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Resonance expansion versus the rotating-wave approximation

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We propose a general perturbative approach to quantum-optical models without the rotating-wave approximation. We show that a generic Hamiltonian describing interaction between two subsystems can be represented as a series of operators corresponding to different transitions between bare energy levels of the whole system. Under certain relations between frequencies of interacting subsystems one of these transitions becomes resonant. The rotating-wave approximation leads to separation of the resonant transition and to appearance of the integral of motion, which makes the problem exactly solvable in this approximation. (Different resonance conditions lead to different integrals of motion.) All of the other terms in these expansion can be considered as a perturbation. They result in dynamic Stark shifts and small corrections to the integrals of motion. All possible resonances are classified, and approximate integrals of motion are found for each resonance. Examples of field-field, field-atom, and atom-atom interactions are considered.

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I. INTRODUCTION

The rotating-wave approximation (RWA) is widely used in quantum optics. This approximation is applied when the frequency associated with the free evolution of the system is essentially bigger than transition frequencies induced by the interaction between subsystems and/or some external source. The RWA basically consists in neglecting rapidly oscillating (counterrotating) terms in such a way that in the rotating frame the system Hamiltonian becomes time independent or it depends slowly on time. In semiclassical models RWA is commonly used when a quantum oscillator and/or two- (or multilevel) level atoms are excited by an external force [1]. In the quantum domain application of the RWA usually leads to the appearance of an additional integral of motion (and of an Abelian invariance group of the Hamiltonian). The representation space of the whole system is then divided into invariant subspaces, which strongly simplifies the mathematical treatment of the problem and usually leads to the integrability of its classical counterpart.

The perturbative approach to the systems beyond the RWA leads to more complicated mathematical structures. Probably, one of the most studied problem in quantum optics is the famous Jaynes-Cummings model (JCM) without the RWA [3–15], which lately has acquired an additional interest in connection to the problem of cold ions dynamics [16,17]. The semiclassical analog of this problem has been exhaustively studied by Shirley [1], and it was shown that apart from the Bloch-Siegert shift [2] of the atomic frequency some high-order resonances appear. That is, in the first approximation the only effect of the antirotating terms is the atomic frequency shift, and the Hamiltonian thus describes only a single-resonance atomic transition. Nevertheless, in the consecutive approximations high-frequency transitions always appear. A similar situation takes place in the JCM with the quantum field. The perturbative expansion developed in Ref. [3] shows that the analog of the Bloch-Siegert shift in the quantum domain is the intensity-dependent atomic frequency shift (“dynamic Stark shift”) and the high-order resonances manifest themselves in such a way that the terms describing $2n+1$ photon transitions $(n=1,2,\dots)$ appear in the effective Hamiltonian.

The main idea of this paper is to show that a generic Hamiltonian governing interaction of two subsystems beyond the RWA can be represented as a series in operators describing all possible transitions in the system. When certain relations hold between frequencies of interacting subsystems, some of these transitions become resonant. Therefore, one can extend the meaning of the term “rotating-wave approximation” in the following way: the RWA distinguishes the principal (resonant) transition, while all of the other transitions can be considered as a small perturbation. The application of the RWA depends on the relations between transition frequencies of interacting subsystems.

II. THE EFFECTIVE HAMILTONIAN

Let us consider two interacting systems X and Y whose dynamics is governed by the following generic Hamiltonian:

$$H=\omega_1 X_0+\omega_2 Y_0+g(X_+ + X_-)(Y_+ + Y_-), \tag{1}$$

where X_0, Y_0 are free Hamiltonians of these systems and without a loss of generality we suppose that $\omega_1 \geqslant \omega_2$. We also impose the condition $\omega_1, \omega_2 \gg g$, which usually holds in quantum-optical interactions. The rising-lowering operators $X_\pm, Y_\pm$ describe transitions between energy levels of the systems X and Y correspondingly and consequently obey the following commutation relations:

$$[X_0, X_\pm]=\pm X_\pm, \quad [Y_0, Y_\pm]=\pm Y_\pm. \tag{2}$$

We do not impose any condition on the commutators between transition operators, which are generally some functions of diagonal operators and some integrals of motion $[N_1, X_0]=[N_2, Y_0]=0$:

$$[X_+, X_-]=P_1(X_0)=\boldsymbol{\nabla}_{X_0}\phi_1(X_0, N_1),$$

$$[Y_+, Y_-]=P_2(Y_0)=\boldsymbol{\nabla}_{Y_0}\phi_2(Y_0, N_2), \tag{3}$$

where $\phi_1(X_0,N_1)=X_+X_-$, $\phi_2(Y_0,N_2)=Y_+Y_-$ (from now on we omit the dependence on integrals $N_{1,2}$ in the arguments), and

$$\nabla_z\phi(z)=\phi(z)-\phi(z+1).$$

The objects $(X_0,X_\pm)$ and $(Y_0,Y_\pm)$ are known as polynomially deformed algebras $\mathrm{sl}_{pd}(2,R)$ [18–27].

Now, following the method [28,29] (see Appendix A) we can adiabatically remove the counterrotating term $X_+Y_+ + X_-Y_-$ from the Hamiltonian (1) by applying some appropriate Lie-type transformations (“small rotations”). The elimination of the above term leads to the appearance of new elements in the transformed Hamiltonian. All these new terms can be divided into three groups. The first group contains terms of the form $X_+^nY_+^m+\mathrm{H.c.}$, which can always be eliminated under the condition $\omega_2,\omega_1 \gg g$ by applying some suitable transformations. The second group consists of terms that cannot be removed if certain relations between ω_1 and ω_2 hold, since the transformation which eliminates a given term from the Hamiltonian becomes singular. This group contains nondiagonal terms of the form $X_+^nY_-^m+\mathrm{H.c.}$, which describe transitions between energy levels of the whole system. The third group includes the diagonal terms (functions only of X_0,Y_0). Our strategy consists in keeping in the Hamiltonian only diagonal terms and those that cannot be eliminated by nonsingular transformations (resonant terms). Besides, we conserve only the leading-order coefficients in these terms.

As a result (see Appendix B), we obtain the following effective Hamiltonian (which we call “the resonance expansion”):

$$\begin{aligned}H_{eff}\approx{}&\omega_1X_0+\omega_2Y_0+g\varepsilon\Phi(X_0,Y_0,\varepsilon)\\&+g\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-\delta)^{l+k-1}\varepsilon^{l+2(k-1)}}{(k-1)!(l+k-1)!}\\&\times[X_+^kY_-^{2l+k}\{\Theta_{kl}(X_0,Y_0,\varepsilon)+O(\varepsilon^2)\}+\mathrm{H.c.}],\end{aligned}\tag{4}$$

where

$$\varepsilon=\frac{g}{\omega_1+\omega_2}\ll 1,\quad \delta=\frac{g}{2\omega_2}\ll 1\tag{5}$$

are small parameters; the term $\Phi(X_0,Y_0,\varepsilon)$ represents the dynamical Stark shift (or Bloch-Siegert shift) and can be expanded in powers of the small parameter ε as follows:

$$\Phi(X_0,Y_0,\varepsilon)=\nabla_{X_0,Y_0}[\phi_1(X_0)\phi_2(Y_0)]+O(\varepsilon^2),\tag{6}$$

where the generalized displacement operators are defined as

$$\nabla_{mX_0,nY_0}f(X_0,Y_0)=f(X_0,Y_0)-f(X_0+m,Y_0+n),$$

where m and n are integers. The terms of the form $X_+^kY_-^{2l+k}+\mathrm{H.c.}$ describe all the admissible resonances:

$$(2l+k)\omega_2=k\omega_1.\tag{7}$$

In particular, term with $k=1$, $l=0$ represents the principal resonance. The coefficients $\Theta_{kl}(X_0,Y_0,\varepsilon)$ define (intensity-dependent) coupling constant. The coefficient in the principal term has the form

$$\Theta_{10}(X_0,Y_0)=1,$$

and

$$\begin{aligned}\Theta_{kl}(X_0,Y_0)={}&\sum_{j=0}^{l+[k/2]}C^j_{l+[k/2]}[P_1(X_0+k)]^{l+[k/2]-j}\\&\times[-P_1(X_0)]^jR_k(X_0,Y_0-2j),\end{aligned}\tag{8}$$

$$\begin{aligned}R_k(X_0,Y_0)={}&\sum_{j=0}^{(k-1)/2}C^j_{(k-1)/2}[P_1(X_0+k)]^{[(k-1)/2]-j}\\&\times[-P_1(X_0)]^j\nabla_{Y_0}^{k-1}\phi_2(Y_0-2j)\\&\times\prod_{i=1-2j}^{k-2(j+1)}\phi_2(Y_0+i),\end{aligned}$$

for $l\geqslant 1$, where C_k^j are the binomial coefficients and $\nabla_z^0f(z)=1$. Note that the product in the last equation equals to unity if the upper limit is less than the lower one.

In the case when $X_\pm$ and/or $Y_\pm$ are nilpotent operators of qth order, all the powers beginning with $q+1$ turn to zero, which obviously puts a natural restriction on the series of possible resonances.

It is worth noting here that in the case of the principal resonance it is convenient to proceed with further transformations and find the first-order correction to the interaction Hamiltonian, which effectively leads to the renormalization of the interaction constant (making it intensity dependent). The width of the corresponding resonance in Eq. (4) is of order $g\delta^{l+k-1}\varepsilon^{l+2(k-1)}$.

One may observe that among the resonances described by Eq. (7) only the sets (k,l) of coprime numbers produce different resonances. It is important to stress that for a given multiplicity of the resonance only the terms of the lowest order in small parameters are kept in the expansion (4); all the higher-order terms that correspond to the same resonance are to be neglected.

Among resonances described by Eq. (7) there are two “entire” resonances.

(a) Odd resonances (including the principal resonance), corresponding to $k=1, l=0,1,2,3,\ldots,$

$$\omega_1=(2l+1)\omega_2.\tag{9}$$

(b) Even resonances, corresponding to $k=2, l=1,3,5\ldots,$ such that

$$\omega_1=2m\omega_2,\quad m=1,2,\ldots,\tag{10}$$

while all the other resonances are of a “fractional” type

$$\omega_1=(2l/k+1)\omega_2.\tag{11}$$

The effective Hamiltonian (4) is the main result of this paper.

A. The first resonance, $\omega_1 \approx \omega_2$

By keeping only terms $\sim X_+ Y_-$ in Eq. (4) we obtain (in the rotating picture) the effective Hamiltonian describing the principal resonance

$$H_{eff}^{(1)} \approx \Delta_{10} X_0 + g\varepsilon \Phi(X_0, Y_0) + g\{X_+ Y_- [1 + \varepsilon^2 \Psi(X_0, Y_0)] + \text{H.c.}\}, \tag{12}$$

where $\Delta_{10} = \omega_1 - \omega_2$, $\Phi(X_0, Y_0)$ is defined in Eq. (6) and we have included the first correction to the coupling constant [originally absent in Eq. (4)]

$$2\Psi(X_0, Y_0) = \nabla_{2X_0, Y_0}[\phi_1(X_0) P_2(Y_0 - 1)] + \nabla_{X_0, 2Y_0}[P_1(X_0) \phi_2(Y_0 - 1)].$$

The *effective* Hamiltonian (12) obviously admits the integral of motion

$$N_{eff}^{(10)} = X_0 + Y_0.$$

Making an inverse transformation we find that the operator

$$\begin{aligned} N^{(1)} = {} & X_0 + Y_0 + 2\delta(X_+ Y_+ + X_- Y_-) \\ & - 2\delta^2 \nabla_{X_0, Y_0}[\phi_1(X_0)\phi_2(Y_0)] \\ & + 2\delta^2 \{[X_+^2 + X_-^2] P_2(Y_0) + [Y_+^2 + Y_-^2] P_1(X_0)\} \\ & + O(\delta^3) \end{aligned} \tag{13}$$

approximately commutes with the original Hamiltonian (1) in the case $\omega_1 = \omega_2$, $[H, N^{(1)}] = O(\delta^2)$. We note here that the width of the first resonance is of order g.

B. Effect of the higher resonances, $k\omega_1 \approx (2l+k)\omega_2$

Now, let us suppose that the resonance condition (7) is satisfied for a given (coprime) k and l. This means that all the other transitions are suppressed and the operators describing those transitions can be eliminated from the Hamiltonian by applying a corresponding transformation. The effective Hamiltonian (in the rotating frame) takes the form

$$\begin{aligned} H_{eff} \approx {} & \Delta_{kl} X_0 + g\varepsilon \Phi(X_0, Y_0, \varepsilon) + g\epsilon \Xi(X_0, Y_0, \delta) \\ & + \frac{g(-\delta)^{l+k-1} \varepsilon^{l+2(k-1)}}{(k-1)!(l+k-1)!} \\ & \times [X_+^k Y_-^{2l+k} \Theta_{kl}(X_0, Y_0) + \text{H.c.}], \end{aligned} \tag{14}$$

where

$$\Delta_{kl} = k\omega_1 - (2l+k)\omega_2, \tag{15}$$

$$\epsilon = g/(\omega_1 - \omega_2) \ll 1,$$

and the functions $\Phi(X_0, Y_0, \varepsilon), \Theta_{kl}(X_0, Y_0)$ are defined in Eqs. (6)–(8). The third (diagonal) term in the above expression has appeared as a result of elimination of all of the lowest-order transitions [note that $\varepsilon \approx k\delta/(l+k)$ in the vicinity of the resonance],

$$\Xi(X_0, Y_0, \delta) = \nabla_{X_0, -Y_0}[\phi_1(X_0)\phi_2(Y_0 + 1)] + O(\delta).$$

In the most important case of the third-order resonance, the effective Hamiltonian has the form

$$\begin{aligned} H_{eff} \approx {} & \Delta_{11} X_0 + g\varepsilon \nabla_{X_0, Y_0}[\phi_1(X_0)\phi_2(Y_0)] \\ & + g\epsilon \nabla_{X_0, -Y_0}[\phi_1(X_0)\phi_2(Y_0 + 1)] \\ & + g\varepsilon\delta[X_+ Y_-^3 \nabla_{X_0}^2 \phi_1(X_0) + \text{H.c.}], \end{aligned}$$

where Δ_{11} is defined in Eq. (15). It is worth noting that the coefficients of the diagonal terms in Eq. (14) are always much bigger than those of the terms describing the higher resonances. This means that the amplitudes of higher-order transitions could be small due to the presence of the dynamical Stark shift and (diagonal) contributions from lowest resonances. Besides, in general, the functions $\Phi(X_0, Y_0, \varepsilon)$ and $\Xi(X_0, Y_0, \delta)$ are nonlinear, which makes difficult a compensation of their effect by the frequency shift (15). In practice, the higher-order transitions are better observed for initial states which belong to subspaces where the diagonal terms are minimized.

The original Hamiltonian (1) in the case (7) (exact resonance, $\Delta_{kl} = 0$) admits the following approximate integral of motion:

$$\begin{aligned} N^{(kl)} = {} & (2l+k)X_0 + kY_0 + 2k\delta(X_+ + X_-)(Y_+ + Y_-) \\ & - \frac{2k^2\delta^2}{l(k+l)}\{(k+l)\nabla_{X_0, -Y_0}[\phi_1(X_0)\phi_2(Y_0 + 1)] \\ & + l\nabla_{X_0, Y_0}[\phi_1(X_0)\phi_2(Y_0)]\}, \end{aligned} \tag{16}$$

such that $[H, N^{(kl)}] = O(\delta^2)$. It is easy to see that the width of the corresponding resonance is of order $g\delta^{2l+3(k-1)}$.

III. APPLICATIONS

A. Linear systems

Let us consider a Hamiltonian describing interaction of two harmonic oscillators

$$H = \omega_1 n_a + \omega_2 n_b + g(a^\dagger + a)(b^\dagger + b), \tag{17}$$

where $n_a = a^\dagger a, n_b = b^\dagger b$. This system belongs to a class of so-called linear systems, in the sense that the Hamiltonian can be recast as a linear function on generators of the su(2) and su(1,1) algebras. After making identifications

$$Y_0 = n_b, \quad Y_+ = b^\dagger, \quad Y_- = b, \quad X_0 = n_a, \quad X_+ = a^\dagger,$$

$$X_- = a,$$

so that $\phi_1(X_0) = X_0, \phi_2(Y_0) = Y_0$, we obtain $\Theta_{10}(n_a, n_b) = 1$ and the coefficients of higher resonance terms as expected are equal to zero: $\Theta_{kl}(n_a,n_b)=0$, $k>1,l>0$. The effective Hamiltonian acquires the form

$$H_{eff}\approx(\omega_1-g\varepsilon)n_a+(\omega_2-g\varepsilon)n_b+g(1+2g\varepsilon^2)\times(a^\dagger b+ab^\dagger).$$

This confirms a well-known result (see, e.g., Ref. [30]) that the antirotating terms in linear systems (17) result only in frequency shifts and renormalization of the interaction constant. The corresponding integral of motion takes the form (in the exact resonant case, $\omega_1=\omega_2$) (13)

$$N=n_a+n_b+2\varepsilon(a^\dagger b^\dagger+ab)+2\varepsilon^2\times(n_a+n_b-a^{\dagger 2}-b^{\dagger 2}-a^2-b^2).$$

B. Harmonic-oscillator-spin interaction

The Hamiltonian governing the evolution of a spin S interacting with a harmonic oscillator has the form

$$H=\omega_1 S_z+\omega_2 n+g(S_++S_-)(a^\dagger+a), \tag{18}$$

where $n=a^\dagger a$ and $S_{z,\pm}$ are generators of the $(2S+1)$-dimensional representation of the su(2) algebra and we suppose that $\omega_1\geqslant\omega_2$.

The following identifications:

$$X_0=S_z,\quad X_\pm=S_\pm,\quad Y_0=n,\quad Y_+=a^\dagger,\quad Y_-=a,$$

so that $\phi_1(X_0)=C_2-X_0^2+X_0,\phi_2(Y_0)=Y_0$, lead to $\Theta_{kl}(S_z,n)=0,k\geqslant 3$ and the effective Hamiltonian takes the form

$$\begin{aligned}H_{eff}\approx{}&\omega_1 S_z+\omega_2 a^\dagger a+g\varepsilon[S_z^2+(2a^\dagger a+1)S_z-C_2]\\&+g[S_+a\{1-\varepsilon^2(a^\dagger a+4S_z+2)\}+\text{H.c.}]\\&+g\sum_{l=1}^{\infty}\frac{(-2\delta\varepsilon)^l}{l!}[a^{2l+1}S_++\text{H.c.}]\\&-g\varepsilon\sum_{m=1}^{\infty}\frac{(4\delta\varepsilon)^{2m}}{(2m)!}[a^{4m}S_+^2+\text{H.c.}],\end{aligned} \tag{19}$$

where $C_2=(S+1)S$. In particular, the effective Hamiltonian describing the evolution in the first resonance case, $\omega_1=\omega_2+\Delta_{10}$, has the form [31,32] (see also Ref. [33])

$$\begin{aligned}H_{eff}\approx{}&\Delta_{10}S_z+g\varepsilon[S_z^2+(2a^\dagger a+1)S_z-C_2]\\&+g[S_+a\{1-\varepsilon^2(a^\dagger a+4S_z+2)\}+\text{H.c.}],\end{aligned}$$

in which we have taken into account the second-order corrections (12). The integral of motion corresponding to this resonance is

$$\begin{aligned}N^{(1)}={}&S_z+a^\dagger a+2\delta(S_+a^\dagger+S_-a)-2\delta^2(S_z+2a^\dagger a+1)S_z\\&+2\delta^2(2S_z[a^{\dagger 2}+a^2]-[S_+^2+S_-^2])+O(\delta^3).\end{aligned}$$

The Hamiltonian describing the third-order resonance has the form

$$\begin{aligned}H_{eff}\approx{}&\Delta_{11}S_z+g[(\varepsilon+\epsilon)S_z(2a^\dagger a+1)+(\epsilon-\varepsilon)(C_2-S_z^2)]\\&-2g\delta\varepsilon(a^{\dagger 3}S_-+a^3S_+),\end{aligned}$$

where Δ_{11} is defined in Eq. (15). Because the Stark shift contains a term $\sim S_z^2$, the amplitude of the third-order transitions would be small for the spin systems with $S\geqslant 1$. Nevertheless, in the case of $S=1/2$ and the harmonic oscillator initially prepared in highly excited coherent state, the diagonal term can be approximately compensated by the detuning Δ_{11}.

It is worth noting that the condition $\Theta_{kl}(S_z,n)=0,k\geqslant 3$ does not mean that there are no fractional (11) resonances, but only that the terms describing the fractional resonances are suppressed to the order $o(\delta^{2l+1+3(k-1)})$ for the given indices k and l.

The integrals of motion for the series of odd and even (exact) resonances are (16)

$$\begin{aligned}N^{(kl)}={}&(2l+k)S_z+ka^\dagger a+2k\delta(S_++S_-)(a^\dagger+a)\\&-\frac{2k^2\delta^2}{l(l+k)}[(2l+k)S_z(2a^\dagger a+1)-kS_z^2]+O(\delta^3),\end{aligned}$$

where $l=2m+1$ in $N^{(2l)}$, $m=0,1,2,3,\ldots$, and $k=1,2$ for even and odd resonances correspondingly.

Let us recall that the higher resonances appear in the effective Hamiltonian (19) only under the condition $\omega_1\geqslant\omega_2$. In the opposite case, $\omega_1\leqslant\omega_2$, only the principal resonance survives and in the approximation (4) the whole effect of the counterrotating terms reduces to the dynamical Stark shift, which has the same form [see (6)] as in the above case. It happens because $P_1(X_0=n)=-1$ and thus, only $\Theta_{10}(n,S_z)\neq 0$ (note that for $\omega_1\leqslant\omega_2$ we identify the X operators with the oscillator and Y operators with the spin system). This does not imply that there are no higher resonances, but rather that those are essentially suppressed (similar to the fractional resonances in the case $\omega_1\geqslant\omega_2$).

Note that the Hamiltonian (18) also describe interaction of a single mode of quantized field with a collection of $A=2S$ two-level atoms initially prepared in a symmetric superposition (the so-called Dicke model). In this case the operators $S_{z,\pm}$ are defined as the collective atomic operators

$$S_z=\frac{1}{2}\sum_{j=1}^{A}\sigma_z^{(j)},\quad S_\pm=\sum_{j=1}^{A}\sigma_\pm^{(j)},$$

where $\sigma_{z,\pm}^{(j)}$ are the Pauli matrices describing jth two-level atom.

C. Spin-spin interaction

The Hamiltonian describing spin-spin (dipole-dipole) interaction has the following generic form

$$H=\omega_1 S_{1z}+\omega_2 S_{2z}+g(S_{1+}+S_{1-})(S_{2+}+S_{2-}),$$

where $S_{1z,\pm}$ and $S_{2z,\pm}$ are generators of $(2S_1+1)$- and $(2S_2+1)$-dimensional representations of the su(2) algebra correspondingly. The following identifications:

$$X_0=S_{1z}, \quad X_\pm=S_{1\pm}, \quad Y_0=S_{2z}, \quad Y_\pm=S_{2\pm},$$

so that $\phi_1(X_0)=C_2^{(1)}-X_0^2+X_0, \phi_2(Y_0)=C_2^{(2)}-Y_0^2+Y_0$, lead to $\Theta_{kl}(S_{z1},S_{2z})=0, k\geqslant 4$ and the effective Hamiltonian takes the form

$$\begin{aligned}H=&\omega_1 S_{1z}+\omega_2 S_{2z}+2g\varepsilon[S_{1z}(C_2^{(2)}-S_{2z}^2)+S_{2z}(C_2^{(1)}-S_{1z}^2)]\\&+g[S_{1+}S_{2-}\{1+\varepsilon^2(8S_{1z}S_{2z}-3S_{1z}+3S_{2z}+S_{1z}^2+S_{2z}^2\\&-C_2^{(1)}-C_2^{(2)})\}+\text{H.c.}]+g\sum_{l=1}^{S_2}\frac{(-2\delta\varepsilon)^l}{l!}\\&\times[S_{1+}S_{2-}^{2l+1}+\text{H.c.}]+2g\varepsilon\sum_{m=1}^{S_2/2}\frac{(4\delta\varepsilon)^{2m}}{(2m)!}\\&\times[S_{1+}^2S_{2-}^{4m}(S_{2z}+2mS_{z1})+\text{H.c.}]+g\varepsilon^2\sum_{l=1}^{S_2-1}\frac{(-\delta\varepsilon)^{l+2}}{(l+2)!}\\&\times[S_{1+}^3S_{2-}^{2l+3}\Theta_{3l}(S_{z1},S_{2z})+\text{H.c.}],\end{aligned} \tag{20}$$

where

$$\begin{aligned}\Theta_{3l}(S_{z1},S_{2z})=&8\cdot 6^l[9S_{2z}^2+4(l+1)(l+2)S_{z1}^2+12(l+1)\\&\times S_{z1}S_{2z}-6(l+2)S_{1z}+18S_{2z}-9C_2^{(2)}],\end{aligned}$$

$C_2^{(1,2)}=(S_{1,2}+1)S_{1,2}$ are the corresponding Casimir operators and $l\neq 3N$, $N=1,2,\ldots$ in the last sum in Eq. (20). As well as in the Dicke model case, not all of the resonances (7) survive in this model. Nevertheless, in addition to the odd (9) and even (10) resonances the first series of fractional resonances (11) appears with $\omega_1=(2l/3+1)\omega_2, l\neq 3N$, $N=1,2,\ldots$.

In particular, the effective Hamiltonian describing the third-order transitions has the form

$$\begin{aligned}H_{eff}\approx&\Delta_{11}S_{1z}+2g[(\varepsilon+\epsilon)S_{1z}(C_2^{(2)}-S_{2z}^2)+(\varepsilon-\epsilon)S_{2z}\\&\times(C_2^{(1)}-S_{1z}^2)]-2g\,\delta\varepsilon(S_{1+}S_{2-}^3+S_{1-}S_{2+}^3).\end{aligned} \tag{21}$$

This transitions can be easily observed, for example, for $S_1=1$, $S_2=3$ and taking the initial state $|m_1=0,S_1=1\rangle|m_2=0,S_2=3\rangle$, see Fig. 1. Note that the diagonal term is zero in the subspace generated by the application of the Hamiltonian (21) to the state $|0,1\rangle_1|0,3\rangle_2$.

The integral of motion corresponding to the first and highest (exact) resonances are obtained from Eq. (16):

$$\begin{aligned}N^{(1)}=&S_{1z}+S_{2z}+2\delta(S_{1+}S_{2+}+S_{1-}S_{2-})\\&-4\delta^2[S_{1z}(C_2^{(2)}-S_{2z}^2)+S_{2z}(C_2^{(1)}-S_{1z}^2)]\\&+4\delta^2[(S_{1+}^2+S_{1-}^2)S_{2z}+(S_{2+}^2+S_{2-}^2)S_{1z}]+O(\delta^3),\end{aligned}$$

and

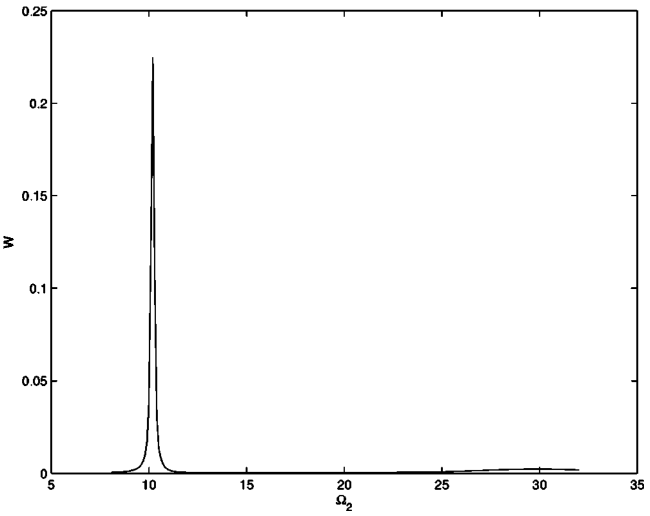

FIG. 1. The third-order resonance in the spin-spin interaction $\omega_1=30, g=1$. The temporal average transition probability $W=\langle|_2\langle -3,3|_1\langle 1,1|U(t)|0,1\rangle_1|0,3\rangle_2|^2\rangle_t$ against ω_2.

$$\begin{aligned}N^{(kl)}=&(k+2l)S_{1z}+kS_{2z}+2k\delta(S_{1+}+S_{1-})(S_{2+}+S_{2-})\\&+\frac{4(k\delta)^2}{l(l+k)}[kS_{2z}(C_2^{(1)}-S_{1z}^2)-(2l+k)S_{1z}\\&\times(C_2^{(2)}-S_{2z}^2)],\end{aligned}$$

where $k=1,2,3$ and when $k=1, l=1,2,3\ldots; k=2, l=1,3,5,\ldots; k=3, l=1,2,4,5,7,8,\ldots$

IV. CONCLUSIONS

We have found an effective Hamiltonian describing interaction between two subsystems beyond the standard RWA. We have shown that this effective Hamiltonian has the form of a series of resonances. We have found and classified all the possible resonance transitions which can take place in a system described by the Hamiltonian (1).

If the system parameters (bare subsystem frequencies) satisfy a resonant condition, Eq. (7), with some specific values of (k,l), only the corresponding resonant term in the effective Hamiltonian (4) survives. It means that in the course of evolution some specific finite-dimensional subspaces in the Hilbert space of the whole system are approximately preserved. For each of these invariant subspaces there exists a corresponding integral of motion. The structure of the effective Hamiltonian (4) essentially depends on the algebraic structure of interacting subsystems [polynomials $\phi_1(X_0), \phi_2(Y_0)$]. In particular, it is reflected in types of resonances which are essential for a given system. In the case of linear systems, when $\phi_2(Y_0)\sim Y_0$ and $\phi_1(X_0)\sim X_0$, no higher resonances appear and the entire effect of counterrotating terms reduces to a renormalization of frequencies and interaction constants. Nevertheless, for the simplest nonlinear system with $\phi_2(Y_0)\sim Y_0$ and $\phi_1(X_0)\sim aX_0^2+bX_0$, two (entire) series of higher-order resonances appear. In a more complicated case, when both $\phi_2(Y_0)$ and $\phi_1(X_0)$ are the second-order polynomials of its arguments, apart from the series of the entire resonances, a series of fractional resonances appears, although the higher-order resonances are suppressed.

On the other hand, if our system is far from any resonance, $\epsilon = g/(\omega_1 - \omega_2) \ll 1$, the effective Hamiltonian becomes diagonal,

$$H_{eff} \approx \omega_1 X_0 + \omega_2 Y_0 + g\varepsilon \Phi(X_0, Y_0) + \epsilon g \nabla_{X_0, -Y_0}[\phi_1(X_0)\phi_2(Y_0+1)],$$

where $\Phi(X_0, Y_0)$ is defined in Eq. (6) and the last term in the above expression results from the elimination of term describing the principal transition $\sim X_+ Y_- + \text{H.c.}$ (note that both diagonal terms can be of the same order of magnitude if we are far from the principal resonance). As expected, the dynamics in this case consists in phase shifts and does not lead to high amplitude transitions. In the examples considered in Sec. III, these "intensity-dependent" shifts have the following forms: $\sim \alpha n_b + \beta n_a$ for linear systems (Sec. III A); $\alpha(2n+1)S_z + \beta S_z^2$ for spin-oscillator interaction (Sec. III B); $\alpha S_{z1}(C_2^{(2)} - S_{z2}^2) + \beta S_{z2}(C_2^{(1)} - S_{z1}^2)$ for spin-spin interaction (Sec. III C), where $\alpha \sim g(\varepsilon + \epsilon), \beta \sim g(\varepsilon - \epsilon)$.

The resonance expansion (4) also could be useful to explain the origin of chaos in the classical counterparts of systems describe by the Hamiltonian (1). In particular, it might be expected that overlapping of different resonances, which takes place with growing ratio g/ω, leads to appearance of unstable and chaotic behavior in the classical region (see, e.g., Refs. [5,11] and for recent discussion see Ref. [34]).

APPENDIX A

In this appendix we briefly describe the method of the Lie-type transformations (which we call small rotations).

The method of small rotations [28,29] provides a *regular* procedure for obtaining approximate Hamiltonians describing effective dynamics of nonlinear quantum systems. The idea of this method is based on the observation that several quantum-optical Hamiltonians can be written in terms of polynomially deformed algebras $\text{sl}_{pd}(2,R)$ [18–27],

$$H_{\text{int}} = \Delta X_0 + g(X_+ + X_-), \tag{A1}$$

where the operators $X_\pm$ and X_0 are generators of the deformed algebra and satisfy the following commutation relations:

$$[X_0, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = P(X_0), \tag{A2}$$

where $P(X_0)$ is a polynomial function of the diagonal operator X_0 with coefficients that may depend on some integrals of motion N_j. If $P(X_0)$ is a linear function of X_0, the usual sl(2,R) or su(2) algebras are restored. If for some physical reason (depending on the particular model under consideration) $\eta = g/\Delta \ll 1$ is a small parameter, the Hamiltonian (A1) is *almost* diagonal in the basis of the eigenstates of X_0 and can be approximately diagonalized by applying in a perturbative manner the following unitary transformation (a *small nonlinear rotation*):

$$U = \exp[\eta(X_+ - X_-)]. \tag{A3}$$

Applying the transformation (A3) to the Hamiltonian (A1) according to the standard expansion

$$e^{\eta A} B e^{-\eta A} = \sum_{k=1}^{\infty} \frac{\eta^k}{k!} \text{ad}_A^k(B),$$

where ad_A is the adjoint operator defined as $\text{ad}_A(B) = [A,B]$, we obtain

$$H_{\text{eff}} = U H_{\text{int}} U^\dagger = \Delta X_0 + g \sum_{k=1}^{\infty} \eta^k \frac{k}{(k+1)!} \text{ad}_T^k (X_+ + X_-), \tag{A4}$$

where $T = X_+ - X_-$ and we have taken into account that, due to Eq. (A2),

$$\text{ad}_T(X_0) = [T, X_0] = -(X_+ + X_-).$$

The effective Hamiltonian acquires the following form:

$$H_{\text{eff}} = \Delta X_0 + g\eta \sum_{k=0}^{\infty}{}' \eta^k [X_+^k f_k(X_0, \eta) + \text{H.c.}],$$

where $f_k(X_0, \eta)$ is a function of the diagonal operator X_0 and can be represented as a series on η:

$$f_k(X_0, \eta) = \frac{2(k+1)}{(k+2)!} \nabla^{k+1} \phi(X_0) + O(\eta),$$

and $\phi(X_0) = X_+ X_-$ is a structural function, $\nabla \phi(X_0) = \phi(X_0) - \phi(X_0 + 1) = P(X_0)$; the prime in the above sum means that the term with $k=0$ is taken with the coefficient 1/2.

By keeping terms up to order η we get

$$H_{\text{eff}} = \Delta X_0 + \eta g \nabla \phi(X_0), \tag{A5}$$

and in the first approximation the resulting effective Hamiltonian is diagonal on the basis of eigenstates of X_0.

The higher-order contributions always have the form $X_+^k f_k(X_0) + \text{H.c.} + g(X_0)$. This makes the procedure of removing the off-diagonal terms somehow trivial at each step, in the sense that it is always obvious which transformation should be applied. For example, to eliminate the terms of the form

$$\eta^l [X_+^k f_k(X_0) + f_k(X_0) X_-^k],$$

it suffices to apply the transformation

$$\exp\left(\frac{\eta^{l+1}}{k} T_k\right), \tag{A6}$$

with $T_k = X_+^k f_k(X_0) - f_k(X_0) X_-^k$, since the first commutator of ΔX_0 with T_k cancels the corresponding term in the Hamiltonian.

Because the transformed Hamiltonian has a form of an expansion in the small parameter η, it is clear that its eigenvalues coincide with those obtained using the standard perturbation theory. The advantage of this method lies in obtaining *effective Hamiltonians* in an operator form, which avoids the necessity of calculating cumbersome series of matrix elements.

This technique also provides corrections to the eigenstates of the Hamiltonian (A1). Indeed, it is easy to see that the eigenstates of the interaction Hamiltonian (A1) can be approximated as

$$|\Psi_m\rangle = U^\dagger |m\rangle,$$

where $|m\rangle$ is an eigenstate of the diagonal operator X_0 and U is the corresponding small rotation. Since the rotation operators and $|m\rangle$ are time independent, the operator U can be applied to $|m\rangle$ in the form of an expansion in η. For example, the eigenstate $|\Psi_m\rangle$ up to order η^2 takes the form

$$|\Psi_m\rangle = |m\rangle - \eta(X_+ - X_-)|m\rangle + \frac{\eta^2}{2}\{(X_+^2 + X_-^2) - [\phi(X_0) + \phi(X_0+1)]\}|m\rangle.$$

This representation is especially advantageous if we construct the space of states of the model as a representation space of the deformed su(2) algebra using the raising operator X_+ [18–23]: $|m\rangle \propto X_+^m |0\rangle$, where $|0\rangle$ is the lowest weight vector fulfilling the standard condition $X_-|0\rangle = 0$.

APPENDIX B

In this appendix we show how the method of small rotations can be applied to the systems which describe virtual processes (where the total number of excitations does not conserve).

The method of small rotations is useful when the interaction Hamiltonian contains some “big” parameters and just a small rotation is needed to transform it into the form which explicitly reveals the existence of an approximate integral of motion. Since generic Hamiltonians describing interactions without the RWA naturally contain a big parameter (the relation between proper transition frequencies of the system and the Rabi frequency), the method of small rotations can be directly applied to this kind of systems leading to essential simplifications in their description. According to this method and taking into account the commutation relations (2) we can eliminate the counterrotating term $X_+Y_+ + X_-Y_-$ from the Hamiltonian (1) by applying the transformation

$$U_1 = \exp[\varepsilon(X_+Y_+ - X_-Y_-)],$$

where

$$\varepsilon = \frac{g}{\omega_1 + \omega_2} \ll 1 \tag{B1}$$

is a small parameter. The resulting Hamiltonian acquires the following form:

$$H_1 = U_1 H U_1^\dagger, \tag{B2}$$

$$H_1 = \omega_1 X_0 + \omega_2 Y_0 + \varepsilon g {\sum_{n=0}^{\infty}}' \varepsilon^n [(X_+Y_+)^n F_n(X_0,Y_0,\varepsilon) + \text{H.c.}] + g \sum_{n=0}^{\infty} \varepsilon^n [X_+Y_-\{(X_+Y_+)^n H_n(X_0,Y_0,\varepsilon) + (X_-Y_-)^n G_n(X_0,Y_0,\varepsilon)\} + \text{H.c.}],$$

where $F_n(X_0,Y_0,\varepsilon)$, $H_n(X_0,Y_0,\varepsilon)$, $G_n(X_0,Y_0,\varepsilon)$ are diagonal operators which can be represented as a series in a small parameter ε:

$$F_n(X_0,Y_0,\varepsilon) = \frac{2(n+1)}{(n+2)!} \boldsymbol{\nabla}^n_{X_0,Y_0} \Phi(X_0,Y_0) + O(\varepsilon),$$

$$\Phi(X_0,Y_0) = \phi_1(X_0)\phi_2(Y_0) - \phi_2(Y_0+1)\phi_1(X_0+1) + O(\varepsilon),$$

$$H_n(X_0,Y_0,\varepsilon) = \frac{1}{n!}\frac{\boldsymbol{\nabla}^n_{Y_0}\phi_2(Y_0)}{\phi_2(Y_0+n)} + O(\varepsilon),$$

$$G_n(X_0,Y_0,\varepsilon) = \frac{1}{n!}\frac{\boldsymbol{\nabla}^n_{X_0}\phi_1(X_0-n+1)}{\phi_1(X_0-n+1)} + O(\varepsilon), \tag{B3}$$

and $\boldsymbol{\nabla}_{X_0,Y_0}\Phi(X_0,Y_0) = \Phi(X_0,Y_0) - \Phi(X_0+1,Y_0+1)$; the prime in the first sum in Eq. (B2) means that the term with $n=0$ is taken with the coefficient 1/2.

The Hamiltonian (B2) still has quite complicated structure and can be simplified under certain resonance conditions. It is easy to observe that all the terms of the form

$$a_n[(X_+Y_+)^n F_n(X_0,Y_0,\varepsilon) + \text{H.c.}] \tag{B4}$$

can be eliminated (in a leading order on ε) by applying transformations

$$U_{11} = \exp\left[\frac{a_n}{n}[(X_+Y_+)^n F_n(X_0,Y_0,\varepsilon) - \text{H.c.}]\right], \quad n \geq 1, \tag{B5}$$

where $a_n \sim \varepsilon^{n+1}$. In a similar way, the terms

$$X_+Y_-[(X_+Y_+)^n H_n(X_0,Y_0,\varepsilon) + (X_-Y_-)^n G_n(X_0,Y_0,\varepsilon)] + \text{H.c.}$$

can be removed from Eq. (B2). The elimination of the above terms leads to the appearance of new elements in the transformed Hamiltonian. The new terms can be divided into three groups: The first group contains terms that can always be eliminated under the condition $\omega_2, \omega_1 \gg g$ by applying some suitable transformations. The second group consists of terms that cannot be removed if certain relations between ω_1 and ω_2 hold, since the rotation which eliminates a given term from the Hamiltonian becomes singular. This group contains certain nondiagonal terms which describe transitions between energy levels of the whole system. For example, the term $X_+Y_- + \text{H.c.}$ cannot be eliminated from the Hamiltonian if $\omega_1 \rightarrow \omega_2$, because the corresponding transformation, having the form $\exp[g(X_+Y_- - \text{H.c.}_-)/(\omega_1 - \omega_2)]$, becomes singular. The third group includes the diagonal terms (functions only of X_0, Y_0). We will keep in the Hamiltonian only diagonal terms and those that cannot be eliminated by nonsingular transformations (resonant terms). Besides, we conserve only the leading-order coefficients in these terms. The resonant terms appear as a consequence of transformation of the terms $X_+ Y_-(X_+ Y_+)^k H_k(X_0, Y_0, \varepsilon) + \text{H.c.}$ with

$$U_{2n} = \exp\left[-\frac{g\varepsilon^n}{(n-1)\omega_1 + (n+1)\omega_2} \times [X_+ Y_-(X_- Y_-)^n G_n(X_0, Y_0, \dot{\varepsilon}) - \text{H.c.}] \right]. \tag{B6}$$

[which is applied to eliminate terms $X_+ Y_-(X_- Y_-)^n G_n(X_0, Y_0, \varepsilon) + \text{H.c.}$]. The leading-order terms come from transformations (B6) with $n=1$ and by taking only leading terms in the expansions (B3):

$$U_{21} = \exp[-\delta\varepsilon(Y_-^2 \nabla_{X_0} \phi_1(X_0) - \text{H.c.})],$$

where $\delta = g/2\omega_2$. As a result, we obtain the following effective Hamiltonian (4).

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