Many-body entanglement in decoherence processes

Helen McAneney, Jinhyoung Lee, and M. S. Kim
School of Mathematics and Physics, Queen's University, Belfast BT7 1NN, United Kingdom
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A pure state decoheres into a mixed state as it entangles with an environment. When an entangled two-mode system is embedded in a thermal environment, however, each mode may not be entangled with its environment by their simple linear interaction. We consider an exactly solvable model to study the dynamics of a total system, which is composed of an entangled two-mode system and a thermal environment. The Markovian interaction with the environment is concerned with an array of infinite number of beam splitters. It is shown that many-body entanglement of the system and the environment may play a crucial role in the process of disentangling the system.

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I. INTRODUCTION

Decoherence has been studied in the context of quantum-classical correspondence, providing a quantum-to-classical transition of a system [1]. A single-mode pure state becomes mixed and loses its quantum nature by decoherence in an environment. Although the dynamics of the system has been studied extensively, there has not been a thorough investigation on the quantum correlation between the system and the environment, which is behind the dynamics of the system. The decoherence process can be understood as a process of entanglement between the system and its environment which is composed of a many (normally, infinite) number of independent modes. The increase of the system entropy may be due to the system-environment entanglement [2,3].

Most of the studies on decoherence have focused on a single-mode or single-particle system [4]. This is because if a many-body pure system is initially separable, its decoherence process is a straightforward extension of a single-body system. However, if there is entanglement in the initial pure system, the decoherence mechanism can be of a different nature. For an entangled two-mode pure system, each mode is generically in a mixed state and its passive linear interaction with an environment, which is normally in a mixed state, does not seem to bring about entanglement between the environment and its interacting mode. What kind of correlation then causes the loss of entanglement initially in the system? In this paper, we answer this question by studying the quantum correlation of a two-mode entangled continuous-variable system with an environment in thermal equilibrium.

A continuous-variable state is defined in an infinite-dimensional Hilbert space and it is convenient to study such a state using its quasiprobability Wigner function [5] $W(\tilde{\mathbf{x}})$ in phase space. For an N-mode field, the coordinates of phase space are composed of quadrature variables, $\tilde{\mathbf{x}} = \{q_1, p_1, \ldots, q_N, p_N\}$. Throughout the paper a vector is denoted in bold face and an operator by a hat. A fiber or a free space, through which a light field propagates, is normally considered a thermal environment. The dynamics of the field mode coupled to the thermal environment is, in the Born-Markov approximation, governed by the Fokker-Planck equation [5]

$$\frac{\partial W(\tilde{\mathbf{x}})}{\partial \tau} = \frac{\gamma}{2} \sum_{i=1}^{2N} \left(\frac{\partial}{\partial \tilde{x}_i} \tilde{x}_i + \frac{\tilde{n}}{4} \frac{\partial^2}{\partial \tilde{x}_i^2} \right) W(\tilde{\mathbf{x}}), \tag{1}$$

where γ is the energy decay rate of the system and $\tilde{n} = 2\overline{n} + 1$ with $\overline{n} = [\exp(\hbar\omega/k_B T) - 1]^{-1}$ is the average number of thermal photons at temperature T. k_B is the Boltzmann constant.

It was shown by one of us that a single-mode Gaussian field interacting with a thermal environment can be modeled by the field passing through an array of infinite beam splitters [6]. A beam splitter is a simple passive linear device which keeps the Gaussian nature of an input field. Each beam splitter has two input ports. As the signal field is injected into one input port, it allows a degree of freedom for the other port where noise is injected. The collection of such degrees of freedom forms the environment. We assume a homogeneous thermal environment of temperature T with all noise modes having the same physical properties. In Ref. [6], the Fokker-Planck equation (1) for a single-mode field was derived using the beam splitter. The model was used to study the dynamics of entanglement between a single-mode field and its environment [7].

Using the Fokker-Planck equation (1), one may study the dynamics of the system. However, it is hard to know the quantum correlation between the system and the environment as Eq. (1) is obtained by tracing over all environmental variables. In this paper, instead of tracing over all the environmental modes, we keep them to study the dynamics of entanglement between the system and the environment. A two-mode squeezed state, which may be generated by a nondegenerated optical parametric amplifier, is the most renowned and experimentally relevant entangled state for continuous variables [8]. Its degree of entanglement increases as the degree of squeezing s increases [9] and it becomes a regularized Einstein-Podolsky-Rosen state when $s \to \infty$ [10]. In order to simplify the problem, we assume that only the mode a_2 of the system modes interacts with the environment, while the other mode a_1 is isolated from it.

II. INTERACTION BETWEEN SYSTEM AND ENVIRONMENT

We consider an exactly solvable model of a two-mode system interacting with a homogeneous thermal environment, which results in the Fokker-Planck equation (1). For the Born-Markov approximation, we employ a time-dependent coupling constant in the model. Let us start with a finite number N of environmental modes interacting with the system. The interaction Hamiltonian, in the interaction picture, is

$$\hat{H}_I(t)=\sum_{m=0}^{N-1} i\lambda_N(t)(\hat{a}_2\hat{b}_m^\dagger-\hat{a}_2^\dagger\hat{b}_m), \tag{2}$$

where $\hat{b}_m$ is the bosonic annihilation operator for environmental mode b_m and the coupling constant $\lambda_N(t)$ is determined so as to reproduce the Fokker-Planck equation (1).

It is convenient to introduce collective modes c_n which are conjugate to b_m under the Fourier transformation such that

$$\hat{c}_n\equiv\sqrt{\frac{2}{N}}\sum_{m=0}^{N-1}\cos\left(\frac{2\pi}{N}nm\right)\hat{b}_m, \tag{3}$$

where $\hat{c}_n$ is an annihilation operator for a collective mode c_n. The collective modes are related with the entangling nature of the modes b_m, for example, the quantum that $\hat{c}_n^\dagger$ creates from a vacuum is in an entangled state of b_m modes.

The collective modes satisfy the boson commutation relation $[\hat{c}_n,\hat{c}_{n'}^\dagger]=\delta_{nn'}$ and carry physical properties as bosonic modes. Using the collective mode, a state $\hat{\rho}$ is described by the characteristic function

$$\chi_c(\mathbf{X})=\mathrm{Tr}\,\hat{\rho}\exp[i\mathbf{X}\cdot\hat{\mathbf{X}}^T], \tag{4}$$

where $\hat{\mathbf{X}}=(\hat{Q}_0,\hat{P}_0,\hat{Q}_1,\hat{P}_1,\ldots,\hat{Q}_{N-1},\hat{P}_{N-1})$ with $\hat{Q}_n=(\hat{c}_n+\hat{c}_n^\dagger)/\sqrt{2}$ and $\hat{P}_n=i(\hat{c}_n^\dagger-\hat{c}_n)/\sqrt{2}$ and $\mathbf{X}=(P_0,-Q_0,P_1,-Q_1,\ldots,P_{N-1},-Q_{N-1})$. It is straightforward to show that, for a given density operator $\hat{\rho}$, χ_c is the same as the usual characteristic function, χ_b, which is obtained in terms of modes b_m: $\chi_c(\mathbf{X})=\chi_b(\mathbf{x})$, where $\mathbf{x}$ is conjugate to $\mathbf{X}$ by Fourier transformation (3). The collective modes c_n provide a different perspective from the modes b_m, preserving all physical properties for a given state.

The time-evolution operator $\hat{U}_I(\tau)$ for the interaction Hamiltonian (2) is equivalent to a beam-splitter operator with the system mode a_2 and a collective mode c_0 as its input ports. That is,

$$\hat{U}_I(\tau)=\exp[\theta(\tau)(\hat{a}_2\hat{c}_0^\dagger-\hat{a}_2^\dagger\hat{c}_0)], \tag{5}$$

where $\theta(\tau)=\sqrt{N/2}\int_0^\tau\lambda_N(t)dt$ determines the transmittivity, $t^2(\tau)=\cos^2\theta(\tau)$. We take the limit, $N\to\infty$, keeping the transmitted energy finite, $t^2(\tau)=\exp(-\gamma\tau)$, in accordance with the Fokker-Planck equation. We find an important fact that the interaction of the system with the infinite modes b_m of the environment can be reduced into the interaction with the single collective mode c_0. The properties of collective mode c_0 changes due to the interaction but each environmental mode b_m hardly changes which is reflected in no change of other collective modes c_n.

III. CORRELATION MATRIX

A Gaussian field has the characteristic function in the form of $\chi(\mathbf{x})=\exp(-\mathbf{x}\mathbf{V}\mathbf{x}^T/4)$, where $\mathbf{V}$ is the correlation matrix whose elements determine the mean quadrature values of the field: $V_{ij}=\langle(\hat{x}_i\hat{x}_j+\hat{x}_j\hat{x}_i)\rangle$. Note that we neglected linear displacement terms in the Gaussian characteristic function as they do not play a crucial role in determining the entanglement. The entanglement nature of a Gaussian field is thus uniquely represented by its correlation matrix $\mathbf{V}$. For a two-mode squeezed state, the correlation matrix $\mathbf{V}_s$ is simply [9]

$$\mathbf{V}_s=\begin{pmatrix}\cosh(2s)\mathbb{1} & \sinh(2s)\boldsymbol{\sigma}_z\\ \sinh(2s)\boldsymbol{\sigma}_z & \cosh(2s)\mathbb{1}\end{pmatrix}, \tag{6}$$

where $\mathbb{1}$ is the 2×2 unit matrix and $\boldsymbol{\sigma}_z$ is the Pauli matrix.

As the system interacts only with the collective mode c_0 in the homogeneous thermal environment, it suffices to consider the correlation matrix of the two system modes a_1 and a_2 and the collective mode c_0. The collective mode c_0 is initially in a thermal state with the average number $\bar{n}$ of the collective bosons. Thus, the correlation matrix of a_1, a_2, and c_0 before the interaction is given by $\mathbf{V}_0=\mathbf{V}_s\oplus\tilde{n}\mathbb{1}$. The evolution operator $\hat{U}_I(\tau)$ is now described by the matrix

$$\mathbf{U}_I=\begin{pmatrix}\mathbb{1} & 0 & 0\\ 0 & t\mathbb{1} & -r\mathbb{1}\\ 0 & r\mathbb{1} & t\mathbb{1}\end{pmatrix}, \tag{7}$$

where $r^2=1-t^2$. Then the correlation matrix for the system and environment after the interaction is obtained as $\mathbf{V}_c=\mathbf{U}_I\mathbf{V}_0\mathbf{U}_I^T$.

A separability condition was derived by Simon [11] that a two-mode Gaussian state is separable if and only if the partially momentum-reversed correlation matrix (or equivalently the partially transposed density operator) satisfies the uncertainty principle. The condition was extended to a biseparability condition between a single mode and a group of N modes by Werner and Wolf [12], which reads that a Gaussian field of $1\times N$ modes is biseparable if and only if

$$\boldsymbol{\Lambda}\mathbf{V}\boldsymbol{\Lambda}-\frac{1}{2}\boldsymbol{\sigma}_y^{\oplus(N+1)}\geqslant0, \tag{8}$$

where $\boldsymbol{\Lambda}$ is a partial momentum-reversal matrix, $\mathbf{V}$ is the correlation matrix of $1\times N$ modes, and $\boldsymbol{\sigma}_y$ is the Pauli matrix. Here, $\boldsymbol{\sigma}_y^{\oplus(N+1)}\equiv\boldsymbol{\sigma}_y\oplus\boldsymbol{\sigma}_y\oplus\cdots\oplus\boldsymbol{\sigma}_y$.

We start with a short discussion on the dynamics of the entanglement for the system. In order to consider quantum statistical properties of the field of modes a_1 and a_2, we trace the total density operator over all environmental modes, which is equivalent to considering the correlation matrix $\mathbf{V}_c$ only for the modes a_1 and a_2,

$$\mathbf{V}_c(a_1,a_2)=\begin{pmatrix}\cosh(2s)\mathbb{1} & t\sinh(2s)\boldsymbol{\sigma}_z\\ t\sinh(2s)\boldsymbol{\sigma}_z & (t^2\cosh(2s)+r^2\tilde{n})\mathbb{1}\end{pmatrix}. \tag{9}$$

It has to be emphasized that this correlation matrix is exactly the same as the solution of the Fokker-Planck equation (1). Using Simon's criterion in Eq. (8) [9,11], the field of modes a_1 and a_2 is separable when the transmittivity of the beam splitter is

$$t^2 \leqslant \frac{\bar{n}}{1+\bar{n}} \equiv t^2_{a_1 a_2} \tag{10}$$

for the squeezing parameter $s \neq 0$. Note that the separability condition does not depend on the initial entanglement of the system as far as there is any entanglement at the initial instance. The separability of the two-mode squeezed state depends only on the temperature of the environment and the overall transmittivity of the beam splitters.

IV. ENTANGLEMENT OF SYSTEM AND ENVIRONMENT

We now study the entanglement of the system and the environment. Here, instead of the entanglement of the system with an individual mode b_m of the environment, we are interested in the biseparability of a system mode and the collection of the environmental modes. Let us first consider the entanglement of the modes a_1 and c_0. The correlation matrix $\mathbf{V}_c(a_1, c_0)$ is equivalent to $\mathbf{V}_c(a_1, a_2)$ in Eq. (9) if r and t are interchanged. The separability condition is found to be

$$t^2 \geqslant \frac{1}{1+\bar{n}} \equiv t^2_{a_1 c_0}, \tag{11}$$

which is again independent from the initial entanglement of the system as far as $s \neq 0$.

The separability of modes a_2 and c_0 is easily discussed using a quasiprobability P function [5], the existence of which is a sufficient condition for separability [13]. Tracing over mode a_1 of the two-mode squeezed state, the other mode a_2 is in a thermal state with the effective number of thermal photons $\bar{n}_s = (\cosh 2s - 1)/2$. A product of thermal states has a positive definite P function and the action of the beam splitter only transforms the coordinates of the input P function, $P_{a_2}(\tilde{x}_1)P_b(\tilde{x}_2) \xrightarrow{bs} P_{a_2}(t\tilde{x}_1 - r\tilde{x}_2)P_b(t\tilde{x}_2 + r\tilde{x}_1)$. This proves that the environment never entangles with its interacting mode by a passive linear interaction.

Figure 1 presents the entanglement structure for a two-mode squeezed state interacting with a thermal environment, where $t^2 = \exp(-\gamma\tau)$ is the transmittivity of the collective beam splitter relating to the interaction time τ. The solid lines are the boundaries of entanglement of the system mode a_1 and the collective mode of the environment c_0 and of the two system modes a_1 and a_2. These lines are obtained by the separability condition (8) in the present exactly solvable model. For the comparison, we consider $N = 100$ beam splitters modeling the interaction with the thermal environment and calculate the biseparability of the 1×100-mode field using computational analysis of Giedke *et al.* [14]. The computational results of entanglement are denoted by circles and dots. We find that the results are also independent of the squeezing parameter. Figure 1 shows that the two methods are exactly consistent.

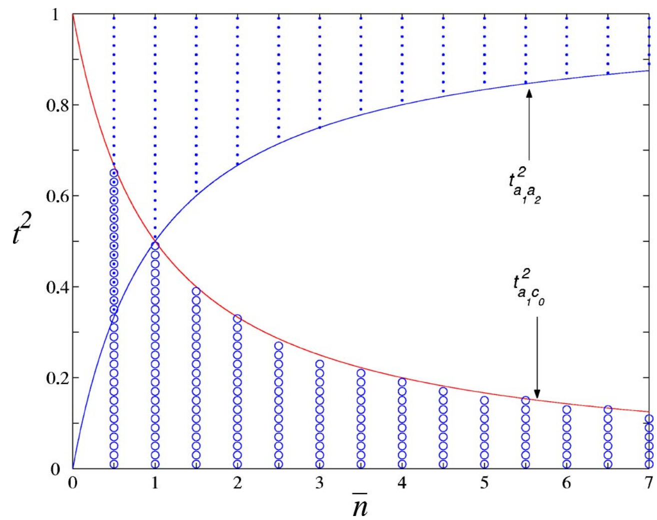

FIG. 1. Nature of entanglement for a two-mode squeezed state interacting with a thermal environment of the average photon number $\bar{n}$. $t^2 = \exp(-\gamma\tau)$ is the transmittivity of the collective beam splitter. The solid lines are the boundaries of entanglement of a_1 and c_0 and of a_1 and a_2, which are obtained by the separability condition. The circles and dots are found by a *computational analysis* with $N = 100$ beam splitters. The circles indicate that the system mode a_1 and the group of environmental modes b_m are entangled and the dots indicate that the system modes a_1 and a_2 are entangled.

The three modes a_1, a_2, and c_0 compose a tripartite system. Many-body entanglement for pure continuous-variables has been studied extensively using beam splitters and single-mode squeezed states [15]. Giedke *et al.* classified types of entanglement for a three-mode Gaussian field in terms of the biseparability [14]. A three-mode field is biseparable when any grouping of three modes into two are separable. When a three-mode field is not biseparable, it is called fully entangled. A fully-entangled tripartite system may be further classified in terms of pairwise entanglement; for qubits, two kinds have been discussed [16,17], one of which is Greenberg-Horne-Zeilinger (GHZ) entanglement [18] and the other is W entanglement. A GHZ-entangled state becomes separable when any one particle is traced out, while a W-entangled state is pairwise entangled for any pair of the three particles. On the other hand, there is another kind of entanglement, two-way entanglement, for a fully-entangled tripartite d-dimensional system [19]. One example for three particles labeled as a, b, and c is pairwise entanglement of a and b and of b and c but the particles a and c are separable.

It is found both computationally, and analytically using the biseparability condition in Eq. (8) that the tripartite system of a_1, a_2, and c_0 is fully entangled if $0 < t^2 < 1$ and $s \neq 0$. This fact is independent of the temperature of the environment so that the tripartite system is fully entangled over the whole region in Fig. 1. The tripartite system is two-way entangled, i.e., a_1-a_2 and a_1-c_0 modes are entangled if $\bar{n} < 1$ and $t^2_{a_1 a_2} < t^2 < t^2_{a_1 c_0}$. There are two regions where one pair of the three-mode field is entangled. In the region of $\bar{n}>1$ and $t^2_{a_1c_0}\leqslant t^2\leqslant t^2_{a_1a_2}$, the three modes are GHZ entangled. Here, we note that GHZ entanglement of an entangled system with an environment is an important source to its loss of the initial pairwise entanglement. This is clearly seen for $\bar{n}>1$ in Fig. 1. The two modes of the system are initially entangled, but as they come to the region of GHZ entanglement, they lose their initial entanglement. Finally, the entanglement is transferred to between the system mode a_1 and the effective environmental mode. For $\bar{n}<1$ the decoherence process is different with initially a_1-a_2 modes being entangled, then two-way entanglement and finally a_1-c_0 being entangled.

V. REMARKS

In this paper, we have studied the decoherence mechanism by highlighting the entanglement of the continuous-variable system with its environment. We showed that the homogeneous thermal environment can be summarized by a single collective mode with respect to the interaction with the system for the study of entanglement. Our decoherence model is composed of a two-mode squeezed state, one mode of which interacts with a thermal environment. We found two entanglement mechanisms between the system and its environment which accompany the decoherence process. When the temperature of the environment is low ($\bar{n}<1$), there is the two-way entanglement. Otherwise, GHZ entanglement causes the system to lose its entanglement. Do the two different entanglement mechanism result in any measurable differences to the system? To find it out we consider the mixedness of the system.

When $\mathrm{Tr}\hat{\rho}^2=1$ the system is pure. For a Gaussian state with correlation matrix $\mathbf{V}$, we found that $\mathrm{Tr}\hat{\rho}^2=1/\sqrt{\det\mathbf{V}}$. We thus define the mixedness of the Gaussian state by

$$M=\sqrt{\det\mathbf{V}}-\mathbf{1}, \tag{12}$$

which is 0 when the state is pure and grows as the system is mixed. This measure is relevant to experiment as all the elements of the correlation matrix are measurable using homodyne detectors [9]. We examine the mixedness of the system at the time when it loses its entanglement, i.e., $t^2=\bar{n}/(1+\bar{n})$:

$$M_e=2\left[\left(2-\frac{1}{\bar{n}+1}\right)\cosh^2 s-1\right]. \tag{13}$$

We see that the mixedness M_e of the system at the moment of disentangling grows with $\bar{n}$ and reaches its half point M_1 when $\bar{n}=1$. This result holds particularly for a two-mode squeezed state but may indeed suggest that the mixedness of the system is a strong indicator of the mechanism of system-environment entanglement. Thus, for $M_e<M_1$, decoherence into two-way entanglement occurs, while for $M_e>M_1$ GHZ entanglement occurs.

We have analyzed the correlation between a system and its environment under the Born-Markov approximation. One may go beyond the Markov approximation and extend the present approach to incorporate N modes $\{\hat{b}_j\}$ of the environment with natural frequencies ω_j that influence the system with arbitrary coupling constants λ_j. The total Hamiltonian ($\hbar=1$) under the rotating-wave approximation is given as

$$\hat{H}=2\sum_{i=1}^{2}\omega_0\hat{L}_{a_i}+2\sum_{j=0}^{N-1}\omega_j\hat{L}_{b_j}+2\sum_{j=0}^{N-1}\lambda_j\hat{M}_{a_2b_j}, \tag{14}$$

where the Hermitian operators $\hat{L}_u=(2\hat{u}^\dagger\hat{u}+1)/4$ and $\hat{M}_{uv}=(\hat{u}^\dagger\hat{v}+\hat{u}\hat{v}^\dagger)/2$ with $u,v=a_1,a_2$ and b_j. Here, the first (second) term is the free Hamiltonian of the two modes of system (the environment modes) and the third term is the interaction Hamiltonian between the second mode of the system and the environment modes. Introducing new Hermitian operators $N_{uv}=-i(\hat{u}^\dagger\hat{v}-\hat{u}\hat{v}^\dagger)/2$ and letting $\hat{L}^{\pm}_{uv}=\hat{L}_u\pm\hat{L}_v$, the set $A=\{L^{\pm}_{uv},M_{uv},N_{uv}\}$ forms a Lie algebra [20] with the commutation relations

$$[\hat{A}_i,\hat{A}_j]=C_{ijk}\hat{A}_k, \tag{15}$$

where $\hat{A}_i\in A$ and C_{ijk} is a structure constant. The evolution operator $\hat{U}=\exp(-i\hat{H}\tau)$ can now be represented by the elements (generators) of the Lie algebra A as

$$\hat{U}=\exp\left(i\sum_i g_i(\tau)\hat{A}_i\right)=\prod_i \exp(ig_i'(\tau)\hat{A}_i), \tag{16}$$

where $g_i(\tau)$ and $g_i'(\tau)$ are real coefficients depending on the time τ. Note that the evolution operator $\hat{U}$ consists of the rotators generated by $\hat{L}^{\pm}_{uv}$ and the beam splitters generated by $\hat{M}_{uv}$ or $\hat{N}_{uv}$. We know that any combination of rotators and beamsplitters does not bring about entanglement in the output fields when the input fields are classical [21] so, even in this case, there is no entanglement between a_2 and environment modes. The separability conditions, Eq. (10) and Eq. (11) within the Non-Markovian environment interaction will be discussed elsewhere.

We investigated the process of disentangling the two-mode squeezed state and found that there are the two distinct routes to disentangling process: two-way and GHZ entanglement. After some algebraic manipulation, we can easily prove that, at the instance of disentanglement, the mutual information between the two modes a_1 and a_2 is higher when it is disentangled through two-way entanglement with the environment than GHZ entanglement. This means that more information on a_1 is gained by the "measurement" of mode a_2 [22].

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