Multipartite entanglement in $2\times2\times n$ quantum systems

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(Received 9 July 2003; published 8 January 2004)

We classify multipartite entangled states in the Hilbert space $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^n$ $(n\geqslant 4)$, for example, the four-qubit system distributed over three parties, under local filtering operations. We show that there exist nine essentially different classes of states, giving rise to a five-graded partially ordered structure, including the celebrated Greenberger-Horne-Zeilinger and *W* classes of three qubits. In particular, all $2\times2\times n$ states can be deterministically prepared from one maximally entangled state, and some applications such as entanglement swapping are discussed.

DOI: 10.1103/PhysRevA.69.012101 PACS number(s): 03.65.Ud, 03.67.−a

I. INTRODUCTION

Entanglement is the key ingredient of all applications in the field of quantum information. Due to the nonlocal character of the correlations that entanglement induces, it is expected that entanglement is especially valuable in the context of many parties. Despite a lot of efforts however, it has proven exceedingly hard to get insight into the structure of multipartite entanglement. Still, the motivation of our work is as follows. In the bipartite (pure) setting, the entanglement present in a Bell-Einstein-Podolsky-Rosen (Bell-EPR) state is essentially unique; i.e., we can evaluate any bipartite entangled state by the number of equivalent Bell pairs, in either a qubits or a qudits system, both in the single-copy and multiple-copy case.

The situation is totally different in the multipartite setting however, where interconvertibility under local operations and classical communication (LOCC) is not expected to hold [1]. Multipartite entanglement exhibits a much richer structure than bipartite entanglement. The first celebrated example thereof was the three-qubit GHZ state, called after Greenberger, Horne, and Zeilinger [2]. This state was introduced because it allows one to disprove the Einstein locality for quantum systems without invoking statistical arguments such as those needed by Bell. Another interesting aspect of multipartite entanglement was discovered by Coffman *et al.* [3]. They showed that a quantum state has only a limited shareability for quantum correlations: more the bipartite correlations in a state, lesser the genuine multipartite entanglement that can be present in the system. This led to the introduction of the so-called three-qubit *W* state [4], which was shown to be essentially different from the GHZ state as they are not interconvertible under LOCC even probabilistically.

In this paper, we will generalize these results and present one of the very few exact and complete results about multipartite quantum systems, by classifying multipartite entanglement in the $2\times2\times n$ cases. Since this includes the four-qubit system distributed over *three parties*, which is the case in, e.g., entanglement swapping, our results will clarify what kinds of essentially different multipartite entanglement there exist in this situation, and give better understanding of multiparty LOCC protocols. More specifically, we will address the stochastic LOCC (SLOCC) classification of entanglement [1,4–13], which is a *coarse-grained* classification under LOCC. Let us consider the single copy of a multipartite pure state $|\Psi\rangle$ on the Hilbert space $\mathcal{H}=\mathbb{C}^{k_1}\otimes\cdots\otimes\mathbb{C}^{k_l}$ (precisely, in abuse of the notation, we would denote a ray on its complex projective space $\mathbb{C}P^{k_1\times\cdots\times k_l-1}$ by $|\Psi\rangle$),

$$|\Psi\rangle=\sum_{i_1,\ldots,i_l=0}^{k_1-1,\ldots,k_l-1}\psi_{i_1\ldots i_l}|i_1\rangle\otimes\cdots\otimes|i_l\rangle, \tag{1}$$

where a set of $|i_1\rangle\otimes\cdots\otimes|i_l\rangle$ constitutes the standard computational basis and often it will be abbreviated to $|i_1\cdots i_l\rangle$. In LOCC, we recognize two states $|\Psi\rangle$ and $|\Psi'\rangle$ which are interconvertible deterministically, e.g., by local unitary operations, as equivalent entangled states. On the other hand in SLOCC, we identify two states $|\Psi\rangle$ and $|\Psi'\rangle$ as equivalent if they are interconvertible probabilistically, i.e., with a nonvanishing probability, since they are supposed to be able to perform the same tasks in quantum information processing but with different success probabilities. Mathematically, $|\Psi\rangle$ and $|\Psi'\rangle$ belong to the same SLOCC entangled class if and only if they can be converted to each other by *invertible* SLOCC operations,

$$|\Psi'\rangle=M_1\otimes\cdots\otimes M_l|\Psi\rangle, \tag{2}$$

where M_i is any local operation having a nonzero determinant on the ith party [4], i.e., M_i is an element of the general linear group $\mathrm{GL}(k_i,\mathbb{C})$ [we do not care about the overall normalization and phase so that we can take its determinant 1, i.e., $M_i\in\mathrm{SL}(k_i,\mathbb{C})$]. It can be also said that an invertible SLOCC operation is a completely positive map followed by

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the postselection of one successful outcome. Mathematically, the SLOCC classification is equivalent to the classification of orbits generated by a direct product of special linear groups $SL(k_1,\mathbb{C})\times\cdots\times SL(k_l,\mathbb{C})$. Note that in the bipartite $l=2$ case, the SLOCC classification means the classification just by the Schmidt rank [or equivalently, the rank of a coefficient "matrix" $\psi_{i_1 i_2}$ in Eq. (1)]. We will also address the question of *noninvertible* SLOCC operations [at least one of the ranks of M_i in Eq. (2) is not full]. The set of invertible and noninvertible SLOCC operations are also called local filtering operations. Consider the bipartite case as an example: SLOCC entangled classes are found to be totally ordered in such a way that an entangled class of the larger Schmidt rank is more entangled than that of the smaller one, because the Schmidt rank is always decreasing under noninvertible local operations.

The paper is organized as follows. In Sec. II, we classify multipartite $2\times2\times n$ pure states under SLOCC, so as to show that nine entangled classes are hierarchized in a five-graded partial order. We discuss the characteristics of multipartite entanglement in our situation in Sec. III, and extend the classification of multipartite pure states to mixed states in Sec. IV. The conclusion is given in Sec. V.

II. CLASSIFICATION OF MULTIPARTITE ENTANGLEMENT

In this section, we give the complete SLOCC classification of multipartite entanglement in $2\times2\times n$ cases. Moreover, we present a convenient criterion to distinguish inequivalent entangled classes by SLOCC invariants.

A. Five-graded partial order of nine entangled classes

We show that there are nine entangled classes and they constitute five-graded partially ordered structure under noninvertible SLOCC operations.

Theorem 1. Consider pure states in the Hilbert space $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^n$ $(n\geqslant4)$, they are divided into nine entangled classes, seen in Fig. 1, under invertible SLOCC operations. These nine entangled classes constitute the five-graded partially ordered structure of Fig. 2, where noninvertible SLOCC operations degrade higher entangled classes into lower entangled ones.

Some remarks are given before its proof. The theorem gives the complete classification of multipartite pure entangled states in $2\times2\times n$ $(n\geqslant4)$ cases. It naturally contains the classification for the $2\times2\times2$ (three-qubit) case [4,5,8] and the $2\times2\times3$ case [5]. We find that SLOCC orbits are added outside the onionlike picture (Fig. 1) and the partially ordered structure (Fig. 2) becomes higher, as the third party Clare has her larger subsystem. Remarkably, for the $2\times2\times n$ $(n\geqslant4)$ cases, the generic class is one "maximally entangled" class located on the top of the hierarchy. This is a clear contrast with the situation of the $2\times2\times2$ and $2\times2\times3$ cases, where there are two different entangled classes on its top. This suggests that even in the multipartite situation, there is a unique entangled class which can serve as resource to create any entangled state, if Clare's subsystem is large enough. This will be proven in Sec. III.

We note that it is sufficient to consider the $2\times2\times4$ case in the proof of the theorem, since Clare can only have support on a four-dimensional subspace. This is an analogy with the bipartite $k\times k'$ $(k<k')$ case whose SLOCC classification is equivalent to that of the $k\times k$ case, because the SLOCC-invariant Schmidt local rank takes at most k. In any $2\times2\times n$ $(n\geqslant4)$ case, the partially ordered structure of multipartite entanglement consists of nine finite classes. Our result not only describes the situation that only Clare has the

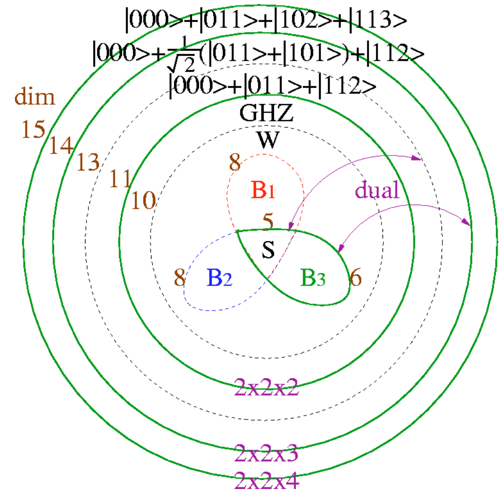

FIG. 1. The onionlike classification of multipartite entangled classes (SLOCC orbits) in the Hilbert space $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^n$ $(n\geqslant4)$. There are nine classes divided by "onion skins" (the orbit closures). The pictures for $2\times2\times n$ $(n>4)$ cases are essentially same although the dimensions of SLOCC orbits are different. These classes merge into four classes, divided by the skins of the solid line, in the "bipartite" (AB)-C picture. Note that although noninvertible SLOCC operations generally cause the conversions inside the onion structure, an outer class cannot necessarily convert into its neighboring inner class (cf. Fig. 2).

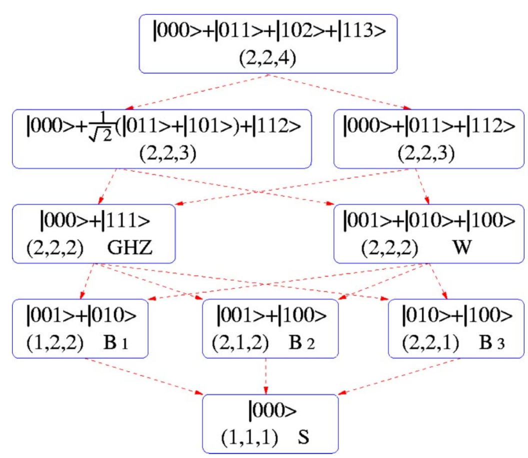

FIG. 2. The five-graded partially ordered structure of nine entangled classes in the $2\times2\times n$ $(n\geqslant4)$ case. Every class is labeled by its representative, its set of local ranks, and its name. Noninvertible SLOCC operations, indicated by dashed arrows, degrade higher entangled classes into lower entangled ones.

abundant resources, but also would be useful in analyzing entanglement of two-qubit mixed states attached with an environment (the rest of the world), which could, e.g., be used to analyze the power of an eavesdropper in quantum cryptography.

Let us consider the situation where Alice and Bob are considered as one party (or, one of Alice and Bob comes to have two qubits) and call it the "bipartite" (AB)-C picture. When two parties have two qubits for each, the onionlike structure of Fig. 1 becomes coarser. The nine entangled classes merge into four classes, and the structure coincides with that of the bipartite 4×4 case. We see that we can perform LOCC operations more freely in the bipartite situation. Likewise, in the bipartite A-(BC) or B-(AC) pictures, the onionlike structure coincides with that of the 2×8 (i.e., 2×2) case so that just two entangled classes, divided by the onion skin of B_1 or B_2, respectively, remain.

On the other hand, it can be said that the SLOCC-invariant onion structure of the $2\times2\times4$ case is a coarse-grained one of the 4-qubit $(2\times2\times2\times2)$ case (see also Refs. [9,11,14]), i.e., the former is embedded into the latter in the same way as the structure of the bipartite 4×4 case is embedded into that of the $2\times2\times4$ case. So, if two four-qubit states belong to different classes in the $2\times2\times4$ classification, these states must be also different in the four-qubit classification. It would be interesting to note that the four-qubit entangled states are divided into infinitely many classes [4,9,5], in comparison with finitely many classes of the $2\times2\times4$ case. In other words, there are infinitely many orbits in the 4-qubit case between some onion skins, while there exists one orbit in the $2\times2\times4$ case. This suggests that a drastic change occurs in the structure of multipartite entanglement even when a party comes to have two qubits in hands [15].

Now, we give the proof of Theorem 1 in two different, algebraic (in Sec. II A) and geometric (in Sec. II B), ways. Readers who are interested just in applying our results can skip to Sec. II C, where a convenient criterion for distinguishing nine classes is given.

Proof. We first give an algebraic proof, utilizing the matrix analysis (cf. Refs. [6–9]). Any state is parametrized by a three-index tensor $\psi_{i_1 i_2 i_3}$ with $i_1, i_2 \in \{0,1\}$ and $i_3 \in \{0,1,2,3\}$. This tensor can be rewritten as a 4×4 matrix $\tilde{\Psi} = (\psi_{(i_1 i_2) i_3})$ by concatenating the indices (i_1, i_2). Next we define the matrix R as

$$R = T\tilde{\Psi}, \tag{3}$$

where T is defined as

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \end{pmatrix}. \tag{4}$$

Let us observe that both 2×2 matrices M_1 and M_2 belong to $\mathrm{SL}(2,\mathbb{C})$ if and only if $O = T(M_1 \otimes M_2)T^\dagger \in \mathrm{SO}(4,\mathbb{C})$ and $\det(M_1) = \det(M_2) = 1$, because of a consequence of an accident in the Lie group theory: $\mathrm{SL}(2,\mathbb{C}) \otimes \mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{SO}(4,\mathbb{C})$ [cf. $\mathrm{SU}(2)\otimes\mathrm{SU}(2) \simeq \mathrm{SO}(4)$]. Accordingly, we see that a SLOCC transformation of Eq. (2) results in a transformation

$$R' = ORM_3^T. \tag{5}$$

Thus, our problem is equivalent to finding appropriate normal forms for the complex 4×4 matrix R under left multiplication with a complex orthogonal matrix $O \in \mathrm{SO}(4,\mathbb{C})$ and right multiplication with an arbitrary 4×4 matrix $M_3^T \in \mathrm{SL}(4,\mathbb{C})$.

If the matrix R has full rank, it is enough to operate M_3 chosen to be $T^\dagger (R^{-1})^T$. As a result, the state $\tilde{\Psi}$ is (proportional to) the identity matrix $\mathbb{1}$, or

$$|000\rangle + |011\rangle + |102\rangle + |113\rangle, \tag{6}$$

the representative of the highest class in the hierarchy.

Suppose however that the rank of R is three. As a first step, R can always be left multiplied by a permutation matrix and right multiplied by M_3^T so as to yield an R of the form

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha & \beta & \gamma & 0 \end{pmatrix}. \tag{7}$$

Suppose $\alpha \neq \pm i$, then it can easily be checked that left multiplication by the complex orthogonal matrix

$$O = \begin{pmatrix} 1/\sqrt{\alpha^2+1} & 0 & 0 & \alpha/\sqrt{\alpha^2+1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\alpha/\sqrt{\alpha^2+1} & 0 & 0 & 1/\sqrt{\alpha^2+1} \end{pmatrix} \tag{8}$$

and right multiplication with

$$M_3^T = \begin{pmatrix} 1 & -\alpha\beta/(\alpha^2+1) & -\alpha\gamma/(\alpha^2+1) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

yield a new R of the form

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \beta' & \gamma' & 0 \end{pmatrix}. \tag{10}$$

Exactly the same can be done in the case where $\beta, \gamma \neq \pm i$, and therefore we only have to consider the case where $\alpha, \beta, \gamma \in \{0, i, -i\}$. It can however be checked that in the case that when 2 or 3 elements α, β, γ are not equal to zero, a new

R can be made where all α,β,γ become equal to zero: this can be done by first multiplying R with orthogonal matrices of the kind

$$O=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1/\sqrt{2}&-1/\sqrt{2}\\0&0&1/\sqrt{2}&1/\sqrt{2}\end{pmatrix},\tag{11}$$

and repeating the procedure outlined above. There remains the case where exactly one of the elements is equal to $\pm i$. Without loss of generality, we assume that $(\alpha,\beta,\gamma)=(i,0,0)$ [this is possible because one can do permutations (with signs) by appropriate $O\in SO(4)$ and M_3]. This case is fundamentally different from the one where all α,β,γ are equal to zero as the corresponding matrix R^TR has rank 2 as opposed to rank 3 of R. There is no way in which this behavior can be changed by left and right multiplying R with appropriate transformations, and we therefore have identified a second class (which is clearly of measure zero: a generic rank-3 state R will also yield a rank-3 R^TR).

It is now straightforward to construct a representative state of each class. As a representative of the major class in the rank-3 R, we choose the state

$$|000\rangle+\frac{1}{\sqrt{2}}(|011\rangle+|101\rangle)+|112\rangle.\tag{12}$$

As a representative of the minor class in the rank-3 R, we choose the state

$$|000\rangle+|011\rangle+|112\rangle,\tag{13}$$

as it makes clear that the states in this class can be transformed to have three terms in some product basis (as opposed to the states in the major class that can be transformed to have four product terms).

The case where R has rank 2 can be solved in a completely analogous way. Exactly the same reasoning leads to the following four possible normal forms for R:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\i&0&0&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&i&0&0\\i&0&0&0\end{pmatrix},\quad\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}.\tag{14}$$

Note that the last two cases cannot be transformed into each other due to the constraint that O has determinant $+1$. The corresponding representative states are easily obtained by choosing symmetric ones:

$$|000\rangle+|111\rangle,\tag{15}$$

$$|001\rangle+|010\rangle+|100\rangle,\tag{16}$$

$$|000\rangle+|011\rangle,\tag{17}$$

$$|000\rangle+|101\rangle.\tag{18}$$

The first state is the celebrated Greenberger-Horne-Zeilinger (GHZ) state, the second one the W state named in Ref. [4] for the three-qubit case, and the remaining ones represent biseparable $B_i(i=1,2)$ states with only bipartite entanglement between Bob and Clare, or Alice and Clare, respectively.

As a last class, we have to consider the one where R has rank equal to 1. This leads to the following two possible normal forms for R:

$$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\i&0&0&0\end{pmatrix}.\tag{19}$$

The corresponding states are given by

$$|000\rangle+|110\rangle,\tag{20}$$

$$|000\rangle,\tag{21}$$

which are the biseparable B_3 state and the completely separable S state, respectively. This ends the complete classification.

It remains to be proven that any state that is higher in the hierarchy of Fig. 2 can be transformed to all the other ones that are strictly lower. The first step downwards is evident from the fact that right multiplication of a rank-4 R with a rank deficient M_3 can yield whatever R of rank-3. In going from a rank 3 R of the major class to a rank-2 one, the state $|000\rangle+(|011\rangle+|101\rangle)/\sqrt{2}+|112\rangle$ can be transformed into the GHZ state by a projection of Clare on the subspace $\{|0\rangle,|2\rangle\}$ and into the W state by Clare implementing the positive operator-valued measure (POVM) element

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&i&0&0\\0&0&0&0\end{pmatrix}.\tag{22}$$

From a rank-3 R of the minor class, the GHZ state can easily be constructed by a projection of Clare on her $\{|1\rangle,|2\rangle\}$ subspace, while the W state is obtained by Clare projecting on her $\{|0\rangle,|1\rangle+|2\rangle\}$ subspace. Finally, the conversion of the GHZ and W states to the Bell state among two parties (the biseparable state), as well as that of the Bell state to the completely separable state, is straightforward.

The proof not only gives a constructive transformations to representatives of nine entangled classes, but also suggests a very simple way of determining to which class a given state belongs. One has to calculate the rank $r(\cdot)$ of the matrices R

[see Eq. (3)], of R^TR, and of the reduced density matrix ρ_1. One gets the following classification:

Class	$r(R)$	$r(R^TR)$	$r(\rho_1)$
$\vert 000\rangle+\vert 011\rangle+\vert 102\rangle+\vert 113\rangle$	4	4	2
$\vert 000\rangle+\frac{1}{\sqrt{2}}(\vert 011\rangle+\vert 101\rangle)+\vert 112\rangle$	3	3	2
$\vert 000\rangle+\vert 011\rangle+\vert 112\rangle$	3	2	2
$\vert 000\rangle+\vert 111\rangle$	2	2	2
$\vert 001\rangle+\vert 010\rangle+\vert 100\rangle$	2	1	2
$\vert 000\rangle+\vert 101\rangle$	2	0	2
$\vert 000\rangle+\vert 011\rangle$	2	0	1
$\vert 000\rangle+\vert 110\rangle$	1	1	2
$\vert 000\rangle$	1	0	1

(23)

Note that the representative states in the GHZ-type classes were chosen to be the ones with *maximal* entanglement: following Ref. [6], the states with maximal entanglement in a SLOCC class are the ones for which all local-density operators are proportional to the maximally mixed state. This is in accordance with the intuition that the local disorder or entropy is proportional to the entanglement present in the (pure) state.

B. Geometry of nine entangled classes

We explore how the whole Hilbert space is geometrically divided into different nine classes, drawn in the onionlike picture Fig. 1. This section can be seen as an alternative proof of the theorem in Sec. II A by a geometric way.

We utilize a duality between the set of separable states and the set of entangled states in order to classify multipartite entangled states under SLOCC [5]. The set S of completely separable states is the smallest closed subset, as seen in Fig. 1. In many cases (such as the l-qubit cases) of interest to quantum information, its dual set is the largest closed subset which consists of all degenerate entangled states and is given by the zero hyperdeterminant $\mathrm{Det}\Psi=0$. We readily see that, in the bipartite $k\times k$ case, the set S is the smallest subset of the Schmidt rank 1, while its dual set is the largest subset where the Schmidt rank is not full (i.e., $\det\Psi=0$).

However, the entangled states in $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^n$ $(n\geqslant4)$ have a peculiar structure from a geometric viewpoint. It is not the case here that the largest subset is dual to the smallest subset S. Indeed, the largest subset is dual to (the closure of) the set B_3 of the biseparable states, i.e., the second smallest closed subset of dimension 6 in Fig. 1. The dual set of S is the second largest subset of dimension 13. The reason will be explained later. Significantly, this suggests that for the $2\times2\times n$ $(n\geqslant4)$ cases, there are no hyperdeterminants in the Gelfand *et al.*'s sense; in other words, the onion structure will not change anymore for $n\geqslant4$. This is intuitively because the subsystem of one party is too large, compared with the subsystems of the other parties. Remember that it is again an analogy to the bipartite $k\times k'$ case $(k<k')$, where there is no determinant but its onion structure remains unchanged from that of the $k\times k$ case.

In general, the hyperdeterminants can be defined for $\mathcal{H}=\mathbb{C}^{k_1}\otimes\cdots\otimes\mathbb{C}^{k_l}$, if and only if

$$k_i-1\leqslant\sum_{j\neq i}(k_j-1)\ \forall\ i=1,\ldots,l \tag{24}$$

are satisfied [5,16]. Of course, in the bipartite cases, this condition suggests that the determinants can be defined just for square $(k_1=k_2)$ matrices as usual. Instead, in the $2\times2\times4$ case, the zero locus of the ordinary determinant of degree 4 for the "flattened" matrix $\tilde{\Psi}$,

$$\begin{vmatrix}\psi_{000}&\psi_{001}&\psi_{002}&\psi_{003}\\ \psi_{010}&\psi_{011}&\psi_{012}&\psi_{013}\\ \psi_{100}&\psi_{101}&\psi_{102}&\psi_{103}\\ \psi_{110}&\psi_{111}&\psi_{112}&\psi_{113}\end{vmatrix}(=\det\tilde{\Psi}), \tag{25}$$

gives the equation of the largest closed subset. Note that it is the SLOCC invariant for the bipartite 4×4 format as well as the tripartite $2\times2\times4$ format. It means that the largest subset is dual to the set B_3 of the biseparable states, i.e., the set of the separable states in the "bipartite" (AB)-C picture. We should stress that this duality itself is valid in any $2\times2\times n(n\geqslant4)$ case, regardless of the absence of the (hyper)determinant.

Next, let us show that the dual set of S is the second largest subset for the $2\times2\times4$ case. In order to decide the dual set of S, we seek the state $|\Psi\rangle$ included in the hyperplane (the orthogonal one-codimensional subspace) tangent at a completely separable state $|x\rangle$ (see Ref. [5] in detail). Mathematically speaking, we should decide the condition for $|\Psi\rangle$ such that a set of equations

$$\begin{gathered}F(\Psi,x)=\sum_{i_1,i_2,i_3=0}^{1,1,3}\psi_{i_1i_2i_3}x^{(1)}_{i_1}x^{(2)}_{i_2}x^{(3)}_{i_3}=0,\\ \frac{\partial}{\partial x^{(j)}_{i_j}}F(\Psi,x)=0\quad\forall\ \ j,i_j\end{gathered} \tag{26}$$

has at least a nontrivial solution $x=(x^{(1)},x^{(2)},x^{(3)})$ of every $x^{(j)}\neq0$. For simplicity, let us suppose that the point of tangency is the completely separable state $|000\rangle$ (i.e., $x^{(1)}_0=x^{(2)}_0=x^{(3)}_0=1$, others$=0$), the corresponding state $|\Psi\rangle$ should satisfy

$$|\Psi\rangle\in\{\psi_{000}=\psi_{100}=\psi_{010}=\psi_{001}=\psi_{002}=\psi_{003}=0\}, \tag{27}$$

according to Eq. (26). We find that the state $|\Psi\rangle$ should belong to the class of dimension 13, because any state,

$$\begin{aligned}|\Psi\rangle=&\psi_{011}|011\rangle+\psi_{012}|012\rangle+\psi_{013}|013\rangle+\psi_{101}|101\rangle\\&+\psi_{102}|102\rangle+\psi_{103}|103\rangle+\psi_{110}|110\rangle+\psi_{111}|111\rangle\\&+\psi_{112}|112\rangle+\psi_{113}|113\rangle,\end{aligned} \tag{28}$$

in Eq. (27) can convert to its representative $|011\rangle+|102\rangle+|113\rangle$ under invertible SLOCC operations.

In brief, we find that the 14-dimensional largest subset is the dual set of the biseparable states B_3, and the 13-dimensional second largest subset is the dual set of the completely separable states S. Moreover, we note that the inside of the largest subset, given by zero locus of Eq. (25), is equivalent to the structure of the $2\times2\times3$ case (since the local rank for Clare should be less than or equal to 3), which has already been clarified in Ref. [5]. That is how we obtain the onionlike picture of Fig. 1. In general, we can take advantage of all kinds of the dual pairs for sets (typically, one is a large set and the other is a small set), in order to distinguish inequivalent entangled classes. This strategy will be explored elsewhere [17].

C. Convenient criterion to distinguish nine entangled classes

We give a convenient criterion to distinguish nine entangled classes by a complete set of SLOCC invariants. Let us denote local ranks of the reduced density matrices ρ_1, ρ_2, and ρ_3 such as

$$\rho_i=\mathrm{tr}_{\forall j\neq i}(|\Psi\rangle\langle\Psi|),\quad i=1,2,3, \tag{29}$$

by the 3-tuples (r_1,r_2,r_3). These local ranks are always useful SLOCC invariants. In the bipartite setting, the 2-tuples (r_1,r_2) are enough to distinguish entangled classes, for both r_1 and r_2 are indeed nothing but the Schmidt rank. In the multipartite setting, however, we need more SLOCC invariants in addition to the set of the local ranks.

The proof of Theorem 1 in Sec. II A has suggested that a complete set of SLOCC invariants is the rank of R in Eq. (3) (i.e., r_3), rank of R^TR, and r_1 (alternatively, r_2). Although we have successfully found the rank of R^TR as an additional SLOCC invariant, this is specific to the substructure associated with 2 qubits, i.e., to a homomorphism $SL(2,\mathbb{C})\otimes SL(2,\mathbb{C})\simeq SO(4,\mathbb{C})$.

In the following, we introduce another complete set of SLOCC invariants, since it also gives an insight about how entanglement measures, distinguishing entangled classes, are derived in general. The set consists of polynomial invariants (hyperdeterminants [5,16]) adjusted to smaller formats, as well as 3-tuples (r_1,r_2,r_3) of the local ranks. The criterion reflects the onion structure drawn in Fig. 1, and suggests that we can utilize the results of the SLOCC classification for smaller formats recursively as if we were skinning the onion recursively.

Any pure state in $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^n$ is written in the form

$$|\Psi\rangle=\sum_{i_1,i_2,i_3=0}^{1,1,n-1}\psi_{i_1i_2i_3}|i_1\rangle\otimes|i_2\rangle\otimes|i_3\rangle. \tag{30}$$

First we calculate a set (r_1,r_2,r_3) of the SLOCC-invariant local ranks of the reduced density matrices.

(i) In the (2,2,4) case, we find that the state $|\Psi\rangle$ belongs to the generic class of dimension 15 (the dimension is indicated for readers' convenience, but it is the one for the $2\times2\times4$ case.).

(ii) In the (2,2,3) case, there are two possibilities. Changing the local basis for Clare, we can always choose all new $\psi_{i_1i_2i_3}=0(i_3\geqslant3)$. We evaluate the hyperdeterminant of degree 6 for the new, $2\times2\times3$ formatted $\psi_{i_1i_2i_3}$,

$$\mathrm{Det}\Psi_{2\times2\times3}=\begin{vmatrix}\psi_{000}&\psi_{001}&\psi_{002}\\ \psi_{010}&\psi_{011}&\psi_{012}\\ \psi_{100}&\psi_{101}&\psi_{102}\end{vmatrix}\begin{vmatrix}\psi_{010}&\psi_{011}&\psi_{012}\\ \psi_{100}&\psi_{101}&\psi_{102}\\ \psi_{110}&\psi_{111}&\psi_{112}\end{vmatrix}-\begin{vmatrix}\psi_{000}&\psi_{001}&\psi_{002}\\ \psi_{010}&\psi_{011}&\psi_{012}\\ \psi_{110}&\psi_{111}&\psi_{112}\end{vmatrix}\begin{vmatrix}\psi_{000}&\psi_{001}&\psi_{002}\\ \psi_{100}&\psi_{101}&\psi_{102}\\ \psi_{110}&\psi_{111}&\psi_{112}\end{vmatrix}. \tag{31}$$

If $\mathrm{Det}\Psi_{2\times2\times3}\neq0$, then $|\Psi\rangle$ belongs to the major class of dimension 14. Otherwise (i.e., $\mathrm{Det}\Psi_{2\times2\times3}=0$), it belongs to the minor class of dimension 13.

(iii) In the (2,2,2) case, there are also two possibilities. Changing the local basis for Clare, we can always choose all new $\psi_{i_1i_2i_3}=0(i_3\geqslant2)$. We evaluate the hyperdeterminant of degree 4 (its absolute value is also known as the 3-tangle [3]) for the $2\times2\times2$ formatted $\psi_{i_1i_2i_3}$,

$$\begin{aligned}\mathrm{Det}\Psi_{2\times2\times2}=&\,\psi_{000}^2\psi_{111}^2+\psi_{001}^2\psi_{110}^2+\psi_{010}^2\psi_{101}^2+\psi_{100}^2\psi_{011}^2\\&-2(\psi_{000}\psi_{001}\psi_{110}\psi_{111}+\psi_{000}\psi_{010}\psi_{101}\psi_{111}\\&+\psi_{000}\psi_{100}\psi_{011}\psi_{111}+\psi_{001}\psi_{010}\psi_{101}\psi_{110}\\&+\psi_{001}\psi_{100}\psi_{011}\psi_{110}+\psi_{010}\psi_{100}\psi_{011}\psi_{101})\\&+4(\psi_{000}\psi_{011}\psi_{101}\psi_{110}+\psi_{001}\psi_{010}\psi_{100}\psi_{111}).\end{aligned} \tag{32}$$

Likewise, if $\mathrm{Det}\Psi_{2\times2\times2}\neq0$, then $|\Psi\rangle$ belongs to the GHZ class of dimension 11. Otherwise, it belongs to the W class of dimension 10.

(iv) In the (1,2,2), (2,1,2), and (2,2,1) cases, $|\Psi\rangle$ belongs to the biseparable B_1, B_2, and B_3 class of dimension 8, 8, and 6, respectively.

(v) In the (1,1,1) case, $|\Psi\rangle$ belongs to the completely separable class S of dimension 5.

In this manner, we can immediately check which class a given state $|\Psi\rangle$ belongs to. We remark that the representatives of nine entangled classes in previous subsections have been chosen with the help of hyperdeterminants; the "GHZ-like" representatives are chosen to maximize the absolute value of (hyper)determinants in Eqs. (25), (31), and (32), which are entanglement monotones under general LOCC [5,6] (cf. Refs. [18,19]).

III. CHARACTERISTICS OF MULTIPARTITE ENTANGLEMENT

A. LOCC protocols as noninvertible flows

The recent trend of experimental quantum optics reaches the stage that we can manipulate two Bell states collectively.

LOCC protocols involving local collective operations over two Bell states are key procedures in, for example, entanglement swapping [20,21] (a building block of quantum communication protocols such as quantum teleportation [22] and the quantum repeater [23]) and the creation of multipartite GHZ and *W* states. Although there appear four particles (qubits), these can be seen as LOCC operations in three parties ($\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^4$) because the third party Clare has initially two particles, each of which is in a Bell state with another particle on Alice's or Bob's side, respectively, and locally performs collective operations on them.

Entanglement swapping is the LOCC protocol where the initial state is prepared as two Bell pairs shared among Alice, Bob, and Clare in the manner described above. We note that two Bell pairs are equivalent to the representative of the generic entangled class of dimension 15,

$$|2\text{Bell}\rangle=(|00\rangle+|11\rangle)_{AC_1}\otimes(|00\rangle+|11\rangle)_{BC_2}$$
$$=|00(00)\rangle+|01(01)\rangle+|10(10)\rangle+|11(11)\rangle_{ABC_{12}}. \tag{33}$$

$|2\text{ Bell}\rangle$ is also equivalent to $\Sigma_{i=0}^{3}|\Phi_i\rangle_{AB}\otimes|\Phi_i\rangle_{C_{12}}$, where a set of $|\Phi_i\rangle$ ($i=0,1,2,3$) is the standard Bell basis. So, this protocol can create the biseparable B_3 state which contains maximal entanglement (a Bell pair) between Alice and Bob,

$$(|00\rangle+|11\rangle)_{AB}\otimes(|(00)\rangle+|(11)\rangle)_{C_{12}}, \tag{34}$$

by Clare's local collective Bell measurement [any $|\Phi_i\rangle_{AB}$ corresponding to the outcome *i* of her Bell measurement is equivalent to $(|00\rangle+|11\rangle)_{AB}$ under LOCC]. Thus, entanglement swapping can be seen as a protocol creating isolated (maximal) entanglement between Alice and Bob from generic entanglement. In other words, it is given by a downward flow in Fig. 2 from the generic class to the biseparable class B_3. Now, we readily find that the entanglement swapping protocol is (probabilistically) successful even when we initially prepare other four-qubit entangled states in the generic class.

On the other hand, two Bell pairs can create two different kinds of genuine three-qubit entanglement, GHZ and W by Clare's local collective operations. These LOCC protocols are given by the downward flow, in Fig. 2, from the generic classes to the GHZ and *W* class, respectively.

That is how we see that important LOCC protocols in quantum information are given as noninvertible (downward) flows in the partially ordered structure, such as Fig. 2, of multipartite entangled classes. So, we expect that the SLOCC classification can give us an insight in looking for LOCC protocols by means of several entangled states over multiparties.

B. Two Bell pairs create any state with certainty

We show that two Bell pairs are powerful enough to create any state *with certainty* in our $2\times2\times n$ cases. We find that this is also the case when one of the multiparties has half of the total Hilbert space.

Theorem 2. Consider pure states in the Hilbert space $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^n$. Two Bell pairs, the representative of the generic class, can create any state $|\Psi\rangle$ with probability 1 by means of a local POVM measurement M_i on Clare followed by local unitary operations $U_A(i)$ and $U_B(i)$ on Alice and Bob, respectively.

Proof. We prove that we can always choose a local POVM M_i on Clare, local unitary operations $U_A(i)$ and $U_B(i)$ on Alice and Bob (depending on the outcome *i* of the POVM M_i), such that

$$|\Psi\rangle=U_A(i)\otimes U_B(i)\otimes M_i$$
$$\times(|000\rangle+|011\rangle+|102\rangle+|113\rangle)\quad\forall\ i, \tag{35}$$

where $\Sigma_i M_i^\dagger M_i=\mathbb{1}$. In terms of the "flattened" matrix form $\tilde{\Psi}$ where the indices (i_1,i_2) are concatenated, Eq. (35) is rewritten as

$$\tilde{\Psi}=[U_A(i)\otimes U_B(i)]\mathbb{1}M_i^T\quad\forall\ i. \tag{36}$$

By choosing $M_i^T=(M_i^*)^\dagger=[U_A(i)\otimes U_B(i)]^\dagger\tilde{\Psi}$, it should be satisfied that

$$\mathbb{1}=\sum_i (M_i^*)^\dagger M_i^*$$
$$=\sum_i [U_A(i)\otimes U_B(i)]^\dagger\tilde{\Psi}\tilde{\Psi}^\dagger[U_A(i)\otimes U_B(i)]. \tag{37}$$

Such a local POVM M_i always exists, because we can depolarize any $\tilde{\Psi}\tilde{\Psi}^\dagger$ to the identity $\mathbb{1}$ by random local unitary operations $U_A(i)\otimes U_B(i)$ on Alice and Bob [24,25]. This randomization can be alternatively achieved by applying a set of 16 local unitary operations $\sigma_A^\mu\otimes\sigma_B^\nu$ with equal probabilities, where σ^μ and σ^ν ($\mu,\nu=0,1,2,3$) are the Pauli matrices. This completes the proof. ■

Theorem 3. Consider *l*-partite pure states in the Hilbert space $\mathcal{H}=\mathbb{C}^{k_1}\otimes\mathbb{C}^{k_2}\otimes\cdots\otimes\mathbb{C}^{k_{l-1}}\otimes\mathbb{C}^{k_1\times k_2\times\cdots\times k_{l-1}}$, the maximally entangled state, which is the $(k_1\times\cdots\times k_{l-1})\times(k_1\times\cdots\times k_{l-1})$ identity matrix $\mathbb{1}$ in concatenating the indices $(i_1,\ldots,i_{l-1})$, can create any state with probability 1 by means of a local POVM on the *l*th party followed by local unitary operations on the rest of the parties.

Proof. The generalization of the proof in the $2\times2\times n$ case is straightforward. ■

These theorems suggest that when one of multiparties holds at least half of the total Hilbert space, the situation is somehow analogous to the bipartite cases. The maximally entangled state, i.e., the representative of the generic class, can create any state with certainty.

IV. EXTENSION TO MIXED STATES

In this section, we extend the onionlike SLOCC classification of pure states in Sec. II to mixed states.

A multipartite mixed state ρ can be written as a convex combination of projectors onto pure states (extremal points),

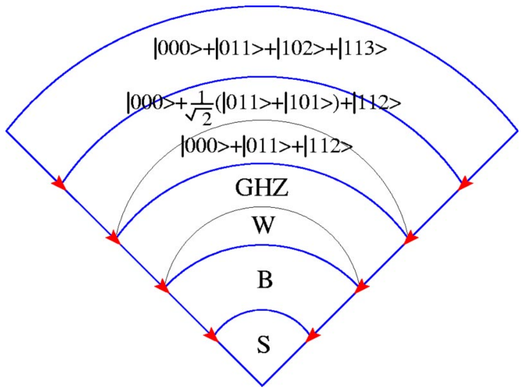

FIG. 3. The SLOCC classification of multipartite mixed states in the $2\times2\times n$ $(n\geqslant4)$ cases. Mixed states in the class, labeled by $|\Psi(\mathcal{O}_{\max})\rangle\in\{|000\rangle+|011\rangle+|102\rangle+|113\rangle,\ldots,S\}$, are convex combinations of pure states *inside* the "onion skin" of $|\Psi(\mathcal{O}_{\max})\rangle$ in Fig. 1. So the outer the class is, the more the kinds of multipartite entangled pure states the mixed states contain. The edges of the "fan" reflect the structure of extremal points (pure states), and non-invertible SLOCC operations can never upgrade an inner class to its outer classes.

$$\rho=\sum_i p_i|\Psi_i(\mathcal{O}_i)\rangle\langle\Psi_i(\mathcal{O}_i)|,\quad p_i>0, \tag{38}$$

where each pure state $|\Psi_i(\mathcal{O}_i)\rangle$ belongs to one of the SLOCC entangled classes (i.e., an SLOCC orbit $\mathcal{O}_i$). Our idea is to discuss, in Eq. (38), how ρ needs at least an outer entangled class $\mathcal{O}_{\max}$, among the set $\{\mathcal{O}_i\}$, in the onion structure of Fig. 1. That is, we are interested in the minimum of $\mathcal{O}_{\max}$ for all possible decomposition of ρ. Because the onion picture is divided by every SLOCC-invariant closed subset (i.e., every SLOCC orbit closure) of pure states, their convex combination in Eq. (38) constitutes the SLOCC-invariant closed convex subsets of mixed states (see Fig. 3). Note that, in the onion picture of the multipartite pure cases, there can be "competitive" closed subsets which never contain nor are contained by each other. An example is the closures of three biseparable classes B_i in Fig. 1. So, in the extension to mixed states, we should assemble all subsets of mixed states which require at most these biseparable classes B_i into one biseparable convex subset by their convex hull. (The argument is similar to the classification of three-qubit mixed states in Ref. [26].)

We find that these entangled classes constitute a totally ordered structure, seen in Fig. 3, where noninvertible SLOCC operations can never upgrade an inner class to its outer classes. For instance, we see that the closure of W_3 class of mixed states (labeled by $|000\rangle+|011\rangle+|112\rangle$) is included in the closure of GHZ_3 class (labeled by $|000\rangle+(1/\sqrt{2})(|011\rangle+|101\rangle)+|112\rangle$). This classification reflects a diversity of multipartite pure entangled states a mixed state ρ consists of: the outer the class of ρ is, the more the kinds of resources it contains. Needless to say, it is very difficult to give the criterion to distinguish convex subsets, even to distinguish the separable convex subset (i.e., the separability problem), since we face trouble evaluating all possible decompositions in Eq. (38) for a given ρ. Let us however prove that the convex combination of nine classes of pure states gives rise to convex sets that are not of measure zero, in contrast with the pure case (cf. Ref. [26]). This can easily be established with the help of the following lemma:

Lemma 1. Given two matrices A,B with corresponding ordered singular values $\{\sigma_i^{A,B}\}$, denote the ordered singular values of the matrices A^TA and B^TB as $\{\tau_i^{A,B}\}$. Then the Hilbert-Schmidt norm

$$\|A-B\|_2=\sqrt{\mathrm{tr}[(A-B)^\dagger(A-B)]}$$

is lower bounded by

$$\|A-B\|_2\geqslant\sqrt{\sum_i(\sigma_i^A-\sigma_i^B)^2},$$

$$\|A-B\|_2\geqslant\frac{\|A\|_2}{2(1+\|A\|_2)}\sqrt{\sum_i(\tau_i^A-\tau_i^B)^2},$$

where we assumed that $\|A\|_2\geqslant\|B\|_2$.

Proof. The first inequality can readily be proven using standard results of linear algebra [27]. The second inequality can be proven as follows. Define $X=A-B$; then

$$\|A^TA-B^TB\|=\|XA^T+AX^T-X^TX\|\leqslant2\|X\|\|A\|+\|X\|^2. \tag{39}$$

The left term of this inequality is bounded below by

$$\|A^TA-B^TB\|\geqslant\sqrt{\sum_i(\tau_i^A-\tau_i^B)^2}. \tag{40}$$

The second inequality of the lemma can now be checked by making use of straightforward algebra. ■

The fact that a structure of convex sets as depicted in Fig. 3 is obtained can now be proven by combining the previous lemma with the results of the table in Eq. (23): indeed, it can easily be shown that whenever there exists a pure state in one class that is separated from all pure states in another class with a finite nonzero Hilbert-Schmidt distance, then the corresponding class for mixed states is absolutely separated from the other one. The previous lemma guarantees that the Hilbert-Schmidt norm will be nonzero for all states having a different rank for the matrices R or R^TR [see the table in Eq. (23)]. More specifically, all the W classes are embedded in the respective GHZ classes, and the convex structure as depicted in Fig. 3 is obtained.

V. CONCLUSION

In this paper, we have the following.

(i) We give the complete classification of multipartite entangled states in the Hilbert space $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^n$ under stochastic local operations and classical communication (SLOCC). Our study can be seen as the first example of the SLOCC classification of multipartite entanglement where one of multiparties has more than one qubits. We show that nine classes constitute the five-graded partially ordered structure of Fig. 2. Remarkably, a unique maximally entangled class lies on its top, in contrast with the l-qubit ($l\geqslant3$) cases. We also present a convenient criterion to distinguish these classes by SLOCC-invariant entanglement measures.

(ii) We illustrate that important LOCC protocols in quantum information processing are given as noninvertible (downward) flows between different entangled classes in the partially ordered structure of Fig. 2. In particular, we show that two Bell pairs are powerful enough to create any state with certainty in our situation. Based on these observations, we suggest that SLOCC classifications can be useful in looking for new prototypes of LOCC protocols.

ACKNOWLEDGMENTS

A.M. would like to express his sincere gratitude to J.I. Cirac, M. Lewenstein, and the members of these groups for stimulating discussions and warm hospitality during his visit. He also thanks K. Matsumoto for the discussions on Sec. II B, as well as the members of both the University of Tokyo and the ERATO Project for helpful comments. A.M. acknowledges financial support by JSPS. The work of F.V. was supported in part by the E.C. (projects RESQ and QUPRODIS) and the Kompetenznetzwerk ''Quanteninformationsverarbeitung'' der Bayerischen Staatsregierung.

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[14] Industrious readers might wonder whether there is any relationship between nine classes in this paper and those in Ref. [9], which addressed the SLOCC classification of four-qubit entangled pure states. But, this is just an accident, and there is no correspondence.
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