Multiqubit *W* states lead to stronger nonclassicality than Greenberger-Horne-Zeilinger states

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The N-qubit states of the W class, for $N>10$, lead to more robust (against noise admixture) violations of local realism, than the Greenberger-Horne-Zeilinger (GHZ) states. These violations are most pronounced for correlations for a pair of qubits, conditioned on specific measurement results for the remaining N-2 qubits. The considerations provide us with a *qualitative* difference between the W state and GHZ state in the situation when they are separately sent via depolarizing channels. For sufficiently high amount of noise in the depolarizing channel, the GHZ states cannot produce a distillable state between two qubits, whereas the W states can still produce a distillable state in a similar situation.

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I. INTRODUCTION

The Greenberger-Horne-Zeilinger (GHZ) states [1] give the maximal violation of correlation function Bell inequalities [2,3]. The second best known multiqubit states is the W family [4,5]. Their behavior is opposite in some respects to the GHZ family. For example, in the three qubit case, the W state has the maximal bipartite entanglement among all three qubit states [5], whereas the GHZ state has no bipartite entanglement (cf. Refs. [6,7]).

In this paper, we exhibit a kind of complementarity between the N-qubit W states and GHZ states from the perspective of robustness of the nonclassical correlations against white noise admixture [8–11]. For the N-qubit W state

$$|W_N\rangle = \frac{1}{\sqrt{N}}(|100\ldots00\rangle + |010\ldots00\rangle \ldots + |000\ldots01\rangle)$$

diluted by white noise, if measurements in a Bell-type experiment at $N-2$ parties are made in the computational basis and all yield the -1 result (associated with $|0\rangle$), the remaining pair of observers is left with a mixture of a 2-qubit Bell state with a substantially reduced amount of white noise. The probability of such a chain of events for the $N-2$ observers is quite low. Nevertheless, we shall show that such an event contradicts very strongly any local realistic description. In the case of the N-qubit GHZ state diluted by white noise, scenario of this kind can lead with unit probability to a 2-qubit Bell state without reduction of noise. Because of that, the N-qubit states of the W class, for $N>10$, can lead to more robust, against noise admixture, violations of local realism, than the N-qubit states of the GHZ class. We also show that if an N-qubit W state and a GHZ state are separately sent through similar depolarizing channels, and the 2-qubit state conditioned on measurements at the $N-2$ parties is considered, the rate of obtaining singlets (in specific distillation protocols) can be higher in the case of W state for a certain range of noise in the depolarizing channel. Importantly, in such scenarios, the W state performs better for *higher* levels of noise, and this feature grows with N. These results may be of importance in quantum cryptography and communication complexity [13,14]. We also show that these considerations lead to a relatively efficient entanglement witness.

II. VIOLATION OF LOCAL REALISM BY *W* STATES

To analyze how strongly the N-qubit W states violate local realism, we use the recently found multiqubit correlation function Bell inequalities which form a necessary and sufficient condition for the existence of a local realistic model for the correlation function in experiments with *two* local settings for each of the N observers [2,3].

An N-qubit state ρ can always be written down as

$$\frac{1}{2^N}\sum_{x_1,\ldots,x_N=0,x,y,z} T_{x_1\ldots x_N}\sigma_{x_1}^{(1)}\ldots\sigma_{x_N}^{(N)},$$

where $\sigma_0^{(k)}$ is the identity operator and $\sigma_{x_i}^{(k)}$'s ($x_i=x,y,z$) are the Pauli operators of the kth qubit. The coefficients

$$T_{x_1\ldots x_N}=\mathrm{tr}(\rho\sigma_{x_1}^{(1)}\ldots\sigma_{x_N}^{(N)}), \quad (x_i=x,y,z)$$

are elements of the N-qubit correlation tensor $\hat{T}$ and they fully define the N-qubit correlation function [3]. A sufficient condition for the N-qubit correlation function to satisfy all correlation function Bell inequalities is that for *any* set of local coordinate systems, one must have [3]

$$\sum_{x_1,\ldots,x_N=x,y} T^2_{x_1\ldots x_N}\leq 1, \tag{1}$$

the sum being taken over *any* set of orthogonal pairs of axes of the local coordinate systems of all observers.

Consider a mixture ρ_N^W, of W_N with white noise $\rho_{noise}^N = I^{(N)}/2^N$, where $I^{(N)}$ is the unit operator in the tensor product of Hilbert spaces of the qubits:

$$\rho_N^W = p_N |W_N\rangle\langle W_N| + (1-p_N)\rho_{noise}^{(N)}. \tag{2}$$

The parameter p_N will be called here "visibility." It defines to what extent the quantum processes associated with W_N are "visible" in those given by ρ_N^W. If the N-qubit correlations of a pure state $|\psi\rangle$ are represented by a correlation tensor $\hat{T}$, then the correlation tensor of a mixed state

$$p_N|\psi\rangle\langle\psi| + (1-p_N)\rho_{noise}^{(N)}$$

is given by $\hat{T}' = p_N \hat{T}$.

A. The case of three qubits

Consider now the mixture of the 3-qubit state $|W_3\rangle$ (of visibility p_3) with white noise. The correlation tensor of the pure state $|W_3\rangle$ is

$$\begin{aligned}\hat{T}^{W_3} &= \vec{z}_1\otimes\vec{z}_2\otimes\vec{z}_3 \\ &\quad -\tfrac{2}{3}(\vec{z}_1\otimes\vec{x}_2\otimes\vec{x}_3 + \vec{x}_1\otimes\vec{z}_2\otimes\vec{x}_3 + \vec{x}_1\otimes\vec{x}_2\otimes\vec{z}_3) \\ &\quad -\tfrac{2}{3}(\vec{z}_1\otimes\vec{y}_2\otimes\vec{y}_3 + \vec{y}_1\otimes\vec{z}_2\otimes\vec{y}_3 + \vec{y}_1\otimes\vec{y}_2\otimes\vec{z}_3),\end{aligned} \tag{3}$$

where $\vec{x}_i, \vec{y}_i, \vec{z}_i$ forms a Cartesian coordinate system with $\vec{z}_i$ defining the computational basis.

Let us find the maximal value of the left-hand side (lhs) of Eq. (1) for W_3. We will show that in an arbitrary set of local coordinate systems,

$$\sum_{i,j,k=x,y} T_{ijk}^2 \leqslant \frac{7}{3}.$$

The value of the lhs of Eq. (1), for the case when we are in the coordinate system in which Eq. (3) is written down, is zero. Moving into a different coordinate system can be always done for each observer using three Euler rotations [15]. Therefore, we shall assume that first all three observers perform a rotation about the axes $\vec{z}_i$, then around the new $\vec{x}_i'$ directions and finally around $\vec{z}_i'$ directions.

The set of first Euler rotations leaves the value of the lhs of Eq. (1) at zero. The second rotation around $\vec{x}_i'$ axes leads to following values of the components of the correlation tensor in the xy sector: T''_{xxx} is vanishing, whereas

$$T''_{yyx} = \tfrac{2}{3}(\sin\phi_{23}\sin\theta_1\cos\theta_2 - \sin\phi_{31}\cos\theta_1\sin\theta_2),$$

$$T''_{yxy} = \tfrac{2}{3}(\sin\phi_{32}\sin\theta_1\cos\theta_3 - \sin\phi_{21}\cos\theta_1\sin\theta_3),$$

$$T''_{xyy} = \tfrac{2}{3}(\sin\phi_{21}\sin\theta_3\cos\theta_2 - \sin\phi_{13}\cos\theta_3\sin\theta_2),$$

$$T''_{xxy} = \tfrac{2}{3}\cos\phi_{12}\sin\theta_3, \quad T''_{xyx} = \tfrac{2}{3}\cos\phi_{13}\sin\theta_2,$$

$$T''_{yxx} = \tfrac{2}{3}\cos\phi_{23}\sin\theta_1$$

(T''_{yyy} is not explicitly needed), where ϕ_i's are local angles of the first rotations and θ_i's are those for the second one and $\phi_{ij} = \phi_i - \phi_j$. Employing $(A\cos\eta + B\sin\eta)^2 \leqslant A^2 + B^2$ and that $T''^2_{yyy} \leqslant 1$, one gets

$$\begin{aligned}\sum_{i,j,k=x,y} T''^2_{ijk} \leqslant 1 + \tfrac{4}{9}(&\sin^2\phi_{31}\cos^2\theta_1 + \sin^2\phi_{23}\sin^2\theta_1 \\ &+ \sin^2\phi_{21}\cos^2\theta_1 + \sin^2\phi_{32}\sin^2\theta_1 \\ &+ \sin^2\phi_{13}\cos^2\theta_3 + \sin^2\phi_{21}\sin^2\theta_3 \\ &+ \cos^2\phi_{12}\sin^2\theta_3 + \cos^2\phi_{13}\sin^2\theta_2 \\ &+ \cos^2\phi_{23}\sin^2\theta_1).\end{aligned}$$

The right-hand (rhs) side of this inequality is a linear function in $\cos^2\theta_i$'s and $\cos^2\phi_{jk}$'s. Thus its maximal value is for extreme values of these parameters, and

$$\sum_{i,j,k=x,y} T''^2_{ijk} \leqslant \frac{7}{3}.$$

The last set of Euler rotations around the $\vec{z}_i'$ axes cannot change the value of the lhs of this relation. So, for any set of local coordinate systems, one has $\Sigma_{i,j,k=x,y} T_{ijk}^2 \leqslant 7/3$. This inequality is saturated, as in the system of coordinates in which Eq. (3) is written down,

$$\sum_{i,j,k=x,z} T_{ijk}^2 = \frac{7}{3}.$$

(Note that y is replaced by z, in the last equation.)

For the noisy 3-qubit W state, one has

$$\max \sum_{i,j,k=x,y} T_{ijk}^2 = \frac{7}{3} p_3^2,$$

and thus there is no violation of the correlation function Bell inequalities [2,3] for

$$p_3 \leqslant \sqrt{\frac{3}{7}} \approx 0.654\,654.$$

Note that at least one of the correlation function Bell inequalities is violated (as checked numerically) for $p_3 \geqslant 0.65\,664$ [16].

Surprisingly, if one takes a second look at the data, that can be acquired in a 3-qubit correlation experiment, with the noisy W_3 state, one can lower the bound for p_3 which allows for a local realistic description (cf. Ref. [17]). Suppose that in the Bell experiment, the observer 3 chooses as her/his measurements as follows: the first observable is the $\sigma_z^{(3)}$ operator, and the second one something else. The computational basis for the third qubit is the eigenbasis of $\sigma_z^{(3)}$. The measurements of $\sigma_z^{(3)} = |1\rangle\langle 1| - |0\rangle\langle 0|$ will cause collapses of the full state ρ_3^W [cf. Eq. (2)] into new states. Whenever the result is -1, the emerging state is of the following form: $\rho_{(3)}^{W_2} \otimes |0\rangle\langle 0|$, where the Werner state $\rho_{(3)}^{W_2}$ reads

$$\rho_{(3)}^{W_2} = p_{(3\to 2)}|W_2\rangle\langle W_2| + (1-p_{(3\to 2)})\rho_{noise}^{(2)}, \tag{4}$$

with $|W_2\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. The new visibility parameter is given by

$$p_{(3\to 2)}=\frac{4p_3}{3+p_3}\geqslant p_3. \quad (5)$$

The 2-qubit state $\rho_{(3)}^{W_2}$ has no local realistic description for

$$1\geqslant p_{(3\to 2)}\geqslant \frac{1}{\sqrt{2}}.$$

Therefore, for this range of $p_{(3\to 2)}$, the results received by the observers 1 and 2, which are conditioned on observer 3 getting the -1 result (when she/he measures $\sigma_z^{(3)}$) cannot have a local realistic model. We have a subset of the data in the full experiment with no local realistic interpretation. Surprisingly, the critical value of p_3 above which this phenomenon occurs is lower than $\sqrt{3/7}$,

$$p_3^{crit}=\frac{3}{4\sqrt{2}-1}\approx 0.644212<\sqrt{\frac{3}{7}}<0.65664, \quad (6)$$

and can be obtained by putting $p_{(3\to 2)}=1/\sqrt{2}$ in Eq. (5).

The correlation functions are averages of products of the local results, and as such do not distinguish the situation when, e.g., local results are $+1,+1,-1$ (for the respective observers), with the one when the results are $+1,-1,+1$. Therefore, an analysis of the results of the first two parties conditioned on the third party receiving, e.g., -1 can lead to a more stringent constraint on local realism. And this is exactly what we have received here. Note that no sequential measurements are involved (the third party may perform just one measurement). Communication between the parties is only after the experiment, just to collect the data.

For the W_3 state, this refinement of data analysis seems to be optimal. To test the ultimate critical value for p_3, we employed the numerical procedure based on linear optimization, which tests whether the full set of *probabilities* involved in an N-particle experiment admits an underlying local realistic model. The procedure has been described in many works [12], and therefore will not be given here. Since the program analyzes the full set of *probabilities*, its verdict is based on the *full* set of data available in the Bell-experiment. The program has found that Bell type experiments on a noisy W_3 state have always a local realistic description for p_3 below the numerical threshold of 0.644 212. That is, we have a full agreement, up to the numerical accuracy, with the result given in Eq. (6).

B. The case of N qubits

For $N>3$, the above phenomenon gets even more pronounced. The sufficient condition for the local realistic description (1) can be shown to be satisfied for

$$p_N\leqslant\sqrt{\frac{N}{3N-2}}. \quad (7)$$

The method that we have used to get Eq. (7) is the straightforward generalization of the Euler rotations method to the N-qubit case. Since

$$\lim_{N\to\infty}\sqrt{\frac{N}{3N-2}}=\frac{1}{\sqrt{3}},$$

explicit local realistic models for the N-qubit correlation functions (in a standard Bell experiment involving pairs of alternative observables at each site) exists for large N for p_3 as high as $1/\sqrt{3}$. We have also numerically found the threshold value of p_N, above which at least one of the correlation function Bell inequalities is violated. It does not differ too much from the right-hand side of Eq. (7) and, e.g., for $N=4$, it reads 0.634 08 and for $N=11$, it is 0.598 97.

However, the more refined method of data analysis leads to different results. Imagine now that the last $N-2$ observers have the σ_z observable within their local pair of alternative observables in the Bell test. Then, if all of them get the -1 result, the collapsed state will be given by

$$\rho_{(N)}^{W_2}\otimes(\otimes_{i=3}^{N}|0\rangle_{ii}\langle 0|),$$

with the 2-qubit state $\rho_{(N)}^{W_2}$ being

$$\rho_{(N)}^{W_2}=p_{(N\to 2)}|W_2\rangle\langle W_2|+(1-p_{(N\to 2)})\rho_{noise}^{(2)}, \quad (8)$$

where

$$p_{(N\to 2)}=\frac{p_N}{p_N+(1-p_N)\frac{N}{2^{N-1}}}. \quad (9)$$

Since for $p_{(N\to 2)}>1/\sqrt{2}$, no local realistic description of this subset of data is possible, the critical visibility p_N, which does not allow a local realistic model reads

$$p_N^{crit}=\frac{N}{(\sqrt{2}-1)2^{N-1}+N}. \quad (10)$$

Note that $p_N^{crit}\to 0$ when $N\to\infty$. For sufficiently large N the decrease has an exponential character! This behavior is strikingly different than the one for the threshold value of p_N which is sufficient to satisfy the N-qubit correlation function Bell inequalities (Fig. 1).

For low N, the value of p_N^{crit} was confirmed by the numerical procedure [12] mentioned earlier (which analyzes the full set of data for the problem). The critical numerical values are $p_4^{thr}=0.546\,918$, which is exactly equal to the 6-digit approximation of the p_4^{crit} value given by Eq. (10) and $p_5^{thr}=0.4300$, which is equal to the 4-digit approximation of p_5^{crit} of Eq. (10). This suggests a conjecture that Eq. (10) is the real threshold. However, if this is untrue, the decrease in p_N must be even bigger!

III. COMPARISON OF GHZ AND *W* STATES FROM THE PERSPECTIVE OF VIOLATION OF LOCAL REALISM

The GHZ states exhibit maximal violations of the correlation function Bell inequalities [2,3], and the violations

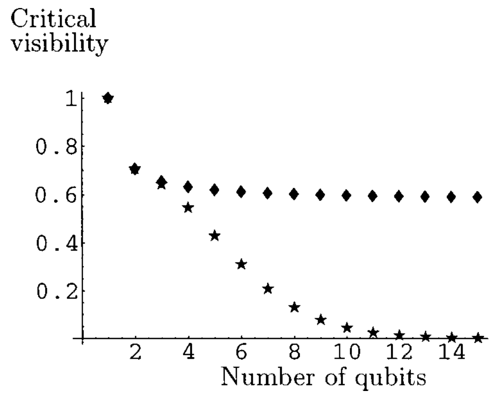

FIG. 1. The stars are a plot of critical visibility (a higher visibility gives violation of local realism) p_N^{crit} obtained by the method of projections and the diamonds are a plot of the values $\sqrt{N/(3N-2)}$ of p_N, below which exist a local hidden variable description of the N-qubit correlation function, for the noisy W state of N qubits.

when measured by the threshold visibility p_N^{GHZ} exhibit an exponential behavior. Let us now compare these two families. A noisy N-particle GHZ state, given by

$$\rho_N^{GHZ}=p_N'|GHZ\rangle\langle GHZ|+(1-p_N')\rho_{noise}^{(N)}$$

violates correlation function Bell inequalities whenever

$$p_N'>p_N^{GHZ}=\frac{1}{\sqrt{2^{N-1}}}.$$

We have performed for $N=3,4,5$, the numerical analysis of the possibility of the existence of a local realistic model for the full set of data for GHZ correlations, and the returned numerical critical values fully agree with this value. For $N \geqslant 11$, p_N^{crit} of Eq. (10) for the W states is lower than the one for GHZ ones. The 11 or more qubit W states violate local realism more strongly than their GHZ counterparts, and this increases exponentially.

Violation of local realism using functional Bell inequalities

It may seem that the GHZ states will regain their glory of being the most nonclassical ones, if one introduces more alternative measurements, than just two, for each observer. However, let us note that there exists a sequence of functional Bell inequalities [18] for N qubits, which involve the entire range of local measurements in one plane. We will now show that even if we consider these functional Bell inequalities, the W states remain more nonclassical than the GHZ states. But before that, we first briefly discuss the functional Bell inequalities.

1. The functional Bell inequalities

The functional Bell inequalities [18] essentially follow from a simple geometric observation that in any real vector space, if for two vectors h and q one has $\langle h|q\rangle<\|q\|^2$, then this immediately implies that $h\neq q$. In simple words, if the scalar product of two vectors has a lower value than the length of one of them, then the two vectors cannot be equal.

Let ϱ_N be a state shared between N separated parties. Let O_n be an arbitrary observable at the nth location ($n=1,\ldots,N$). The quantum-mechanical prediction E_{QM} for the correlation in the state ϱ_N, when these observables are measured, is

$$E_{QM}(\xi_1,\ldots,\xi_N)=\mathrm{tr}(O_1\ldots O_N\varrho_N), \qquad (11)$$

where ξ_n is the aggregate of the local parameters at the nth site. Our object is to see whether this prediction can be reproduced in a local hidden variable theory. A local hidden variable correlation in the present scenario must be of the form

$$E_{LHV}(\xi_1,\ldots,\xi_N)=\int d\lambda\,\rho(\lambda)\Pi_{n=1}^N I_n(\xi_n,\lambda), \qquad (12)$$

where $\rho(\lambda)$ is the distribution of the local hidden variables and $I_n(\xi_n,\lambda)$ is the predetermined measurement result of the observable $O_n(\xi_n)$ corresponding to the hidden variable λ.

Consider now the scalar product

$$\langle E_{QM}|E_{LHV}\rangle=\int d\xi_1\ldots d\xi_N E_{QM}(\xi_1,\ldots,\xi_N)\times E_{LHV}(\xi_1,\ldots,\xi_N) \qquad (13)$$

and the norm

$$\|E_{QM}\|^2=\int d\xi_1\ldots d\xi_N[E_{QM}(\xi_1,\ldots,\xi_N)]^2. \qquad (14)$$

If we can prove that a strict inequality holds, namely, for all possible E_{LHV}, one has

$$\langle E_{QM}|E_{LHV}\rangle\leqslant B, \qquad (15)$$

with the number $B<\|E_{QM}\|^2$, we will immediately have $E_{QM}\neq E_{LHV}$, indicating that the correlations in the state ϱ_N are of a different character than in any local realistic theory. We then could say that the state ϱ_N violates the "functional" Bell inequality (15), as this Bell inequality is expressed in terms of a typical scalar product for square integrable functions. Note that the value of the product depends on a continuous range of parameters (of the measuring apparatus) at each site.

2. Comparison of W state with GHZ state when the latter violates functional Bell inequalities

The critical visibility for which the GHZ state violates the functional Bell inequalities [18] is lower than that for which it violates the multiqubit two-settings Bell inequalities [2,3]. Precisely, the critical visibility above which a functional Bell inequality is violated by an N-qubit GHZ state is $2(2/\pi)^N$ [18,19]. This is obtained for measurement settings in the $x-y$ planes for all observers sharing the GHZ state. This critical visibility is better than that for violation of the multiqubit Bell inequalities for $N\geqslant 4$. Nevertheless, the W family leads to stronger violations of local realism for $N\geqslant 15$ [compare with Eq. (10)]. Interestingly, the W states do not violate the functional Bell inequalities involving all settings in one plane for $N>3$.

IV. THE CASE OF G STATES

It may seem that the results obtained in Sec. III depend on the fact that the GHZ state has an equal number of $|0\rangle$s and $|1\rangle$s, when expressed in the σ_z basis, while the W state has an asymmetry in this respect. In this section, we show that this is not the case.

Consider for example, the N ($\geqslant 3$) qubit state [20]

$$|G_N\rangle = \frac{1}{\sqrt{2}}(|W_N\rangle + |\overline{W}_N\rangle),$$

where $|\overline{W}_N\rangle$ is obtained by interchanging $|0\rangle$ and $|1\rangle$ in $|W_N\rangle$. For $N=2$, we define $|G_2\rangle = |W_2\rangle = |\overline{W}_2\rangle$. This state ($|G_N\rangle$) has an equal number of $|0\rangle$s and $|1\rangle$s, just as in the GHZ states. Now consider the state G_N admixed with white noise,

$$\rho_N^G = q_N|G_N\rangle\langle G_N| + (1-q_N)\rho_{noise}^N.$$

In a similar process as described before, if $N-2$ parties make measurements in the σ_z basis and when all of them obtain $+1$ or all of them obtain -1, the state obtained at the remaining two parties is the (2-qubit) Werner state [similarly as in Eq. (8)]

$$\rho_{(N)}^{G_2} = q_{(N\to 2)}|G_2\rangle\langle G_2| + (1-q_{(N\to 2)})\rho_{noise}^{(2)}$$

with

$$q_{(N\to 2)} = \frac{q_N}{q_N + (1-q_N)\dfrac{N}{2^{N-2}}}.$$

We can then proceed just as we did in Sec. III, in the case of W state.

The state $\rho_{(N)}^{G_2}$ has no local realistic description for $q_{(N\to 2)} > 1/\sqrt{2}$ and this implies that the state ρ_N^G cannot have a local realistic model for

$$q_N > q_N^{crit} \equiv \frac{N}{N + (\sqrt{2}-1)2^{N-2}}.$$

For $N\geqslant 13$,

$$q_N^{crit} \leqslant p_N^{GHZ}.$$

Therefore, for $N\geqslant 13$, the G states also violate local realism more strongly than GHZ states. However the nonclassicality in the G states is less pronounced than that in the W states. For the W states, the crossover was at $N=11$ (see Sec. III).

V. COMPARISON BETWEEN GHZ AND W STATES WITH RESPECT TO YIELD OF SINGLETS

The method of data analysis presented above implies another difference between the W states and the GHZ states. For a noisy GHZ state, if one of the observers performs a measurement in the basis $\{(|0\rangle \pm |1\rangle)/\sqrt{2}\}$, the projected state, has for the other $N-1$ observers again the form of a noisy GHZ state, and the visibility parameter is the *same* as before. Further, after $N-2$ observers perform measurements in this basis, whatever are their results, the last two observers share a noisy Bell state, with the same visibility as the original noisy GHZ state. In contrast, noisy (N-qubit) W states, upon a measurement of σ_z, by the first observer, resulting in -1, lead to a noisy W state ρ_{N-1}^W, for the remaining $N-1$ observers, of *increased visibility*, namely to

$$p'|W_{N-1}\rangle\langle W_{N-1}| + (1-p')\rho_{noise}^{(N)}$$

with

$$p' \equiv p_{(N\to N-1)} = \frac{p_N}{p_N + (1-p_N)\dfrac{N}{2(N-1)}} \geqslant p_N.$$

The other result $+1$ leads to a separable mixture of $|00\ldots 0\rangle$ with white noise.

If one's aim is to get a Bell state in the hands of just two observers of the required (high) visibility, say at least p_2^R, then this can always be achieved with some probability by taking a noisy W state of sufficiently many qubits, and performing $N-2$ measurements of σ_z on $N-2$ qubits. The success is conditioned on all results being -1. For the given p_2^R, the noisy W state of visibility p_N must be for the number of qubits N for which

$$p_2^R > \frac{p_N}{p_N + (1-p_N)\dfrac{N}{2^{N-1}}}.$$

Therefore, if one's aim is to have a Bell state between two observers with as high visibility as possible, one can send through a noisy channel a W_N state. From the noisy W_N state, in a probabilistic way, one can extract a high visibility Bell state. No such possibility exists for the GHZ state.

Consider now the situations when a large number of copies of the N-qubit W state and GHZ state are separately sent through a depolarizing channel of visibility p [21]. For W_N, using the method of projections, one is able to obtain with probability

$$\mathbb{P} = \frac{2p}{N + (1-p)/2^{N-2}},$$

a Bell state of visibility [see Eq. (9)]

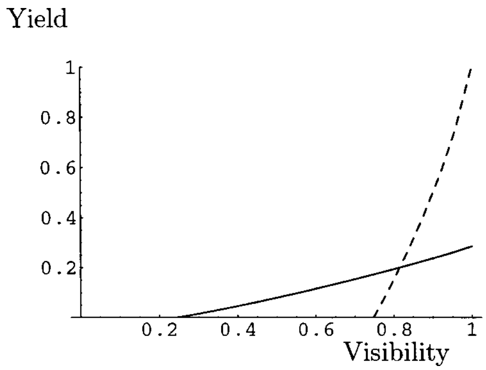

FIG. 2. Plot of visibility in the depolarizing channel vs the yield in the one-way hashing protocol. The undashed line is for the state W_7 while the dashed line is for GHZ_7.

$$v=\frac{p}{p+(1-p)\frac{N}{2^{N-1}}}. \tag{16}$$

For the state GHZ_N, one obtains with unit probability a Bell state with the original visibility *p*. We will now compare the yield of singlets in these two situations, by using (in both cases) two different protocols for distilling the resulting 2-qubit state, obtained after the specific projections on the multiqubit states.

A. One-way hashing protocol for distillation

Using the one-way hashing protocol for distillation [22], the per copy yield of singlets (of arbitrarily high fidelity) is

$$[1-S(\varrho(v))]\mathbb{P}$$

for the state W_N and $1-S(\varrho(p))$ for the state GHZ_N, where

$$\varrho(x)=x|W_2\rangle\langle W_2|+(1-x)\rho^{(2)}_{noise}$$

and

$$S(\eta)=-\mathrm{tr}(\eta\log_2\eta)$$

is the von Neumann entropy of η. As shown in Fig. 2 (for $N=7$), the yield for the *W* state is better for a large range of *p* than that for the GHZ state. This feature remains for all *N* and gets pronounced with increasing *N*. Even for $N=3$, there is a small range of *p*, in which the yield of singlets is higher for the *W* state. And importantly, the ranges in which the *W* states are better are for *higher* levels of noise.

B. Two-way recurrence-hashing distillation protocol

Similar features are obtained for *two-way* distillable entanglement. For example, although one-way distillable entanglement is vanishing for $\varrho(1/2)$, the two-way recurrence-hashing distillation protocol gives a positive yield [22]. To get $\varrho(1/2)$ from, say W_7, the visibility in the depolarizing channel can be as low as

$$0.098\,592.$$

[This value is obtained by putting $v=1/2$ and $N=7$ in Eq. (16).] For the same visibility in the channel, GHZ_7 produces the *separable* state $\varrho(0.098\,592)$ [23]. We therefore obtain a *qualitative* difference between the *W* state and the GHZ state in this respect.

VI. A SIMPLE ENTANGLEMENT WITNESS

Looking at the projected state can also serve as a simple entanglement witness. The state

$$\rho=\varepsilon\rho'+\frac{(1-\varepsilon)}{2^N}I_{2^N},$$

where ρ' is a normalized density matrix is separable for [24]

$$\varepsilon<\frac{1}{1+\frac{2^N}{2}}.$$

Choosing ρ' as the *N*-qubit *W* state and using Eq. (9), one obtains an upper bound

$$\frac{1}{1+\frac{2^N}{N}}$$

of the radius ε of the separable ball (as a noisy Bell state is separable for $p_{(N\rightarrow 2)}\leqslant 1/3$ [23]), which is of the same order as obtained in Ref. [24].

It is interesting whether states of other families have similar surprising properties, which can emerge after specific measurements by some of the observers.

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