Entanglement sharing in the two-atom Tavis-Cummings model

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Individual members of an ensemble of identical systems coupled to a common probe can become entangled with one another, even when they do not interact directly. We investigate how this type of multipartite entanglement is generated in the context of a system consisting of two two-level atoms resonantly coupled to a single mode of the electromagnetic field. The dynamical evolution is studied in terms of the entanglements in the different bipartite partitions of the system, as quantified by the I tangle. We also propose a generalization of the so-called residual tangle that quantifies the inherent three-body correlations in our tripartite system. This enables us to completely characterize the phenomenon of entanglement sharing in the case of the two-atom Tavis-Cummings model, a system of both theoretical and experimental interest.

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I. INTRODUCTION

The control of quantum systems through active measurement and feedback has been developing at a rapid pace. In a typical scenario, a single atom is monitored indirectly through its coupling to a traveling probe such as a laser beam. The scattered beam and the system become correlated, and a subsequent measurement of the probe leads to backaction on the system. A coherent drive applied to the system can then be made conditional on the measurement record, leading to a closed-loop control model [1,2]. Such a protocol has been implemented to control a single-mode electromagnetic field in a cavity [3], and has been envisioned for controlling a variety of systems such as the state of a quantum dot in a solid [4], the state of an atom coupled to a cavity mode [5], and the motion of a micromechanical resonator coupled to a Cooper pair charge box [6].

A common theme in the examples given above is that measurements are made on *single* copies of the quantum system of interest. However, in many situations one does not have access to an individually addressable system. In a gas, for example, preparing and/or addressing individual atoms is extremely difficult. In situations such as this, it is useful to think of the entire ensemble as a single many-body system. Indeed, recent experiments [7,8] and theoretical proposals [9] have explored the control of such ensembles from the point of view of the Dicke model [10], where a collection of N two-level atoms is treated as a pseudospin with $J=N/2$.

Measurement backaction on the pseudospin can lead to squeezing of the quantum fluctuations [7–9], which may be enhanced through active closed-loop control [1,2]. This squeezing can reduce the quantum fluctuations of an observable as in, for example, the reduction of ''projection noise'' leading to enhanced precision measurements in an atomic clock [11]. Moreover, spin squeezing is related to quantum entanglement between the atomic members of the ensemble [12]. This entanglement arises not through direct interaction between the atoms, but through their coupling to a common ''quantum bus'' in the form of an applied probe.

Measures of entanglement associated with these spin-squeezed states have been studied by Stockton *et al.* [10] under the assumption that all of the atoms in the ensemble are symmetrically coupled to the bus. However, completely quantifying entanglement in the most general cases is extremely difficult, and as yet, an unsolved problem [13]. In this paper we consider the simplest possible ensemble consisting of two two-level atoms. Although at first sight this might appear trivial, when such a system is coupled to a quantum bus a rich structure emerges. Again, we consider the simplest realization of the bus—a single-mode quantized electromagnetic field. The resulting physical system then corresponds to the two-atom Tavis-Cummings model (TCM) [14]. A thorough understanding of the dynamical evolution of the TCM has obvious implications for the performance of quantum information processing [15–17], as well as for our understanding of fundamental quantum mechanics [15,18]. Bipartite entanglement has been investigated in this system for the one-atom case, known as the Jaynes-Cummings model, for initial pure states [19] and mixed states [20,21] of the field.

Taken as a whole, the two-atom TCM in an overall pure state constitutes a *tripartite* quantum system in a Hilbert space with tensor product structure $2\otimes 2\otimes\infty$. Entanglement in tripartite systems has been studied by Coffman *et al.* [22] for the case of three qubits. They found that such quantum correlations cannot be arbitrarily distributed amongst the subsystems; the existence of three-body correlations constrains the distribution of the bipartite entanglement which remains after tracing over any one of the qubits. For ex-

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ample, in a GHZ state, $|\mathrm{GHZ}\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$, tracing over one qubit results in a maximally mixed state containing no entanglement between the remaining two qubits. In contrast, for a *W* state, $|W\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$, the average remaining bipartite entanglement is maximal [23]. Coffman *et al.* analyzed this phenomenon of *entanglement sharing* [22], using an entanglement monotone known as the tangle, a simple generalization of the more familiar concurrence [24,25]. They also introduced a new quantity known as the residual tangle, in order to quantify the irreducible tripartite correlations in a three-qubit system [22].

In this paper we extend the analysis of entanglement sharing to the case of the two-atom TCM. This has implications for the study of quantum control of ensembles. For example, if we imagine that the quantum bus is measured, e.g., the field leaks out of the cavity and is then detected, then the degree of correlation between the field and one of the atoms determines the degree of backaction on *one* atom. We can then quantify the degree to which one can perform quantum control on a single member of an ensemble even when one couples only *collectively* to the entire ensemble. We will accomplish this by extending the residual tangle formalism of Coffman *et al.* to our $2 \otimes 2 \otimes \infty$ system.

The remainder of this paper is organized as follows. The important features of the TCM are reviewed in Sec. II, and the applicable measures of entanglement are introduced in Sec. III. With this formalism in hand, we calculate the tangle for each of the bipartite partitions of this tripartite system in Sec. IV. We will find an approximate analytic expression for the tangle between the field and the ensemble in the limit of large average photon number and in the Markoff approximation which provides further insight into these results. In Sec. V, we study the irreducible tripartite correlations in the system using our proposed generalization of the residual tangle. Finally, we summarize our results in Sec. VI.

II. THE TAVIS-CUMMINGS MODEL

The Tavis-Cummings model (TCM) [14] (also known as the ''Dicke model'' [26]) describes the simplest fundamental interaction between a single mode of the quantized electromagnetic field and a collection of *N* atoms under the usual two-level and rotating wave approximations. The two-atom ($N=2$) TCM is governed by the Hamiltonian

$$\begin{aligned} H &= H_0 + H_{int} \\ &= \hbar\omega(a^\dagger a + \tfrac{1}{2}\sigma_z^{(1)} + \tfrac{1}{2}\sigma_z^{(2)}) \\ &\quad + \hbar g[(\sigma_-^{(1)} + \sigma_-^{(2)})a^\dagger + (\sigma_+^{(1)} + \sigma_+^{(2)})a], \end{aligned} \tag{1}$$

where $\sigma_\pm^{(i)}$ and $\sigma_z^{(i)}$ display a local su(2) algebra for the *i*th atom in the two-dimensional subspace spanned by the ground and excited states $\{|g\rangle, |e\rangle\}$, and $a(a^\dagger)$ is bosonic annihilation (creation) operator for the monochromatic field. The Hilbert space $\mathcal{H}$ of the joint system is given by the tensor product $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_F$ where $\mathcal{H}_{A_1}(\mathcal{H}_{A_2})$ denotes the Hilbert space of atom one (two) and $\mathcal{H}_F$ is the Hilbert space of the electromagnetic field.

The total number of excitations $K = a^\dagger a + \frac{1}{2}(\sigma_z^{(1)} + \sigma_z^{(2)} + 2)$ is a conserved quantity which allows one to split the Hilbert space $\mathcal{H}$ into a direct sum of subspaces, i.e., $\mathcal{H} = \Sigma_{K=0}^{\infty} \oplus \Omega_K$, with each subspace Ω_K spanned by the eigenvectors $\{|ee,k-2\rangle, |eg,k-1\rangle, |ge,k-1\rangle, |gg,k\rangle\}$ of K with eigenvalue *k*. The analytic form for the time evolution operator within a subspace Ω_K is given in Zeng *et al.* [27].

It is assumed throughout that the initial state of the TCM system is pure. Furthermore, we consider only the effects of the unitary evolution generated by Eq. (1), i.e., we do not include the effects of measurement, nor of mixing due to environment-induced decoherence [28], so that our system remains in an overall pure state at all times. Finally, by assuming an identical coupling constant *g* between each of the atoms and the field, the Hamiltonian is symmetric under atom exchange. This invariance under the permutation group was used by Stockton *et al.* [10] to analyze the entanglement properties of very large ensembles. We will also make use of this fact in order to reduce the number of different partitioning schemes that one needs to consider when studying entanglement sharing in the two-atom TCM.

III. THE TANGLE FORMALISM

The tangle between two qubits in an arbitrary state is defined in terms of the concurrence [24,25]. For a pure state $|\psi\rangle$ of two qubits, the concurrence is given by $C(\psi) \equiv |\langle\psi|\tilde{\psi}\rangle|$, where $|\tilde{\psi}\rangle$ represents the ''spin flip'' of $|\psi\rangle$, i.e., $|\tilde{\psi}\rangle \equiv \sigma_y \otimes \sigma_y |\psi^*\rangle$, and the asterisk denotes complex conjugation in the standard basis.

The generalization of the concurrence to a mixed state ρ of two qubits is defined as the infimum of the average concurrence over all possible pure state ensemble decompositions of ρ, defined as convex combinations of pure states $S_i = \{p_i, \psi_i\}$, such that $\rho = \Sigma_i p_i |\psi_i\rangle\langle\psi_i|$. In this way, $C(\rho) \equiv \inf_{S_i} \Sigma_i p_i C(\psi_i) = \inf_{S_i} \Sigma_i p_i |\langle\psi_i|\tilde{\psi}_i\rangle|$. Wootters succeeded in deriving an analytic solution to this difficult minimization procedure in terms of the eigenvalues of the non-Hermitian operator $\rho\tilde{\rho}$, where the tilde again denotes the spin flip of the quantum state. The closed-form solution for the concurrence of a mixed state of two qubits is given by $C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where the λ_i's are ordered in decreasing order [25]. Rungta *et al.* extended this formalism by introducing an analytic form for the concurrence of a bipartite system *AB*, with *arbitrary* dimensions D_A and D_B, in an overall pure state [29]. The analytic form of this quantity, dubbed the *I* concurrence, is given by $C(\psi_{AB}) = \sqrt{2\nu_A\nu_B[1 - \mathrm{tr}(\rho_A^2)]}$, where ρ_A is the marginal density operator obtained by tracing the joint pure state over subsystem *B*, and ν_A and ν_B are arbitrary scale factors.

Coffman *et al.* defined the tangle τ_2 for a system of two qubits as the square of the concurrence [22], i.e.,

$$\tau_2(\rho) \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}^2. \tag{2}$$

Indeed, we may extend this definition directly to the result of Rungta *et al.* in order to obtain an analytic form for the tangle τ of a bipartite system in a pure state with arbitrary subsystem dimensions,

$$\tau(\psi_{AB}) \equiv C^2(\psi_{AB}) = 2\nu_A\nu_B[1-\mathrm{tr}(\rho_A^2)]. \tag{3}$$

However, when extending this definition to apply to a bipartite *mixed state* ρ_{AB} with arbitrary subsystem dimensions, one must find the infimum of the average squared pure state concurrence [30],

$$\tau(\rho_{AB}) \equiv \inf_{S_i} \sum_i p_i C^2(\psi_{AB}^{(i)})$$

$$= 2\nu_A\nu_B \inf_{S_i} \sum_i p_i\{1-\mathrm{tr}[(\rho_A^{(i)})^2]\}, \tag{4}$$

where we have used Eq. (3) for the pure state tangle with $\rho_A^{(i)}$ as the marginal state for the ith term in the ensemble decomposition.

At this point we note that the scale factors ν_A and ν_B in Eqs. (3) and (4), which may in general depend on the dimensions of the subsystems D_A and D_B respectively, are usually set to one so that agreement with the two-qubit case is maintained, and so that the addition of extra unused Hilbert-space dimensions has no effect on the value of the concurrence [29]. We will find in Sec. V, when we attempt our own further generalization of the tangle formalism, that it is useful to take advantage of this scale freedom. For now, however, we adopt the usual convention both for the sake of clarity and to demonstrate exactly where in our proposed generalization this freedom is required.

Using the definition given by Eq. (4), dubbed the *I* tangle in reference to the work of Rungta *et al.*, Osborne derived an analytic form for $\tau(\rho_{AB})$ in the case where the rank of ρ_{AB} is no greater than 2,

$$\tau(\rho_{AB}) = \mathrm{tr}(\rho_{AB}\tilde{\rho}_{AB}) + 2\lambda_{min}^{(AB)}[1-\mathrm{tr}(\rho_{AB}^2)], \tag{5}$$

where $\tilde{\rho}_{AB}$ now represents the universal inversion [29] of ρ_{AB} and $\lambda_{min}^{(AB)}$ is the smallest eigenvalue of the M matrix defined by Osborne [30]. The important point is that Eq. (5) yields a closed form which, as we will see in Sec. IV C, is directly applicable to a specific bipartite partition of the two-atom TCM.

IV. BIPARTITE TANGLES IN THE TWO-ATOM TCM

Let the two atoms in the ensemble be denoted by A_1 and A_2, respectively, and the field, or quantum bus, by F. Because of the assumed exchange symmetry, there are four nonequivalent partitions of the two-atom TCM into tensor products of bipartite subsystems: (i) the field times the two-atom ensemble, $F\otimes(A_1A_2)$; (ii) one atom times the remaining atom and the field, $A_1\otimes(A_2F)\equiv A_2\otimes(A_1F)$; (iii) the two atoms taken separately, having traced over the field, $A_1\otimes A_2$; and (iv) one of the atoms times the field, having traced over the other atom, $A_1\otimes F\equiv A_2\otimes F$. We calculate how the tangle for each of these partitions evolves as a function of time under TCM Hamiltonian evolution using the formalism reviewed in Sec. II. We take the initial state to be a pure product state of the field with the atoms. We will capture the key features of the tangle evolution by considering three classes of initial state vectors:

$$|e\rangle_{A_1}\otimes|e\rangle_{A_2}\otimes|n\rangle_F \equiv |ee,n\rangle, \tag{6a}$$

$$|ee,\alpha\rangle \quad \text{or} \quad |gg,\alpha\rangle, \tag{6b}$$

and

$$\frac{1}{\sqrt{2}}(|eg\rangle+|ge\rangle)\otimes|\alpha\rangle \quad \text{or} \quad \frac{1}{\sqrt{2}}(|gg\rangle+|ee\rangle)\otimes|\alpha\rangle, \tag{6c}$$

where $|g(e)\rangle$ denotes the ground (excited) state of the atom, $|n\rangle$ denotes a Fock state field with n photons, and $|\alpha\rangle$ denotes a coherent-state field with an average number of photons given by $\langle n\rangle$. The alternatives in Eqs. (6b) and (6c) arise from the fact that, in the limit of large $\langle n\rangle$, the evolution of *all* the tangles are found to be identical for the two different initial atomic conditions, as we will see below.

A. Field-ensemble and one-atom-remainder tangles

Under the assumption that the system is in an overall pure state, we may easily calculate the tangles in partitions (i) and (ii) above by applying Eq. (3), with $\nu_A=\nu_B=1$. Specifically,

$$\tau_{F(A_1A_2)} = 2[1-\mathrm{tr}(\rho_F^2)] = 2[1-\mathrm{tr}(\rho_{A_1A_2}^2)] \tag{7}$$

and

$$\tau_{A_1(A_2F)} = 2[1-\mathrm{tr}(\rho_{A_1}^2)] = 2[1-\mathrm{tr}(\rho_{A_2F}^2)], \tag{8}$$

where we have used the fact that the (nonzero) eigenvalue spectra of the two marginal density operators for a bipartite division of a pure state are identical [15,31] in obtaining the rightmost equalities. These tangles have implications for the quantum control of atomic ensembles. Because the overall system is pure, any correlation between the field and the ensemble is necessarily in one-to-one correspondence with the amount of entanglement between these two subsystems. The quantum backaction on the ensemble due to measurement of the field is thus quantified by Eq. (7). Alternatively, a measurement of one atom leads to backaction on the remaining subsystem as described by Eq. (8).

The time evolutions for each of the different tangles, corresponding to the initial conditions given by Eqs. (6a)–(6c), are shown in Figs. 1(a)–3(a), respectively. Figures 1(b)–3(b) show the time evolution of the atomic inversion, defined as the probability of finding both atoms in the excited state minus the probability of finding both atoms in the ground state, for reference purposes. We find, under certain conditions, that the two stretched states in Eq. (6b) lead to identical evolution for the tangles in all of the bipartite partitions of the system, corresponding to the curves shown in Fig. 2(a). Similarly, the two symmetric states given in Eq. (6c) both yield the curves shown in Fig. 3(a). This behavior can be derived under a set of highly accurate approximations. In the limit of large average photon number, an initial coherent-state field with zero phase will remain approximately sepa-

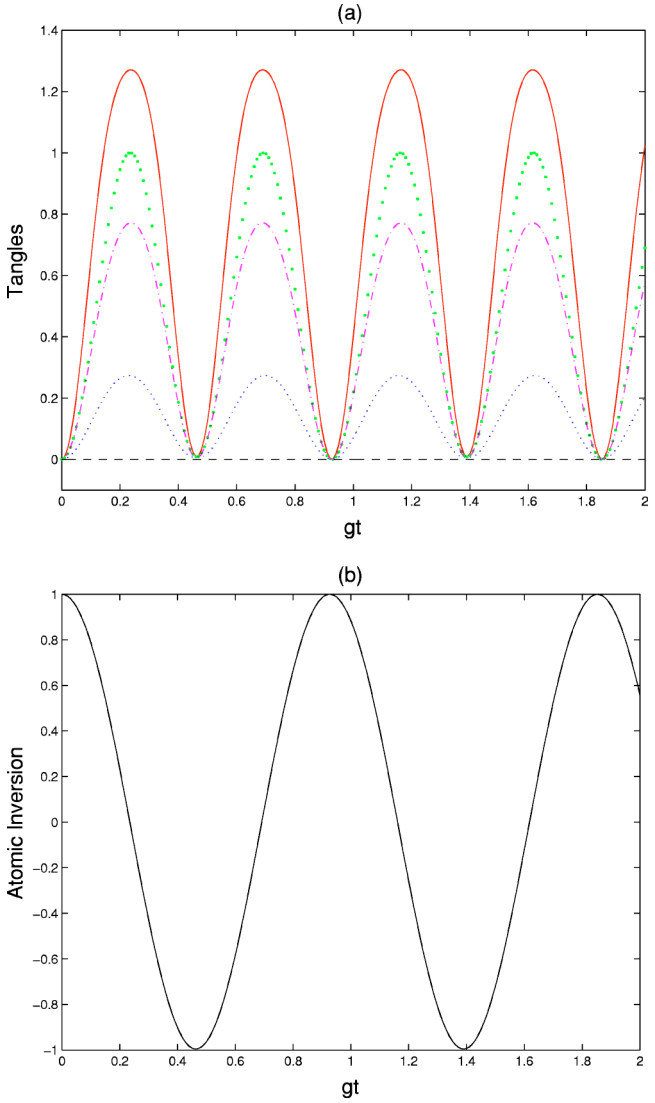

FIG. 1. (Color online) TCM evolution for both atoms initially in the excited state and the field in an initial Fock state with $n=10$. (a) Solid curve (red), field-ensemble tangle $\tau_{F(A_1A_2)}$; large-dotted curve (green), one-atom-remainder tangle $\tau_{A_1(A_2F)}$; dashed curve (black), atom-atom tangle $\tau_{A_1A_2}$ (note that the atom-atom tangle is always zero for the given initial condition); dot-dashed curve (pink), single-atom-field tangle τ_{A_1F}; and dotted curve (blue), residual tangle $\tau_{A_1A_2F}$. (b) Atomic inversion of the ensemble.

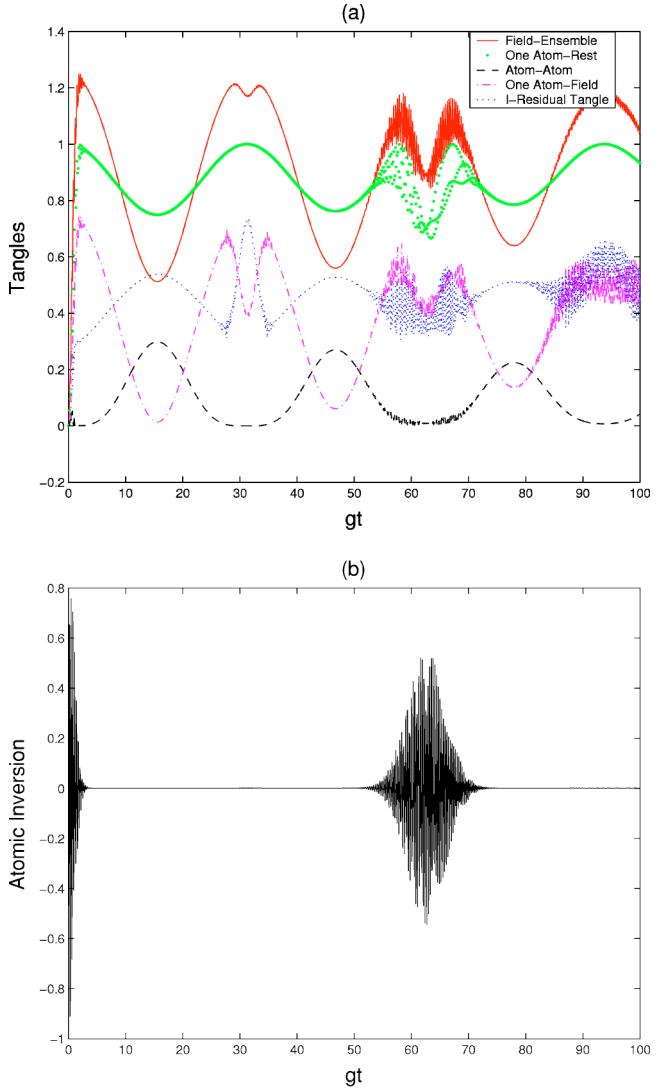

FIG. 2. (Color online) TCM evolution for both atoms initially in a stretched state and the field in an initial coherent state with $\langle n\rangle = 100$. (a) Solid curve (red), field-ensemble tangle $\tau_{F(A_1A_2)}$; large-dotted curve (green), one-atom-remainder tangle $\tau_{A_1(A_2F)}$; dashed curve (black), atom-atom tangle $\tau_{A_1A_2}$; dot-dashed curve (pink), single-atom-field tangle τ_{A_1F}; and dotted curve (blue), residual tangle $\tau_{A_1A_2F}$. (b) Atomic inversion of the ensemble.

rable from the atomic ensemble in an eigenstate of $J_x \equiv J_+ + J_-$ up to times of the order of $\langle n\rangle/g$ where, in the pseudospin picture, $J_\pm \equiv \sigma_\pm^{(1)} + \sigma_\pm^{(2)}$. This follows immediately from the time evolution operator generated by H_{int} in Eq. (1) in the interaction picture. The key observation is that, for a macroscopic field, the removal or addition of a single photon has a negligible effect. This allows one to approximate the time evolution operator by $\exp^{-iH_{int}t/\hbar} \approx \exp^{-ig\sqrt{\langle n\rangle}J_x t}$. Thus, the eigenstates of J_x form a convenient basis to use in describing the state of the atomic ensemble. This approach was taken by Gea-Banacloche in analyzing the behavior of the single-atom Jaynes-Cummings model [32] and the generation of macroscopic superposition states [33], and extended to the multiatom TCM by Chumakov *et al.* [34–36].

We take as the appropriate basis the three symmetric eigenstates of J_x, which we label by $m=-1$, 0, and 1; the singlet state $J=0$ is a dark state and thus does not couple to the field. Writing the initial state of the system as

$$|\psi(0)\rangle = \sum_{m=-1}^{1} d_m |m\rangle \otimes |\alpha\rangle, \tag{9}$$

and using the factorization approximation [34], we find that the state of the system up to times of the order of $\langle n\rangle/g$ is given by

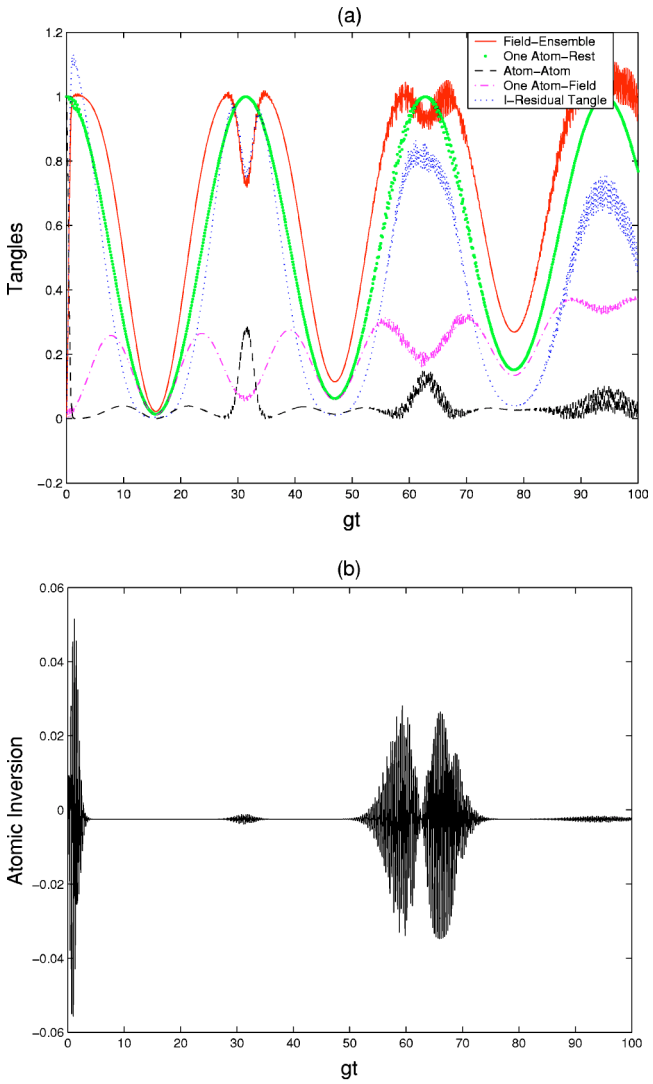

FIG. 3. (Color online) TCM evolution for the atoms initially in one of the symmetric states and the field in an initial coherent state with $\langle n\rangle = 100$. (a) Solid curve (red), field-ensemble tangle $\tau_{F(A_1A_2)}$; large-dotted curve (green), one-atom-remainder tangle $\tau_{A_1(A_2F)}$; dashed curve (black), atom-atom tangle $\tau_{A_1A_2}$; dot-dashed curve (pink), single-atom-field tangle τ_{A_1F}; and dotted curve (blue), residual tangle $\tau_{A_1A_2F}$. (b) Atomic inversion of the ensemble.

$$|\psi(t)\rangle \approx \sum_{m=-1}^{1} d_m |A_m(t)\rangle \otimes |\phi_m(t)\rangle, \tag{10}$$

where $|A_m(t)\rangle$ and $|\phi_m(t)\rangle$ are the time-evolved atomic and field states, respectively. The marginal density operator for the two atoms is then

$$\rho_{A_1A_2}(t) \approx \sum_{l,m} d_l^* d_m |A_m(t)\rangle\langle A_l(t)| f_{ml}(gt,\langle n\rangle), \tag{11}$$

where $f_{ml}(gt,\langle n\rangle) \equiv \Sigma_n \langle n|\phi_m(t)\rangle\langle\phi_l(t)|n\rangle$. We find that this function has "memory" only for $t \ll \sqrt{\langle n\rangle}/g$, and behaves very much like a δ function for longer time scales. Effectively, the large dimensional Hilbert space of the field acts as a broadband reservoir for the atoms—the generalization of the familiar "collapse" phenomenon in the Jaynes-Cummings model. This "Markoff" approximation is valid up to times of the order of $2\pi\sqrt{\langle n\rangle}/g$, corresponding to the well-known revival time in the Jaynes-Cummings model [32]. Making this approximation in Eq. (11), the states $|A_m(t)\rangle$ act effectively as a "pointer basis" for decoherence [28] of the atomic density matrix, i.e.,

$$\rho_{A_1A_2}(t) \approx \sum_m |d_m|^2 |A_m(t)\rangle\langle A_m(t)|. \tag{12}$$

Substituting this formula into Eq. (7) yields

$$\tau_{F(A_1A_2)}(t) \approx 2\{1 - \tfrac{1}{4}[c - h(t')]\}, \tag{13}$$

where

$$c \equiv 4(|d_{-1}|^4 + |d_0|^4 + |d_1|^4) + 2|d_0|^2|d_1|^2 + |d_{-1}|^2(2|d_0|^2 + 3|d_1|^2) - 4|d_{-1}|^2|d_1|^2, \tag{14}$$

$$h(t') \equiv 2|d_0|^2(|d_{-1}|^2 + |d_1|^2)\cos(4t') + |d_{-1}|^2|d_1|^2\cos(8t'), \tag{15}$$

and

$$t' \equiv \frac{gt}{2\sqrt{\langle n\rangle - \frac{N}{2} + \frac{1}{2}}}. \tag{16}$$

Under the factorization and Markoff approximations, the field-ensemble tangle is given by a constant term c that depends only on the *initial* probabilities to find the atomic ensemble in each of the J_x eigenstates, and a time-dependent piece $h(t')$. These probabilities depend solely on the absolute squares of the expansion coefficients of the initial atomic state given by Eq. (9). It is now clear why certain initial atomic conditions result in identical evolution for the different tangles. For example, the initial atomic states $|gg\rangle$ and $|ee\rangle$ both satisfy

$$|d_{-1}| = |d_1| = \frac{1}{2} \quad \text{and} \quad |d_0| = \frac{1}{\sqrt{2}}, \tag{17}$$

corresponding to identical evolution for all of the tangles shown in Fig. 2(a). Similarly, the initial atomic states $1/\sqrt{2}(|eg\rangle + |ge\rangle)$ and $1/\sqrt{2}(|ee\rangle + |gg\rangle)$ both satisfy

$$|d_{-1}| = |d_1| = \frac{1}{\sqrt{2}} \quad \text{and} \quad |d_0| = 0, \tag{18}$$

corresponding to the curves shown in Fig. 3(a). More generally, this property holds for any class of initial states $|\psi^{(i)}(0)\rangle$ having the form of Eq. (9) such that $|d_m^{(i)}| = |d_m^{(j)}|$, $m \in \{-1,0,1\}$. One immediate consequence of this result is

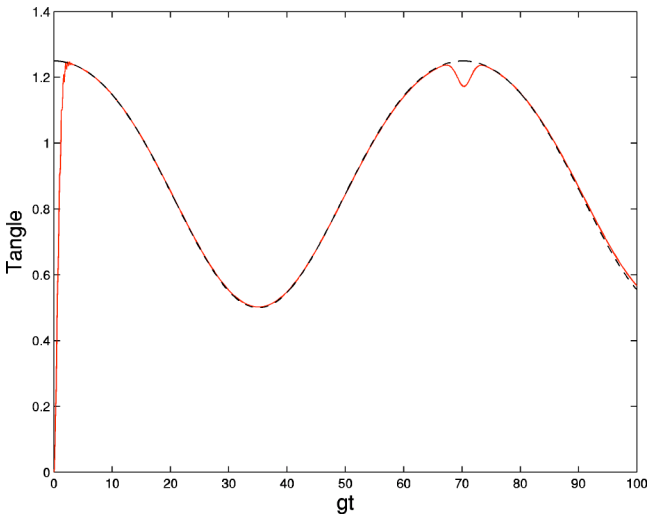

FIG. 4. (Color online) Exact field-ensemble tangle, solid (red) curve, and approximate formula, dashed (black) curve, as functions of time.

that the relative phase information encoded in the initial state of the atomic system is irrelevant to the evolution of the field-ensemble tangle.

The field-ensemble tangle calculated according to Eq. (7) and the approximation given by Eq. (13) for an initial stretched atomic state and an initial coherent-state field with $\langle n\rangle = 500$ are shown by the solid (red) and dashed (black) curves in Fig. 4, respectively. The approximation is seen to track the exact evolution extremely well over the range of its validity. The discrepancy at very small times is explained by the fact that at these times the Markoff approximation breaks down. It is also seen that our approximate solution does not capture the small dip in the field-ensemble tangle occurring at $t = \pi\sqrt{\langle n\rangle}/g$. The absence of this feature can be explained by noting that in making the Markoff approximation we have effectively wiped out any information regarding the initial coherence between the $|m=-1\rangle$ and $|m=1\rangle$ states. The presence of this dip is then seen to be dependent upon the existence of this coherence. This is borne out by the fact that the dip in the field-ensemble tangle in Fig. 2(a) is much shallower than that in Fig. 3(a), where the initial atomic expansion coefficients are given by Eqs. (17) and (18), respectively.

B. Atom-atom tangle

Given an initial state, we time evolve the system according to the dynamics governed by Eq. (1) and then trace over the field subsystem. The tangle of the two-atom mixed state $\rho_{A_1A_2}(t)$ may then be calculated according to Eq. (2). The resulting atom-atom tangles corresponding to the initial conditions in Eqs. (6a)–(6c) are depicted by the dashed (black) curves in Figs. 1(a)–3(a), respectively. These curves yield direct insight into the state of the atomic ensemble as a function of time. Specifically, the atom-atom tangle quantifies the degree to which the ensemble behaves as a collective entity, rather than as two individual particles.

It is somewhat surprising that for the initial condition given by Eq. (6a), i.e., when the field is initially in a *Fock state* with any value for n, the atom-atom tangle remains zero at all times, whereas the evolution of the atom-atom tangle resulting from an initial *coherent-state* field is nontrivial and, in general, nonzero. As a first step towards understanding these observations, we have studied the evolution of the atom-atom tangle for other initial conditions. When the field is initially in a Fock state and both atoms start in the ground state, the loss of an excitation in the field can result in the creation of an excitation in the atomic ensemble. This produces entanglement between the field and the ensemble and in the single-atom-field and one-atom-remainder partitions. Since it is not possible to distinguish in which atom the excitation is created, the two atoms become entangled with each other as well. It is found that the atom-atom entanglement falls off as $1/n^2$ so that, in the limit of a highly excited Fock state, these initial conditions yield results reminiscent of those found in Fig. 1(a). Specifically, we find that the entanglement in all of the different subsystem partitions always oscillate in phase at twice the Rabi frequency, and that the atom-atom tangle approaches zero as n becomes large.

Next, we considered the case when both atoms initially reside in a stretched state, and the initial field state consists of a coherent superposition of two neighboring Fock states. We find, on a time scale much longer than that given by the inverse of the associated Rabi frequencies, that the overall behavior again closely resembles the evolution seen for an initial field consisting of a single Fock state. Specifically, we find that the general features of all of the different bipartite tangles oscillate in phase with one another. However, on much shorter time scales, the effects of dephasing between the two Rabi frequencies become apparent, yielding the first clues regarding how the observed coherent-state behavior arises in terms of initial Fock state superpositions. For example, it seems likely that these observations provide insight into the Fock-like behavior seen in Fig. 2(a) at the revival time, when there is a partial rephasing of the Rabi frequencies. At this time all of the bipartite tangles decrease simultaneously, while at other times the tangles in certain bipartite partitions may be completely out of phase with one another. We are currently performing a detailed study of the entanglement that can be dynamically generated between the two atoms under TCM evolution for the most general initial conditions in order to better understand this behavior.

C. Single atom-field tangle

The final bipartite partition of the two-atom TCM is that consisting of a single atom, say A_1, as one subsystem and the field F as the second subsystem. Again by exchange symmetry $\tau_{A_1F} = \tau_{A_2F}$, so we need to calculate only one of these quantities. Because the tripartite system is in an overall pure state, the Schmidt decomposition theorem [15,31] implies that the marginal density operator ρ_{A_1F} has at most rank 2. The rank of the reduced density matrix is set by the dimension of the smallest subsystem, which in this case is a two-level atom. This is exactly the scenario envisioned by Osborne [30], as described in Sec. III. The tangle corresponding

to this partition, $A_1 \otimes F$, is computed by first tracing over the state of the remaining atom, A_2, and then applying Eq. (5).

Employing this procedure,

$$\tau_{A_1F} = \mathrm{tr}(\rho_{A_1F}\tilde{\rho}_{A_1F}) + 2\lambda_{min}^{(A_1F)}[1 - \mathrm{tr}(\rho_{A_1F}^2)], \qquad (19)$$

where $\lambda_{min}^{(A_1F)}$ represents the minimum eigenvalue of the Osborne M matrix [30] generated from the marginal density operator ρ_{A_1F}. The dot-dashed (pink) curves in Figs. 1(a)–3(a) give the time evolution of the single-atom-field tangle for the different initial conditions considered.

We are now in possession of closed forms for the tangles of all bipartite partitions of the two-atom TCM. Any other entanglement that the system may possess must necessarily be in the form of irreducible three-body quantum correlations. In Sec. V we review the *residual tangle* formalism introduced by Coffman *et al.* in order to quantify this type of tripartite entanglement in a system of three qubits. We then propose a generalization of this quantity that is applicable to a $2 \otimes 2 \otimes D$ system in an overall pure state. This extension of the tangle formalism allows us to study the phenomenon of entanglement sharing in the two-atom TCM.

V. ENTANGLEMENT SHARING AND THE RESIDUAL TANGLE

The concept of entanglement sharing studied in Refs. [22,37] refers to the fact that entanglement cannot be freely distributed among subsystems in a multipartite, i.e., tripartite or higher, system. Rather, the distribution of entanglement in these systems is subject to certain constraints. As a simple example, consider a tripartite system of three qubits labeled A, B, and C. Suppose that qubits A and B are known to be in a maximally entangled pure state, e.g., the singlet state, given by $|\psi_{AB}\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$ when written in the logical basis. In this case, it is obvious that the overall system ABC is constrained such that no entanglement may exist either between A and C or between B and C. Otherwise, tracing over subsystem C would necessarily result in a *mixed* marginal density operator for AB in contradiction to the known purity of the singlet state.

Coffman *et al.* analyze the phenomenon of entanglement sharing for a system of three qubits in an overall pure state in full generality by introducing a quantity known as the residual tangle [22]. This definition is motivated by the observation that the tangle of A with B plus the tangle of A with C cannot exceed the tangle of A with the joint subsystem BC, i.e.,

$$\tau_{AB} + \tau_{AC} \leqslant \tau_{A(BC)}. \qquad (20)$$

Here, τ_{AB} and τ_{AC} are calculated according to Eq. (2), and $\tau_{A(BC)}$ may be obtained from Eq. (3).

The original proof [22] of the inequality in Eq. (20), which forms the heart of the phenomenon of entanglement sharing for the case of three qubits, may be substantially simplified by making use of certain results due to Rungta *et al.* Specifically, we note that [29]

$$\mathrm{tr}(\rho_{xy}\tilde{\rho}_{xy}) = 1 - \mathrm{tr}(\rho_x^2) - \mathrm{tr}(\rho_y^2) + \mathrm{tr}(\rho_{xy}^2) \geqslant 0 \qquad (21)$$

for subsystems x and y having arbitrary Hilbert-space dimensions. Under the assumption that x and y are in an overall pure state with a third subsystem z, Eq. (21) may be rewritten

$$\mathrm{tr}(\rho_{xy}\tilde{\rho}_{xy}) = 1 - \mathrm{tr}(\rho_x^2) - \mathrm{tr}(\rho_y^2) + \mathrm{tr}(\rho_z^2) \geqslant 0, \qquad (22)$$

where we have used the equality of the nonzero eigenvalue spectra of ρ_{xy} and ρ_z. Then, by the observation [22] that for an arbitrary state of two qubits A and B, the following upper bound on the tangle defined by Eq. (2) holds:

$$\tau_2(\rho_{AB}) \leqslant \mathrm{tr}(\rho_{AB}\tilde{\rho}_{AB}), \qquad (23)$$

and by Eq. (3) with $\nu_A = \nu_B = 1$, the inequality in Eq. (20) follows immediately.

Subtracting the terms on the left-hand side of Eq. (20) from that on the right-hand side yields a positive quantity referred to as the residual tangle τ_{ABC}, i.e.,

$$\tau_{ABC} \equiv \tau_{A(BC)} - \tau_{AB} - \tau_{AC}. \qquad (24)$$

The residual tangle is interpreted as quantifying the inherent tripartite entanglement present in a system of three qubits, i.e., the entanglement that cannot be accounted for in terms of the various bipartite tangles. This interpretation is given further support by the observation that the residual tangle is invariant under all possible permutations of the subsystem labels [22].

We wish to generalize the residual tangle, defined for a system of three qubits, to apply to a $2 \otimes 2 \otimes D$ quantum system in an overall pure state so that we may study entanglement sharing in the two-atom TCM. Note that we already have all of the other tools needed for such an analysis. Specifically, from Sec. IV, we know the analytic forms for all of the different possible bipartite tangles in such a system.

Any proper generalization of the residual tangle must, at a minimum, be a positive quantity, and be equal to zero if and only if there is no tripartite entanglement in the system, i.e., if and only if all of the quantum correlations can be accounted for using only bipartite tangles. It should also reduce to the definition of the residual tangle in the case of three qubits. Further it is reasonable to require, if this is to be a true measure of irreducible three-body correlations, that symmetry under permutation of the subsystems be preserved, and that it remains invariant under local unitary operations. Finally, we conjecture that this quantity satisfies the requirements for being an entanglement monotone [38,39] under the set of stochastic local operations and classical communication (SLOCCs) or, equivalently, under the set of invertible local operations [23]. We limit the monotonicity requirement to this restricted set of operations since, in the context of entanglement sharing, we are only concerned with LOCCs that preserve the local ranks of the marginal density operators such that all subsystem dimensions remain constant.

Let A and B again be qubits, and let C now be a D-dimensional system with the composite system ABC in an overall pure state. We note that, under these assumptions, we are still capable of evaluating each of the terms on the right-

hand side of Eq. (24) analytically using the results of Sec. IV. However, we cannot simply use the definition of the residual tangle (with C now understood to represent a D-dimensional system) as the proper generalization for two reasons. First, since the three subsystems are no longer of equal dimension, symmetry under permutations of the subsystems is lost. However, as we will see, this problem is easily fixed by explicitly enforcing the desired symmetry. The second and more difficult problem to overcome is the fact that the inequality given by Eq. (20) no longer holds for our generalized system because λ_{min} in Eq. (5) can be negative, implying that Eq. (24) can also be negative.

The required permutation symmetry may be restored by taking our generalization of the residual tangle, which we dub the *I-residual tangle* in reference to previous work and continue to denote by τ_{ABC}, to be

$$\tau_{ABC} \equiv \tfrac{1}{3}(\tau_{A(BC)} + \tau_{B(AC)} + \tau_{C(AB)}) - \tfrac{2}{3}(\tau_{AB} + \tau_{AC} + \tau_{BC}). \tag{25}$$

The definition in Eq. (25) is obtained by averaging over all possible relabelings of the subsystems in Eq. (24). By inspection it is obvious that Eq. (25) preserves permutation symmetry. However, it still suffers from the problem that its value can be negative. In order to deal with this difficulty, we make use of the arbitrary scale factors appearing in Eqs. (3) and (4), and discussed in Sec. III.

Let d be the smaller of the two ''dimensions'' of two arbitrary dimensional subsystems x and y, i.e., $d \equiv \min\{D_x, D_y\}$. Note that by dimension we do not necessarily mean the total Hilbert-space dimension of the physical system under consideration, but only the number of different Hilbert-space dimensions *that contribute to the formation* of the overall pure state of the system. This is a subtle but important point which automatically enforces insights such as those due to Rungta *et al.* [29] and Verstraete *et al.* [40], which state that the scale chosen for a measure of entanglement must be invariant under the addition of extra, but unused, Hilbert-space dimensions. The two-atom TCM provides one example of the relevant physics underlying these ideas.

Consider, for example, the bipartite partitioning of the TCM into a field subsystem with $D_F = \infty$, and an ensemble subsystem consisting of the two qubits with $D_{A_1A_2} = 4$. Any entangled state of the overall system has a Schmidt decomposition with at most four terms, implying that the field effectively behaves like a four-dimensional system. Further, since the Tavis-Cummings Hamiltonian given by Eq. (1) does not induce couplings between the field and the singlet state of the atomic ensemble, i.e., the singlet state is a dark state, the field behaves effectively as a three-level system, or *qutrit*, in the context of the TCM. Accordingly, in any entangled state of the field with the ensemble, the field is considered to have a dimension no greater than three. We employ this revised definition of dimension throughout the remainder of the paper.

We now make the choice

$$\nu_A \nu_B = d/2, \tag{26}$$

when calculating each of the bipartite tangles appearing on the right-hand side of Eq. (25). This choice is made for several reasons. First of all, it is in complete agreement with the two-qubit case, yielding $\nu_A \nu_B = 1$ as required. Indeed, when A, B, and C are all qubits, the residual tangle given by Eq. (24) is recovered. Second, it takes differences in the Hilbert-space dimensions of the subsystems into account when setting the relevant scale for each tangle. This is important since, in order to study the phenomenon of entanglement sharing, the tangles for each of the different bipartite partitions must be compared on a common scale. It is reasonable that this scale be a function of the smaller of the two subsystem dimensions since, for an overall pure state, it is this quantity that limits the number of terms in the Schmidt decomposition. Finally, it is conjectured that a proper rescaling of the various tangles will result in the positivity of Eq. (25).

Note that when applying the proposed rescaling to the terms on the right-hand side of Eq. (25), the only term affected is $\tau_{C(AB)}$, which is rescaled by one-half of the smaller of the two subsystem dimensions D_C and D_{AB}. Each of the other terms remains unaltered since, in each case, at least one of the two subsystems involved is a qubit. The net effect of this rescaling is to increase the ''weight'' of the tangle between C and AB relative to that of the rest of the tangles. This is reasonable when one recognizes that both AB, a system of two qubits, and C, a D-dimensional system (in the case $D > 2$), have *entanglement capacities* [37] exceeding that of a single qubit.

The requirement that the I residual tangle be invariant under local unitary operations follows trivially, since each term on the right-hand side of Eq. (25) is known to satisfy this property individually. It is still an open question as to whether or not the proposed rescaling is sufficient to preserve positivity when generalizing the residual tangle, Eq. (24), to the I residual tangle, Eq. (25). However, numerical calculations give strong evidence that this is the case. The I residual tangle has been calculated for over 200×10^6 randomly generated pure states of a $2 \otimes 2 \otimes 3$ system and of a $2 \otimes 2 \otimes 4$ system, the only nontrivial possibilities. In each instance the resulting quantity has been positive. We conjecture that the I residual tangle satisfies the requirements of positivity and monotonicity under SLOCC not only for a $2 \otimes 2 \otimes D$ system, where closed forms currently exist for all of the terms on the right-hand side of Eq. (25), but for the most general $D_A \otimes D_B \otimes D_C$ dimensional tripartite system in an overall pure state [with the proper scaling of each term again given by Eq. (26)]. The I residual tangle arising in the context of the two-atom TCM is shown by the blue curves in Figs. 1(a)–3(a).

The residual tangle, as well as our proposed generalization of this quantity, may be interpreted as the irreducible tripartite entanglement in a system since it cannot be accounted for in terms of any combination of bipartite entanglement measures [22]. A slightly different and possibly more enlightening interpretation is that the I residual tangle quantifies the amount of ''freedom'' that a system has in satisfying the constraints imposed by the phenomenon of entanglement sharing. If the I residual tangle of a tripartite system is zero, then each bipartite tangle is uniquely deter-

mined by the values of all of the other bipartite tangles. Alternatively, if τ_{ABC} is strictly greater than 0, then the bipartite tangles enjoy a certain latitude in the values that each may assume while still satisfying the positivity criterion. The larger the value of the *I* residual tangle, the more freedom the system has in satisfying the entanglement sharing constraints. This reasoning highlights the relationship between entanglement sharing and the *I* residual tangle.

Finally, we may interpret the *I* residual tangle as the *average fragility* of a tripartite state under the loss of a single subsystem. That is, if one of the three subsystems is selected at random and discarded (or traced over), then the *I* residual tangle quantifies the amount of *bipartite* entanglement that is lost, on average. It is the existence of physically meaningful interpretations such as these which prompt us to postulate this measure of tripartite entanglement for a $2\otimes 2\otimes D$ system in an overall pure state, rather than to rely on previously defined measures based on normal forms [40] or on the method of hyperdeterminants [41], for example. At this point it is unclear what, if any, connection these entanglement monotones have to the entanglement that exists in different bipartite partitions of the system, a key ingredient in any discussion of entanglement sharing.

The constraint imposed by entanglement sharing on the values of the various bipartite tangles, each of which is known to be a positive function, is simply that Eq. (25) cannot be negative. It then follows that the strongest constraint of this form is placed on the two-atom TCM when the *I* residual tangle is equal to zero. This occurs (to a good approximation) periodically in Fig. 1(a) for the initial condition given by Eq. (6a). It is at these points that each bipartite tangle is uniquely determined in terms of the values of all of the other bipartite tangles. Conversely, at one-half of this period when the *I* residual tangle achieves its maximum value, the various bipartite partitions enjoy their greatest freedom with respect to how entanglement may be distributed throughout the system while still satisfying the entanglement sharing constraints. The distribution of correlations is, of course, still determined by the initial state of the system and by the TCM time evolution, both of which we consider to be separate constraints.

Similarly, the dotted (blue) curves in Figs. 2(a) and 3(a) show the evolution of the residual tangle for the initial states given by Eqs. (6b) and (6c), respectively. Note how the more complicated behavior resulting from an initial coherent-state field arises from a specific superposition of Fock states, the tangles of which all have a simple oscillatory evolution. This suggests that the phenomenon of entanglement sharing may offer a useful perspective from which to investigate the way in which the coherent-state evolution results from a superposition of Fock state evolutions.

The fact that the TCM Hamiltonian leads to a nonzero *I* residual tangle is interesting in its own right. Inspection of Eq. (1) shows that this model does not include a physical mechanism, e.g., a dipole-dipole coupling term enabling direct interaction between the two atoms in the ensemble, but only for coupling between the field and the atoms. Consequently, all interactions between the atoms are mediated by the electromagnetic field via the exchange of photons, and are in some sense indirect. This, however, turns out to be sufficient to allow genuine tripartite correlations to develop in the system, as evidenced by values of the *I* residual tangle that are strictly greater than zero.

Finally, we note that we have considered an alternative approach to understanding the constraints on the distribution of entanglement among the different subsystem partitions of the two-atom TCM by using the relative entropy of entanglement [38] as our entanglement measure. This quantity generalizes in a straightforward manner to the multipartite case [42], and has a clear physical interpretation relating the amount of entanglement in a state to its distance from the set of separable states [38]. The existence of upper and lower bounds in the tripartite case [43] yields another method by which to investigate the genuine three body entanglement arising in the two-atom TCM. The results of the relative entropy of entanglement approach will be presented in a subsequent paper.

VI. SUMMARY AND FUTURE DIRECTIONS

The two-atom Tavis-Cummings model provides the simplest example of a collection of two-level atoms, or qubits, sharing a common coupling to the electromagnetic field. A detailed understanding of the evolution of entanglement in different bipartite partitions of this model is valuable for both fundamental theoretical investigations, and for accurately describing the behavior of certain nontrivial, yet experimentally realizable systems. Our proposed generalization of the residual tangle augments the current formalism, and allows one to analyze the irreducible three-body correlations that arise in a broader class of tripartite systems, providing a tool useful for studying the phenomenon of entanglement sharing in the context of a physically relevant and accessible system.

In future work we hope to generalize this analysis to include ensembles with an arbitrary number of atoms. This will require further extensions of the tangle formalism in order to quantify both the entanglement in a mixed state of a bipartite system of arbitrary dimensions having a local rank greater than 2, and the multipartite entanglement in a system composed of more than three subsystems. Ultimately, we hope to connect this analysis to the phenomenon of quantum backaction on individual particles when the whole ensemble is measured. The tradeoff between the information gained about a system and the disturbance caused to that system is certainly fundamental to quantum mechanics [44–46]. However, the relationship of this tradeoff to multiparticle entanglement is far from clear. Such an understanding would not only be a crucial step in designing protocols for the quantum control of ensembles, but would also provide deeper insight into the nature of the correlations at the heart of quantum mechanics [47].

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