad\forall\ \ k\neq l.\end{aligned} \tag{5}$$

These conditions are *necessary* for a complete projection measurement onto the basis $\{|\chi_k\rangle\}$. However, if the entire set of conditions is satisfied, this is in general also a *sufficient* condition, since $e^{i\vec{c}^\dagger(D-\tilde{D})\vec{c}}$ is an analytic function of the relative phases $\vec{y}$. Note that orthogonality $\langle\chi_k|\chi_l\rangle=0$, $\forall\ k\neq l$, is the "zeroth-order condition."

By exploiting the fact that $(\hat{c}^\dagger\hat{c})^n$ is of the form $\Sigma_{m=1}^n d_m(\hat{c}^\dagger)^m\hat{c}^m$ with some coefficients d_m and that $[\hat{c}_j^\dagger,\hat{c}_{j'}]=0$ for $j\neq j'$, the higher-order conditions can be rewritten in an equivalent normally ordered form, provided the lower-order conditions are satisfied. This leads to the hierarchy of conditions

$$\begin{aligned}\langle\chi_k|\hat{c}_j^\dagger\hat{c}_j|\chi_l\rangle&=0\quad\forall\ \ j,\\ \langle\chi_k|\hat{c}_j^\dagger\hat{c}_{j'}^\dagger\hat{c}_j\hat{c}_{j'}|\chi_l\rangle&=0\quad\forall\ \ j,j',\\ \langle\chi_k|\hat{c}_j^\dagger\hat{c}_{j'}^\dagger\hat{c}_{j''}^\dagger\hat{c}_j\hat{c}_{j'}\hat{c}_{j''}|\chi_l\rangle&=0\quad\forall\ \ j,j',j'',\\ \vdots\quad&=\quad\vdots\quad\forall\ \ k\neq l.\end{aligned} \tag{6}$$

In this form, one can directly see that the hierarchy breaks off for higher-order terms if the number of photons in the states $\{|\chi_k\rangle\}$ is bounded. Hence, for finite photon number, we end up having a finite hierarchy of necessary and sufficient conditions for complete projective measurements. The states of an orthogonal set $\{|\chi_k\rangle\}$ are, in principle, exactly distinguishable via a *fixed array of linear optics* represented by $\vec{c}=U\vec{a}$, if and only if these conditions hold for the complete set of modes.

The subset of conditions referring only to a particular mode operator $\hat{c}_j$ represents *necessary* conditions for exact discrimination based on *conditional dynamics* after detecting that mode j. They are given by

$$\langle\chi_k|(\hat{c}_j^\dagger)^n(\hat{c}_j)^n|\chi_l\rangle=0\quad\forall\ \ n\geqslant1,\quad\forall\ \ k\neq l. \tag{7}$$

Already the failure to find some $\hat{c}_j$ satisfying Eq. (7) means that, as soon as one output mode is selected and measured, this will make exact discrimination of the states impossible. Conversely, one may also use the conditions of Eq. (7) in a constructive way. The recipe is to find one $\hat{c}_j$ that satisfies Eq. (7), to calculate the corresponding conditional states of the remaining modes, and to test them for their distinguishability. It is instructive to view this in terms of the partially dephased states. After dephasing only one mode j, we obtain

$$\hat{\rho}'^{(j)}_{k,H}=\sum_m\frac{P^{(j)}(m|k)}{m!}(\hat{a}_j^\dagger)^m|0\rangle_j|c_{k,m}^{(j)}\rangle\langle c_{k,m}^{(j)}|_j\langle0|\hat{a}_j^m, \tag{8}$$

where $P^{(j)}(m|k)$ is the probability of finding m photons in the measured mode j for given input state $|\chi_k\rangle$, and $|c_{k,m}^{(j)}\rangle$ is the corresponding (normalized) conditional state of the remaining modes. Failure to satisfy Eq. (7) implies that the conditional states $|c_{k,m}^{(j)}\rangle$ form a nonorthogonal set in k for each fixed combination of (m,j). For such sets, we know that a further exact discrimination is impossible. We will show now that the condition in Eq. (7) for $n=1$ suffices to reproduce easily all known no-go theorems for projective measurements with linear optics, including auxiliary photons and conditional dynamics.

III. EXAMPLES

In this section, we present a few examples that illustrate the simplicity and usefulness of the criteria derived in the preceding section. These examples include general statements on the effect of extra resources on the exact distinguishability of arbitrary quantum states and "back of the envelope" derivations of no-go theorems (some known). Among them, the simplest and most remarkable example is that for a pair of orthogonal two-photon states, because the previously known no-go results apply to sets of at least four orthogonal states (e.g., the Bell states).

We start by investigating the use of auxiliary photons [8]. Splitting the input modes into a set of signal and a set of auxiliary modes allows us to decompose the mode operator $\hat{c}_j=\Sigma_i U_{ji}\hat{a}_i$ from Eq. (7) into two corresponding parts as (we drop the index j) $\hat{c}=b_s\hat{c}_s+b_{\rm aux}\hat{c}_{\rm aux}$, with real coefficients b_s and $b_{\rm aux}$, so that $\hat{c}_s|\mathbf{0}\rangle\otimes|\psi_{\rm aux}\rangle=\hat{c}_{\rm aux}|s_k\rangle\otimes|\mathbf{0}\rangle=0$. Now we find

$$\begin{aligned}\langle\chi_k|\hat{c}^\dagger\hat{c}|\chi_l\rangle=&b_s^2\langle s_k|\hat{c}_s^\dagger\hat{c}_s|s_l\rangle+b_sb_{\rm aux}\langle s_k|\hat{c}_s|s_l\rangle\\&\times\langle\psi_{\rm aux}|\hat{c}_{\rm aux}^\dagger|\psi_{\rm aux}\rangle+b_sb_{\rm aux}\langle s_k|\hat{c}_s^\dagger|s_l\rangle\\&\times\langle\psi_{\rm aux}|\hat{c}_{\rm aux}|\psi_{\rm aux}\rangle+b_{\rm aux}^2\langle s_k|s_l\rangle\\&\times\langle\psi_{\rm aux}|\hat{c}_{\rm aux}^\dagger\hat{c}_{\rm aux}|\psi_{\rm aux}\rangle.\end{aligned}\tag{9}$$

The last term always vanishes for $k\neq l$, since the $|s_k\rangle$ are orthogonal. In any situation where either the signal states or the auxiliary state have a fixed photon number, the two middle terms vanish, and the first-order condition $\langle\chi_k|\hat{c}^\dagger\hat{c}|\chi_l\rangle=0$ depends only on the signal states alone, $b_s^2\langle s_k|\hat{c}_s^\dagger\hat{c}_s|s_l\rangle=0$, $\forall\ k\neq l$. The trivial case $b_s=0$ can be omitted without loss of generality. It is straightforward to extend this derivation to any order in Eq. (7) by inserting a mode operator decomposed into a signal and an auxiliary part. *Hence for signal states with a fixed photon number, auxiliary systems never help, and for signal states with an unfixed number, adding an auxiliary state may help, but only provided the auxiliary state has unfixed number too.*

The no-go theorem for the qubit Bell states [2,3]

$$|\Psi_\pm\rangle=\frac{1}{\sqrt{2}}(\hat{a}_1^\dagger\hat{a}_4^\dagger\pm\hat{a}_2^\dagger\hat{a}_3^\dagger)|\mathbf{0}\rangle,$$

$$|\Phi_\pm\rangle=\frac{1}{\sqrt{2}}(\hat{a}_1^\dagger\hat{a}_3^\dagger\pm\hat{a}_2^\dagger\hat{a}_4^\dagger)|\mathbf{0}\rangle,\tag{10}$$

is obtainable now in a very simple way. In order to check for the existence of a mode j satisfying Eq. (7) for $n=1$, let us again drop the index j and use the ansatz $\hat{c}_s\propto\nu_1\hat{a}_1+\nu_2\hat{a}_2+\nu_3\hat{a}_3+\nu_4\hat{a}_4$, by defining $U_{ji}\equiv\nu_i$. We have six conditions $\langle\chi_k|\hat{c}^\dagger\hat{c}|\chi_l\rangle=0$ for the pairs (k,l),

$$(\Psi_+,\Psi_-),(\Phi_+,\Phi_-):|\nu_1|^2-|\nu_2|^2\mp|\nu_3|^2\pm|\nu_4|^2=0,$$

$$(\Psi_+,\Phi_+),(\Psi_+,\Phi_-):\nu_1\nu_2^*+\nu_3\nu_4^*\pm\nu_1^*\nu_2\pm\nu_3^*\nu_4=0,$$

$$(\Psi_-,\Phi_+),(\Psi_-,\Phi_-):\nu_3\nu_4^*-\nu_1\nu_2^*\pm\nu_1^*\nu_2\mp\nu_3^*\nu_4=0.\tag{11}$$

These conditions imply

$$(\Psi_+,\Psi_-),(\Phi_+,\Phi_-)\Rightarrow|\nu_1|^2=|\nu_2|^2,|\nu_3|^2=|\nu_4|^2,$$

$$(\Psi_+,\Phi_+),(\Psi_+,\Phi_-)\Rightarrow\nu_1\nu_2^*=-\nu_3\nu_4^*,$$

$$(\Psi_-,\Phi_+),(\Psi_-,\Phi_-)\Rightarrow\nu_1\nu_2^*=\nu_3\nu_4^*.\tag{12}$$

It can be easily seen that these conditions have only trivial solutions $\nu_i=0$, $\forall\ i$, which proves the no-go theorem for the Bell states including auxiliary photons and conditional dynamics.

A similar no-go theorem is known [9] for an orthogonal set of separable two-qutrit states [10]:

$$|s_{1,2}\rangle=\frac{1}{\sqrt{2}}\hat{a}_1^\dagger(\hat{a}_4^\dagger\pm\hat{a}_5^\dagger)|\mathbf{0}\rangle,|s_{3,4}\rangle=\frac{1}{\sqrt{2}}\hat{a}_3^\dagger(\hat{a}_5^\dagger\pm\hat{a}_6^\dagger)|\mathbf{0}\rangle,$$

$$|s_{5,6}\rangle=\frac{1}{\sqrt{2}}\hat{a}_4^\dagger(\hat{a}_2^\dagger\pm\hat{a}_3^\dagger)|\mathbf{0}\rangle,|s_{7,8}\rangle=\frac{1}{\sqrt{2}}\hat{a}_6^\dagger(\hat{a}_1^\dagger\pm\hat{a}_2^\dagger)|\mathbf{0}\rangle,$$

$$|s_9\rangle=\hat{a}_2^\dagger\hat{a}_5^\dagger|\mathbf{0}\rangle.\tag{13}$$

The entire set of 36 first-order conditions for one mode j with $\hat{c}=\Sigma_i\nu_i\hat{a}_i$ now leads to

$$|\nu_1|^2=|\nu_2|^2=|\nu_3|^2,\quad|\nu_4|^2=|\nu_5|^2=|\nu_6|^2,$$

$$\nu_1\nu_2^*=\nu_1\nu_3^*=\nu_2\nu_3^*=\nu_4\nu_5^*=\nu_4\nu_6^*=\nu_5\nu_6^*=0.\tag{14}$$

Again, only trivial solutions exist. Going beyond Ref. [9], we can now easily investigate subclasses of the set. The full no-go theorem also applies to the eight states when leaving out state $|s_9\rangle$. For other subclasses, this example illustrates the role of conditional dynamics. For instance, leaving out state $|s_8\rangle$, the conditions remain exactly those in Eq. (14) except that $|\nu_1|^2$ does not occur in the first line. The only nontrivial solution is now where $\nu_1=1$ and $\nu_i=0$, $\forall\ i=2,\ldots,6$. The interpretation is that, in order to enable discrimination of the conditional states for the entire subset, mode 1 must be detected first. This can be seen intuitively in Eq. (13) and in Fig. 1.

With the help of the hierarchy of conditions, one can now easily find other no-go theorems. Consider the orthogonal set of four two-qubit states

$$|s_1\rangle=(\alpha\hat{a}_1^\dagger\hat{a}_4^\dagger+\beta\hat{a}_2^\dagger\hat{a}_3^\dagger)|\mathbf{0}\rangle,$$

$$|s_2\rangle=(\beta^*\hat{a}_1^\dagger\hat{a}_4^\dagger-\alpha^*\hat{a}_2^\dagger\hat{a}_3^\dagger)|\mathbf{0}\rangle,$$

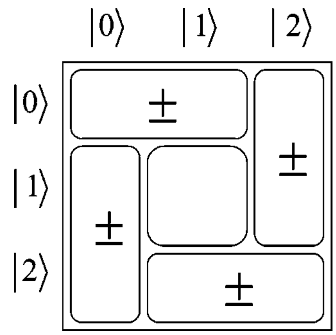

FIG. 1. The nine two-qutrit product states that are undistinguishable via linear optics when encoded into two-photon states. The three logical basis states $\{|0\rangle,|1\rangle,|2\rangle\}$ at each side are then represented by a single photon in one of three modes, for instance, the photonic states $|s_{1,2}\rangle$ from Eq. (13) correspond to the logical states $|0\rangle\otimes(|0\rangle\pm|1\rangle)/\sqrt{2}$.

$$|s_3\rangle=(\gamma\hat{a}_1^\dagger\hat{a}_3^\dagger+\delta\hat{a}_2^\dagger\hat{a}_4^\dagger)|\mathbf{0}\rangle,$$

$$|s_4\rangle=(\delta^*\hat{a}_1^\dagger\hat{a}_3^\dagger-\gamma^*\hat{a}_2^\dagger\hat{a}_4^\dagger)|\mathbf{0}\rangle. \tag{15}$$

If all four states are entangled, $|\alpha\beta|>0$ and $|\gamma\delta|>0$, only trivial solutions exist for the six first-order conditions Eq. (7) with $n=1$. Hence, the full no-go statement applies, including auxiliary photons and conditional dynamics. For only two entangled states, e.g., $|\alpha\beta|>0$ and $\gamma=0$, one mode $\hat{c}_j$ always exists that satisfies Eq. (7). However, there are only trivial solutions to the second-order condition in Eq. (6) for some pairs of modes $\hat{c}_j$ and $\hat{c}_{j'}$ $(j\neq j')$, if the two states are *nonmaximally* entangled. In fact, a fixed array of linear optics is not sufficient in this case, but a conditional-dynamics solution exists. If the two states are *maximally* entangled, any order in Eq. (6) is satisfied with a 50:50 beam splitter.

A particularly interesting example is the following *pair* of orthogonal states:

$$\frac{1}{\sqrt{2}}(|20\rangle\pm|11\rangle), \tag{16}$$

described in the Fock basis. We find that the $n=1$ and $n=2$ conditions of Eq. (7) can be simultaneously satisfied only trivially, $\nu_1=\nu_2=0$. Thus, there is no linear-optical discrimination scheme for the two states of Eq. (16), not even with the help of conditional dynamics and auxiliary photons, since the two states have fixed photon number. In fact, this no-go statement applies to the whole family of pairs of orthogonal states $\alpha|20\rangle+\beta|11\rangle$ and $\beta^*|20\rangle-\alpha^*|11\rangle$ for $|\alpha\beta|>0$.

What about quantitative statements beyond the no-go theorems for exact state discrimination? A linear-optics network with photon counting yields for each input state a classical probability distribution for the pattern of photon detections in the output modes. This distribution can be used to estimate the input state. A possible measure in the context of estimating an input state is the probability of minimum error [11]. For four equally probable output distributions, it can be written as

$$P_{\rm error}^{\rm min}=1-\frac{1}{4}\sum_i \max_k[P(i|k)], \tag{17}$$

where $P(i|k)$ is the conditional probability for obtaining the result i (pattern of the photon detections) given the distribution k. Using the classical distributions of the results i in the totally dephased states with the two-photon Bell states of Eq. (10) as the input states (parametrized by an arbitrary unitary 4×4 matrix U), we found numerically that $P_{\rm error}^{\rm min}\geq 1/4$. This bound can be attained by using a 50:50 beam splitter [12], where

$$\hat{\rho}'_{\Psi_+,{\rm BS}}=\frac{1}{2}(|1100\rangle\langle1100|+|0011\rangle\langle0011|),$$

$$\hat{\rho}'_{\Psi_-,{\rm BS}}=\frac{1}{2}(|1001\rangle\langle1001|+|0110\rangle\langle0110|),$$

$$\hat{\rho}'_{\Phi_\pm,{\rm BS}}=\frac{1}{4}(|2000\rangle\langle2000|+|0200\rangle\langle0200|+|0020\rangle\langle0020| +|0002\rangle\langle0002|), \tag{18}$$

corresponding to the optimal partial Bell measurement without auxiliary photons and conditional dynamics [13].

IV. QUADRATURE MEASUREMENTS

So far, the dephasing approach has been solely used to describe the decohering effect of photon detections, i.e., measurements in the Fock basis. However, it is worth pointing out that this method is applicable to other kinds of measurements too. We may also consider, for example, homodyne detections, i.e., measurements in a continuous-variable basis. In that case, the appropriate replacement in the dephasing formula of Eq. (2) is

$$e^{i\vec{a}^\dagger D\vec{a}}=e^{i\sum_j \phi_j\hat{a}_j^\dagger\hat{a}_j}\rightarrow e^{i\sum_j \phi_j\hat{x}_j^{(\theta_j)}}, \tag{19}$$

where $\hat{x}_j^{(\theta_j)}=(\hat{a}_je^{-i\theta_j}+\hat{a}_j^\dagger e^{+i\theta_j})/2$ are the quadratures of mode j. For example, for $\theta_j=0$ and $\theta_j=\pi/2$, we obtain, respectively, the position $\hat{x}$ and momentum $\hat{p}$ associated with the mode's harmonic oscillator. The derivation of a set of necessary and sufficient conditions for exact state discrimination, Eqs. (3)–(5), also follows through with the replacement in Eq. (19). The resulting conditions in that case become (we drop the superscript θ_j)

$$\begin{aligned}\langle\chi_k|\hat{x}_j^c|\chi_l\rangle&=0 \quad \forall\ j,\\ \langle\chi_k|\hat{x}_j^c\hat{x}_{j'}^c|\chi_l\rangle&=0 \quad \forall\ j,j',\\ \langle\chi_k|\hat{x}_j^c\hat{x}_{j'}^c\hat{x}_{j''}^c|\chi_l\rangle&=0 \quad \forall\ j,j',j'',\\ \vdots\quad &=\quad \vdots \quad \forall\ k\neq l,\end{aligned} \tag{20}$$

where $\hat{x}_j^c = (\hat{c}_j e^{-i\theta_j} + \hat{c}_j^\dagger e^{+i\theta_j})/2$ denotes the quadratures of mode j after the linear-optics circuit with $\vec{c} = U\vec{a}$. A continuous-variable Bell measurement discriminates between the two-mode eigenstates of relative position $\hat{x}_1 - \hat{x}_2$ and total momentum $\hat{p}_1 + \hat{p}_2$. This can be achieved with a simple 50:50 beam splitter and subsequent $\hat{x}$ and $\hat{p}$ measurements at the two output ports [14]. Conditional dynamics is not needed. However, in order to satisfy the above conditions for all (that is two) modes, two conjugate quadratures must be detected, for example, $\hat{x}_1^c = (\hat{c}_1 + \hat{c}_1^\dagger)/2 = (\hat{x}_1 - \hat{x}_2)/\sqrt{2}$ and $\hat{x}_2^c = (\hat{c}_2 - \hat{c}_2^\dagger)/2i = (\hat{p}_1 + \hat{p}_2)/\sqrt{2}$. Here, $\hat{x}_j$ and $\hat{p}_j$ are the two conjugate quadratures of the input modes $\hat{a}_j$. Hence, due to the orthogonality of the continuous-variable Bell states, the described scheme represents a solution to the above conditions. In a very intuitive way, this explains why a fixed linear-optics scheme suffices to perform a continuous-variable Bell measurement with arbitrarily high efficiency, in contrast to a qubit Bell measurement: the continuous-variable Bell states are eigenstates of the detected quadratures, whereas the qubit Bell states are no eigenstates of the detected photon numbers.

V. SUMMARY AND OUTLOOK

In summary, we have presented a different approach to describing the processing of quantum states via linear optics including photon counting or other measurements such as homodyne detection. The advantage of this approach is that the detection mechanism is included in the transformation from the input quantum states to the output quantum states. For the case of a complete projection measurement onto a (joint) orthogonal basis, we obtained a hierarchy of necessary and sufficient conditions. When photon counting is considered, this hierarchy breaks off and yields a finite set of simple conditions for states with finite photon numbers. Apart from homodyne detection, our universal approach can also be used to include other "continuous-variable tools" such as displacements and squeezing. It also provides a promising method to treat more general scenarios, e.g., the realization of general measurements [positive operator valued measures (POVM's)] with linear optics [15]. Any POVM can be described via Naimark extension as an orthogonal von Neumann measurement in a larger Hilbert space. The extended signal states may then be analyzed using the criteria derived in this paper. This generalization is particularly significant, because it would extend our approach from qualitative statements on exact projection measurements to quantitative statements on approximate projection measurements.

Although progress is being made in enhancing the effective strength of nonlinear optical interactions, it appears reasonable to exploit the entire toolbox of linear optics first and explore it, in order to be aware of its capabilities, but also its limitations. In the recent work of Ref. [4], the authors demonstrate that the capabilities of linear optics are unexpectedly broad; however, unfeasibly, many extra resources may be needed for a good performance. We hope that the question of the trade-off between these extra resources and the performance can be attacked utilizing our criteria.

ACKNOWLEDGMENTS

We are grateful to John Calsamiglia and Bill Munro for useful comments. We also acknowledge the financial support of the DFG under the Emmy-Noether program, the EU FET network RAMBOQ (Grant No. IST-2002-6.2.1), and the network of competence QIP of the state of Bavaria.

[1] C.H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
[2] N. Lütkenhaus, J. Calsamiglia, and K.A. Suominen, Phys. Rev. A **59**, 3295 (1999).
[3] L. Vaidman and N. Yoran, Phys. Rev. A **59**, 116 (1999).
[4] E. Knill, R. Laflamme, and G.J. Milburn, Nature (London) **409**, 46 (2001).
[5] J.D. Franson *et al.*, Phys. Rev. Lett. **89**, 137901 (2002).
[6] M. Reck *et al.*, Phys. Rev. Lett. **73**, 58 (1994).
[7] P. Törmä, S. Stenholm, and I. Jex, Phys. Rev. A **52**, 4853 (1995).
[8] A. Carollo and G.M. Palma, J. Mod. Opt. **49**, 1147 (2002).
[9] A. Carollo *et al.*, Phys. Rev. A **64**, 022318 (2001).
[10] C.H. Bennett *et al.*, Phys. Rev. A **59**, 1070 (1999).
[11] C.A. Fuchs, Ph.D. thesis, University of New Mexico, 1995.
[12] H. Weinfurter, Europhys. Lett. **25**, 559 (1994); S.L. Braunstein and A. Mann, Phys. Rev. A **51**, R1727 (1995).
[13] J. Calsamiglia and N. Lütkenhaus, Appl. Phys. B: Lasers Opt. **72**, 67 (2001).
[14] S.L. Braunstein and H.J. Kimble, Phys. Rev. Lett. **80**, 869 (1998).
[15] J. Calsamiglia, Phys. Rev. A **65**, 030301(R) (2002).